Fluctuations in irreversible quantum Otto engines

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We derive the general probability distribution function of stochastic work for quantum Otto engines in which both the isochoric and driving processes are irreversible due to finite time duration. The time-dependent work fluctuations, average work, and thermodynamic efficiency are explicitly obtained for a complete cycle operating with an analytically solvable two-level system. The effects of the irreversibility originating from finite-time cycle operation on the thermodynamic efficiency, work fluctuations, and relative power fluctuations are discussed.

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I. INTRODUCTION

The second law of thermodynamics tells us that any heat engines working between a hot and a cold thermal bath of constant inverse temperatures β_h and β_c (with $\beta = 1/T$ and $k_B \equiv 1$) are not able to run more efficiently than a reversible Carnot cycle with its efficiency $\eta_C = 1 - \beta_h/\beta_c$. When a cyclic heat engine runs at the Carnot efficiency, its cycle operation consisting of consecutive thermodynamic processes must be reversible and thus the power output becomes null. Practically, heat engines must proceed in a finite-time period and produce finite power, and they do not attain the maximum efficiency due to entropy production quantifying irreversibility [1–13]. The performance in finite time was intensively studied for both quantum [5,11–14] and classical [2,3,7] engines. For an adequate description of heat engines, the effects of the irreversibility on the machine performance have to be considered by involving both heat-transfer and thermodynamic adiabatic processes.

The irreversibility in the classical and quantum engines basically comes from two important generic sources: finite-rate heat transfer and friction. For an endoreversible (classical [1] or quantum [5,6]) engine, which is modeled as being internally reversible, the irreversibility is exclusively induced by finite-rate heat transfer between the working substance and a heat reservoir. In both classical and quantum thermodynamics, the friction can be classified into external friction and internal friction [3,15]. While the external friction is associated with the exchange of energy via an external mechanical linkage to the surroundings, the internal friction is related to the dissipation of energy due to the timescale disparity between the internal dynamics and engine operation. A kind of friction can be traced to a quantum phenomenon that the driving Hamiltonian does not commutate with itself at different times [13]. Such friction is exclusively created by the finite-time driving in a unitary evolution process (which is thermodynamic adiabatic but not quantum adiabatic), and it is responsible for transitions among the instantaneous energy eigenstates [16–19] and related to quantum coherence in the energy basis [13]. The quantum engines operating in finite time were proposed for studying the effects of the quantum friction on the engine performance [4,5,18–23]. However, the explicit expressions for the performance parameters with the irreversibility existing in all four finite-time strokes have not been obtained so far.

Unlike in macroscopic systems where both work and heat are deterministic, the work [9,24–31] and heat [32–34] for microscale systems are random due to thermal and quantum fluctuations [14,19,24,35–37]. The statistics of either work or efficiency for heat engines at microscale were examined experimentally and theoretically [14,18,19,25,30,36,38–48], but under the assumption that either adiabatic [14,36] or isothermal strokes [19] are reversible. To our knowledge, the fluctuations have not been examined so far for an irreversible quantum heat engine where the isothermal and thermodynamic adiabatic strokes are of finite time and thus both of them are away from the reversible limit.

In this paper, we derive the general expressions for the probability distribution functions of quantum work and heat in quantum Otto engines [14,17–19,36,49,50] composed of two irreversible finite-time isochoric and two irreversible driving strokes. This distribution function allows us to obtain the quantum work statistics explicitly depending on the time evolution dynamics of the two isochoric and two driving processes. We then analytically examine the work statistics of an Otto cycle working with an exactly solvable two-level system. The effects of irreversibility induced by finite-time duration of either driving or isochoric strokes on machine performance and fluctuations are discussed. We finally demonstrate that irreversibility yields an increase in the work fluctuations and relative power fluctuations, but a decrease in machine efficiency.

II. MODEL

The model of a quantum Otto engine is sketched in Fig. 1. This machine consists of two isochoric branches, one with a

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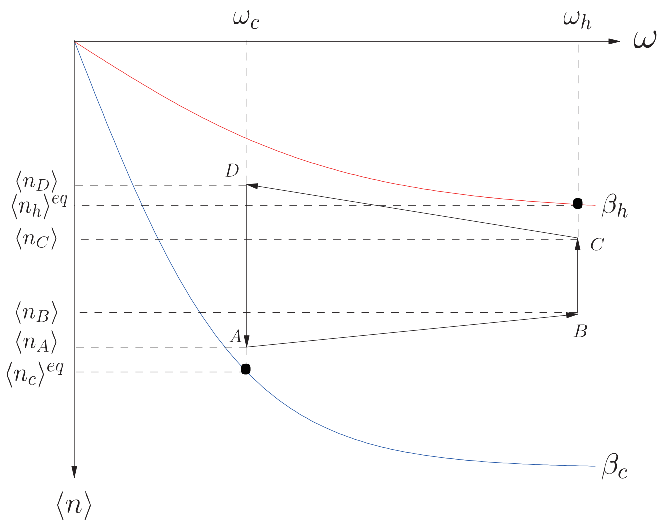

FIG. 1. Schematic diagram of a two-level quantum Otto cycle operating between a hot and a cold thermal bath of inverse constant inverse temperatures (β_h and β_c) in the (ω, $\langle n\rangle$) plane. The cycle consists of the two unitary strokes (connecting states A and B, and C and D), where the system is isolated from the two thermal baths but driven by the external field, and two isochoric strokes (connecting states B and C, and D and A), where the system is kept in thermal contact with the hot and the cold reservoir, respectively. The time durations along these four processes, $A \to B$, $B \to C$, $C \to D$, and $D \to A$, are τ_{ch}, τ_h, τ_{hc}, and τ_c, respectively. The average populations at the four instants A, B, C, D can be expressed as $\langle n_A\rangle = \langle n(0)\rangle$, $\langle n_B\rangle = \langle n(\tau_{ch})\rangle$, $\langle n_C\rangle = \langle n(\tau_{ch} + \tau_h)\rangle$, $\langle n_D\rangle = \langle n(\tau_{cyc} - \tau_c)\rangle$. The mean population $\langle n_A\rangle(\langle n_C\rangle)$ at the end of the cold (hot) isochoric stroke would approach its asymptotic value $\langle n_c\rangle^{\rm eq}$ ($\langle n_h\rangle^{\rm eq}$), when and only when the thermalization is completed.

hot and another with a cold thermal bath where the system frequency is kept constant, and two driving strokes, where the system is isolated from the two heat reservoirs and it undergoes unitary evolution due to external driving. The four branches can be described as follows.

During the adiabatic compression $A \to B$, the system is isolated from the two thermal baths but driven by the external field from time $t = 0$ to $t = \tau_{ch}$, and its Hamiltonian changes from $H(0) = H_c$ to $H(\tau_{ch}) = H_h$. The driving time period τ_{ch} is much shorter than the timescale of typical decoherence, implying that this irreversible expansion can still be described by unitary evolution [18]. We introduce the inverse temperature of the system at initial state (β_A) to express the thermal occupation probabilities at instant A in the canonical form [4,5]

$$p_n^0 = \frac{e^{-\beta_A E_n^c}}{Z_A}, \tag{1}$$

where $Z_A = \sum_n e^{-\beta_A E_n^c}$ is the canonical partition function and E_n^c is the eigenenergy of the system Hamiltonian [$H(0) = H_c$] at instant A. The transition probability from eigenstate $|n\rangle$ to $|m\rangle$ is given by

$$p_{n\to m}^{\tau_{ch}} = |\langle n|U_{\rm com}|m\rangle|^2, \tag{2}$$

where $U_{\rm com}$ denotes the unitary time evolution operator along the compression. For a quantum adiabatic process, the timescale of the change in the state must be larger than that of the dynamical one, $\sim\hbar/\langle H\rangle$, such that the generic quantum adiabatic condition [51] is satisfied. A quantum adiabatic process and a thermodynamic adiabatic process are "adiabatic" in two different senses. Along a quantum adiabatic process, the transitions among instantaneous eigenstates do not happen and the transition probability becomes $p_{n\to m}^{\tau_{ch}} = \delta_{nm}$, with the Kronecker delta function δ, leading to no heat exchange. As a thermodynamic adiabatic process merely indicates that the system is isolated from the external heat reservoir and no heat is exchanged between the system and the heat reservoir, it does not necessarily require the state transitions to be forbidden, thereby indicating that a thermodynamic adiabatic process includes a quantum adiabatic process [52]. Specifically, when the process $A \to B$ as a unitary evolution occurs fast, the quantum adiabatic condition is not satisfied and the internal excitations occur [15] ($0 < p_{n\to m}^{\tau_{ch}} = |\langle n|U_{\rm com}|m\rangle|^2 < 1$). In such a case, no heat is exchanged between the system and the reservoir, but internal excitations related to inner friction accounts for internal irreversibility and entropy production. This irreversible process is not quantum adiabatic but thermodynamic adiabatic.

The next step is the hot isochore $B \to C$, where the system (with constant frequency $\omega = \omega_h$ and constant Hamiltonian $H = H_h$) is in contact with the hot thermal bath of constant inverse temperature β_h in a time duration τ_h. For the stochastic heat q_h absorbed by the hot reservoir, its probability distribution can be determined by the conditional probability to obtain

$$p(q_h) = \sum_{k,l} \delta\big[q_h - \big(E_l^h - E_k^h\big)\big] p_{k\to l}^{\tau_h}, \tag{3}$$

where E_k^h and E_l^h are the respective eigenenergies of the Hamiltonian H at the beginning and at the end of the hot isochoric stroke. As the system occupies state l at end of the isochore, $p_{k\to l}^{\tau_h}$ denotes the probability of the system being in state l. If the time duration τ_h satisfies the complete thermalization condition $\tau_h \gg \tau_{h,\rm relax}$, where $\tau_{h,\rm relax}$ is the relaxation time for the system with the hot reservoir, the system achieves a thermal equilibrium state at the ending instant C after the complete thermalization, and then $p_{k\to l}^{\tau_h} = e^{-\beta_h E_l^h}/Z_h$ with partition function $Z_h = \sum_l e^{-\beta_h E_l^h}$. For the engines under consideration, the finite-time isochoric stroke implies that the complete thermalization condition is not satisfied and thus the working substance cannot achieve thermal equilibrium even at the end of the stroke. For this incomplete thermalization, we let β_C be the effective temperature of the working substance at instant C so that $p_{k\to l}^{\tau_h} = e^{-\beta_C E_l^h}/Z_C$ with partition function $Z_C = \sum_l e^{-\beta_C E_l^h}$. The temperature β_C would approach the value of the reservoir temperature β_h when the thermalization becomes complete.

During the expansion $C \to D$, the system is isolated from these two thermal baths in a time period τ_{hc} while its frequency changes back to ω_h from ω_c. Like in the compression, there is a transition probability from initial state i to final one

j, which reads

$$p_{i\to j}^{\tau_{hc}} = |\langle i|U_{\rm exp}|j\rangle|^2, \tag{4}$$

where $U_{\rm exp}$ is the time evolution operator along the expansion. This transition probability $p_{i\to j}^{\tau_{hc}} = \delta_{ij}$ if and only if the quantum adiabatic condition is satisfied. Here $p_{i\to j}^{\tau_{hc}}$ is situated between $0 < p_{i\to j}^{\tau_{hc}} < 1$ due to finite-time driving, indicating that the finite-time unitary process is thermodynamic adiabatic but not quantum adiabatic. Similar to driving stroke $A \to B$, the transitions among the eigenstates of the system, which are related to inner friction, result in the irreversible entropy production. At the initial instant C the occupation probabilities take the form $p_i^{\tau_{ch}+\tau_h} = \delta_{il} p_{k\to l}^{\tau_h}$, where $p_{k\to l}^{\tau_h}$ was defined below Eq. (3).

On the fourth branch $D \to A$, the system with constant frequency $\omega = \omega_c$ is coupled to a cold reservoir of inverse temperature β_c in a time period τ_c. After a cycle with the total period $\tau_{cyc} = \tau_{ch} + \tau_h + \tau_{hc} + \tau_c$ the system returns to its initial state A, and we can easily determine the quantum heat q_c in a similar way to the quantum heat q_h. As emphasized, after the finite-time system-bath interaction interval, the system cannot reach thermal equilibrium with the cold reservoir, and the occupation probabilities satisfy the form (1) due to incomplete thermalization. If the time duration $\tau_c \gg \tau_{c,\rm relax}$, where $\tau_{c,\rm relax}$ is the relaxation time corresponding to the cold isochore, the system achieves thermal equilibrium at instant A and the occupation probabilities become $p_n^{0,eq} = e^{-\beta_c E_n^c}/Z_c$, with the partition function $Z_c = \sum_n e^{-\beta_c E_n^c}$.

In view of the fact that the work per cycle is produced only in the two driving strokes $A \to B$ and $C \to D$, the quantum work output can be obtained by determining the total work produced along these two microscopic trajectories to arrive at

$$\begin{aligned} &w[n(0) \to m(\tau_{ch}); i(\tau_{ch}+\tau_h) \to j(\tau_{cyc}-\tau_c)] \\ &\quad = \left(E_i^h - E_j^c\right) - \left(E_m^h - E_n^c\right), \end{aligned} \tag{5}$$

where E_n^c and E_m^h (E_i^h and E_j^c) denote the respective energy eigenvalues at the initial and final instants of the compression (expansion). Here $n(t)$ are quantum numbers indicating the states occupied by the system at time t. The work distribution of a quantum heat engine can be derived by using the quantum trajectory [46] or characteristic function [18,48]. The characteristic function of the work probability distribution along the compression can be given by [18] $\chi_{\rm com}(u) = {\rm Tr}[U_{\rm com} e^{-iuH_c}\rho(0)(e^{-iuH_h}U_{\rm com})^\dagger] = \sum_{n,m} p_n^0 p_{n\to m}^{\tau_{ch}} e^{iu(E_m^h - E_n^c)}$, where u is the conjugate variable to w and $\rho(0) = \sum_n p_n^0 |n\rangle\langle n|$ is the density matrix at instant A. Analogously, the expression of $\chi_{\rm exp}(u)$ for the expansion can be written as $\chi_{\rm exp}(u) = {\rm Tr}[U_{\rm exp} e^{-iuH_h}\rho(\tau_{hc}+\tau_h)(e^{-iuH_c}U_{\rm exp})^\dagger] = \sum_{n,m} p_n^{\tau_{ch}+\tau_h} p_{n\to m}^{\tau_{hc}} e^{iu(E_m^c - E_n^h)}$. The states $|n(0)\rangle$ and $|m(\tau_{ch})\rangle$ can be assumed to be independent of $|m(\tau_{ch}+\tau_h)\rangle$ and $|n(\tau_{cyc}-\tau_c)\rangle$, since the system would relax to thermal equilibrium in an isochoric process if its time duration is long enough. The characteristic function for the total work produced can thus be written as the product of characteristic functions for both of the two driving strokes: $\chi(u) = \chi_{\rm com}(u)\chi_{\rm exp}(u)$. Using the inverse Fourier transform of $\chi(u)$, $p(w) = \int du\, \chi(u) e^{iuw}$, we can obtain the main result of this paper, namely, the following expression for the probability distribution of the work w:

$$\begin{aligned} p(w) = &\sum_{m,n,i,j} p_{n\to m}^{\tau_{ch}} p_n^0 p_{i\to j}^{\tau_{hc}} p_i^{\tau_{ch}+\tau_h} \delta\{w - w[n(0) \to m(\tau_{ch}); \\ &\times i(\tau_{ch}+\tau_h) \to j(\tau_{cyc}-\tau_c)]\}. \end{aligned} \tag{6}$$

This probability distribution function for the irreversible Otto engines allows us to determine all moments of quantum work: $\langle w^k\rangle = \int w^k p(w) dw$ $(k = 1, 2, \ldots)$. For an endoreversible model, where the two adiabatic strokes are isentropic but two isochoric processes take finite time, we recover the work fluctuations [14] by setting $p_{n\to m}^{\tau_{ch}} = \delta_{nm}$ and $p_{i\to j}^{\tau_{hc}} = \delta_{ij}$, $p(w) = \sum_{n,i} p_n^0 p_i^{\tau_{ch}+\tau_h} \delta\{w - w[n(0); i(\tau_{ch}+\tau_h)]\}$. The result further reduces to that previously obtained in a quasistatic [40] Otto engine, where $p_i^{\tau_{ch}+\tau_h} = e^{-\beta_h E_i}/Z_h$ and $p_n^0 = e^{-\beta_c E_n}/Z_c$ for complete isochoric thermalization (with $\tau_c \gg \tau_{c,\rm relax}$ and $\tau_h \gg \tau_{h,\rm relax}$). We present it here in a broader context by arguing that irreversibility is unavoidable in a finite-time cyclic engine where both the driving and system-bath interaction steps are away from the quasistatic limit.

III. QUANTUM OTTO ENGINE WORKING WITH A TWO-LEVEL SYSTEM

We now consider a quantum Otto engine that works with a two-level system of the eigenenergies $E_+ = \hbar\omega/2$ and $E_- = -\hbar\omega/2$. As the occupation probabilities at these two eigenstates are $p_+ = e^{-\beta\hbar\omega_c/2}/Z_A$ and $p_- = e^{\beta\hbar\omega_c/2}/Z_A$, where the partition function $Z_A = e^{-\beta_A\hbar\omega_c/2} + e^{\beta_A\hbar\omega_c/2} = 2\cosh\left(\frac{\beta_A\hbar\omega_c}{2}\right)$, the mean population at instant A (with time $t = 0$) can be determined by using $\langle n\rangle = \sum_n n p_n$ to arrive at

$$\langle n(0)\rangle = -\frac{1}{2}\tanh\left(\frac{\beta_A\hbar\omega_c}{2}\right). \tag{7}$$

The mean population at final instant B can then be determined according to $\langle n(\tau_{ch})\rangle = \sum_{n,m} n p_{m\to n}^{\tau_{ch}} p_m^0$, which together with Eqs. (1) and (2) leads to

$$\langle n(\tau_{ch})\rangle = (1 - 2\xi)\langle n(0)\rangle, \tag{8}$$

where $\xi = |\langle\pm|U_{\rm com}|\mp\rangle|^2$. Using $\langle n(\tau_{cyc}-\tau_c)\rangle = \sum_{n,m} n p_{m\to n}^{\tau_{hc}} p_m^{\tau_{ch}+\tau_h}$, for the unitary expansion $C \to D$ there is the relation

$$\langle n(\tau_{cyc}-\tau_c)\rangle = (1 - 2\xi)\langle n(\tau_{ch}+\tau_h)\rangle, \tag{9}$$

where we have introduced the adiabaticity parameter which reads $\xi = |\langle\pm|U_{\rm exp}|\mp\rangle|^2 = |\langle\pm|U_{\rm com}|\mp\rangle|^2$. The parameter ξ represents the probability of transition between states $|+\rangle$ and $|-\rangle$ during the compression or expansion, and the probability of no state transition is accordingly $|\langle\pm|U_{\rm exp}|\pm\rangle|^2 = |\langle\pm|U_{\rm com}|\pm\rangle|^2 = 1 - \xi$. Here ξ depends on the speed at which the driving process is performed [11,17–19], and depends on the form of the driving Hamiltonian that generates coherence in the energy basis for finite-time operation. This adiabaticity parameter ξ decreases with a larger time duration of the driving stroke, although not monotonically [17,18], and it must be vanishing in the quantum adiabatic case when the time duration τ_{ch} or τ_{hc} is long enough in order for quantum adiabatic condition to be satisfied. For the finite-time driving stroke, the rapid change in frequency ω leads to inner friction and results in possible state transitions ($\xi > 0$) [16,17,19–21].

Such nonadiabatic internal dissipation accounts for irreversible entropy production and leads to an increase in mean population $\langle n\rangle$ (see Fig. 1). Since the mean population $\langle n\rangle$ for the two-level system is bounded by $-1/2 < \langle n\rangle \leqslant 0$, we therefore have the relation: $0 \leqslant \xi < 1/2$.

We are interested in the finite-time operation of the Otto engine in which the isochoric processes are far away from quasistatic limit, and thus complete thermalization cannot be achieved for the system. Using the master equation of stochastic thermodynamics, one can find that the mean populations $\langle n(0)\rangle$ and $\langle n(\tau_{ch}+\tau_h)\rangle$ can be expressed in terms of the corresponding asymptotic equilibrium values $\langle n_c\rangle^{\rm eq}$ and $\langle n_h\rangle^{\rm eq}$ (see Appendix A),

$$\langle n(0)\rangle = \langle n_c\rangle^{\rm eq} + [\langle n(\tau_{cyc})\rangle - \langle n_c\rangle^{\rm eq}]e^{-\gamma_c\tau_c}, \tag{10}$$

$$\langle n(\tau_{ch}+\tau_h)\rangle = \langle n_h\rangle^{\rm eq} + [\langle n(\tau_{ch})\rangle - \langle n_h\rangle^{\rm eq}]e^{-\gamma_h\tau_h}. \tag{11}$$

Using Eqs. (8), (9), (10), and (11), $\langle n(0)\rangle$ and $\langle n(\tau_{ch}+\tau_h)\rangle$ can be rewritten as

$$\langle n(0)\rangle = \langle n_c\rangle^{\rm eq} + \Delta_c,\ \langle n(\tau_{ch}+\tau_h)\rangle = \langle n_h\rangle^{\rm eq} + \Delta_h, \tag{12}$$

where

$$\Delta_h = \frac{(2\xi-1)[(2\xi-1)\langle n_h\rangle^{\rm eq} + \langle n_c\rangle^{\rm eq}] - x_c[\langle n_h\rangle^{\rm eq} + (2\xi-1)\langle n_c\rangle^{\rm eq}]}{x_hx_c-(2\xi-1)^2}, \tag{13}$$

$$\Delta_c = \frac{(2\xi-1)[(2\xi-1)\langle n_c\rangle^{\rm eq} + \langle n_h\rangle^{\rm eq}] - x_h[\langle n_c\rangle^{\rm eq} + (2\xi-1)\langle n_h\rangle^{\rm eq}]}{x_hx_c-(2\xi-1)^2}. \tag{14}$$

Here $x_h \equiv e^{\gamma_h\tau_h}$ and $x_c \equiv e^{\gamma_c\tau_c}$ denote the effective time durations along the hot and cold isochoric branches, respectively. Here $\langle n_c\rangle^{\rm eq}$ and $\langle n_h\rangle^{\rm eq}$ are achieved in the reversible, quasistatic limit when $x_h, x_c \to \infty$ leads to $\Delta_{h,c} \to 0$, whether ξ is zero or not. However, Δ_c and Δ_h are still positive for finite values of x_c and x_h even for two reversible driving processes with $\xi \to 0$. These corrections Δ_c and Δ_h indicate how far the heat-transfer processes deviate from the reversible limit, and imply that irreversibility is exclusively caused by heat transferred between the system and the thermal bath. Such a deviation is quite natural in both quantum heat engines and classical context when the heat-transfer processes are irreversible due to finite-time duration.

For this two-level engine, the probability distribution of quantum work Eq. (6) can be analytically obtained as

$$\begin{aligned} p(w) = &\left[\tfrac{1}{2} + 2\langle n(0)\rangle\langle n(\tau_{ch}+\tau_h)\rangle(1-2\xi) - (1-\xi)\xi\right]\delta(w) \\ &+ \tfrac{1}{2}[\{1-2\langle n(0)\rangle\}(1-\xi)\xi]\delta(w+\hbar\omega_c) \\ &+ \tfrac{1}{2}[\{2\langle n(\tau_{ch}+\tau_h)\rangle+1\}(1-\xi)\xi]\delta(w-\hbar\omega_h) \\ &+ \tfrac{1}{4}[\{2\langle n(\tau_{ch}+\tau_h)\rangle-1\}\{2\langle n(0)\rangle-1\}\xi^2]\delta(w+\hbar\omega_h+\hbar\omega_c) \\ &+ \tfrac{1}{4}[1-\{2\langle n(\tau_{ch}+\tau_h)\rangle\}\{2\langle n(0)\rangle+1\}(1-\xi)^2]\delta(w+\hbar\omega_h-\hbar\omega_c) \\ &+ \tfrac{1}{4}[\{2\langle n(\tau_{ch}+\tau_h)\rangle+1\}\{1-2\langle n(0)\rangle\}(1-\xi)^2]\delta(w+\hbar\omega_c-\hbar\omega_h) \\ &+ \tfrac{1}{2}[\{2\langle n(0)\rangle+1\}(1-\xi)\xi]\delta(w-\hbar\omega_c) \\ &+ \tfrac{1}{4}[\{2\langle n(0)\rangle+1\}\{2\langle n(\tau_{ch}+\tau_h)\rangle+1\}\xi^2]\delta(w-\hbar\omega_c-\hbar\omega_h) \\ &+ \tfrac{1}{2}[\{1-2\langle n(\tau_{ch}+\tau_h)\rangle\}(1-\xi)\xi]\delta(w+\hbar\omega_h). \end{aligned} \tag{15}$$

The stochastic work can take nine different discrete values as shown Fig. 2. The following should be noted regarding the stochastic work per cycle: (i) $w=0$ indicates that the stochastic work by the system along the expansion is fully counterbalanced by that during the compression. (ii) $w=\hbar\omega_c-\hbar\omega_h$ ($w=\hbar\omega_h-\hbar\omega_c$) corresponds to the case when the system jumps down (up) from a high energy (low energy) state to a low energy (high energy) one along the driving stroke, namely, $n=m=1/2$ but $i=j=-1/2$ ($n=m=-1/2$ and $i=j=1/2$) in Eq. (6). These values exist in the adiabatic or nonadiabatic driving. Unlike the average work done by the system which must be positive work in an expansion, the quantum work can be negative even in an expansion due to quantum fluctuations. (iii) There are more values of stochastic work in the nonadiabatic driving than in the adiabatic case due to nonadiabatic transitions. (iv) Finally the distribution $p(w)$ is expected to be normalized to 1 for either adiabatic or nonadiabatic driving.

Using Eq. (3), the average heat injection, $\langle q_h\rangle = \int q_hp(q_h)dq_h$, can be obtained as

$$\langle q_h\rangle = \hbar\omega_h[\langle n(\tau_{ch}+\tau_h)\rangle - (1-2\xi)\langle n(0)\rangle]. \tag{16}$$

By using simple algebra (see Appendix B for details), the average work ($\langle w\rangle$) and the work fluctuations ($\delta w^2 = \langle w^2\rangle - \langle w\rangle^2$) can be obtained as

$$\langle w\rangle = \hbar[\omega_c-(1-2\xi)\omega_h]\langle n(0)\rangle + \hbar[\omega_h-(1-2\xi)\omega_c]\langle n(\tau_{ch}+\tau_h)\rangle, \tag{17}$$

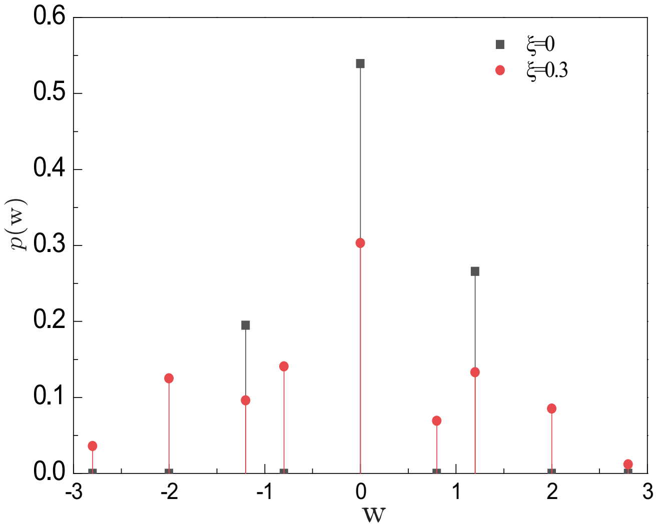

FIG. 2. The probability distribution $p(w)$ of the quantum work for the adiabatic (with $\xi = 0$, black squares) and nonadiabatic (with $\xi = 0.3$, red dots) steps. The parameters are $x_c = x_h = 8$, $\omega_c = 0.4\omega_h = 0.4$, $\beta_h = 0.2\beta_c = 0.2$, and $\hbar = 2$.

$$\langle w\rangle = \hbar(\omega_h - \omega_c)[\langle n(\tau_{ch} + \tau_h)\rangle - \langle n(0)\rangle] + 2\hbar\xi[\omega_h\langle n(0)\rangle + \omega_c\langle n(\tau_{ch} + \tau_h)\rangle], \tag{18}$$

and

$$\begin{aligned}\delta w^2 =& \hbar^2\omega_h^2\left[\tfrac{1}{2} - \langle n(0)\rangle^2(1-2\xi)^2 - \langle n(\tau_{ch}+\tau_h)\rangle^2\right]\\ &+ \hbar^2\omega_c^2\left[\tfrac{1}{2} - \langle n(0)\rangle^2 - (1-2\xi)^2\langle n(\tau_{ch}+\tau_h)\rangle^2\right]\\ &- \hbar^2\omega_c\omega_h(1-2\xi)[1 - 2\langle n(0)\rangle^2 - 2\langle n(\tau_{ch}+\tau_h)\rangle^2].\end{aligned} \tag{19}$$

Taking into account Eq. (18) for $\langle w\rangle > 0$, we find that the positive work condition can be given by

$$\xi < \xi^+ \equiv \frac{1}{2}\frac{(\omega_c - \omega_h)[\langle n(\tau_{ch}+\tau_h)\rangle - \langle n(0)\rangle]}{\omega_c\langle n(0)\rangle + \omega_h\langle n(\tau_{ch}+\tau_h)\rangle}, \tag{20}$$

which must be satisfied in order for the work to be extracted from the heat engines.

From Eqs. (16) and (18), the machine efficiency defined by $\eta = \langle w\rangle/\langle q_h\rangle$ can be expressed as

$$\eta = 1 - \frac{\omega_c}{\omega_h}\frac{\langle n(0)\rangle - (1-2\xi)\langle n(\tau_{ch}+\tau_h)\rangle}{(1-2\xi)\langle n(0)\rangle - \langle n(\tau_{ch}+\tau_h)\rangle}. \tag{21}$$

In case both isochoric and driving processes proceed in finite time, internal dissipation and uncomplete thermalization occur in the system, resulting in the thermodynamic efficiency (21) that depends on the time evolution along each cycle, except if these four processes are infinitely long, making the efficiency reduce to the one for cycles with complete thermalization, $\eta = 1 - \frac{\omega_c}{\omega_h}\frac{(1-2\xi)\langle n_h\rangle^{eq} - \langle n_c\rangle^{eq}}{\langle n_h\rangle^{eq} - (1-2\xi)\langle n_c\rangle^{eq}}$ [11,17,18], or the one for models without internal dissipation [4,5,17], $\eta = 1 - \frac{\omega_c}{\omega_h}$.

The efficiency η and work fluctuations δw^2 as a function of the inverse temperature β_c of the cold bath is shown in Figs. 3(a) and 3(b). When decreasing temperature, the efficiency η increases but the work fluctuations δw^2 decrease. The efficiency at low temperatures is larger than that at high temperatures, showing that the quantum effects which are of significance in the low temperature regime can improve the machine efficiency. Because the quantum fluctuations characterizing the low temperature domain are smaller than the thermal fluctuations dominating the high temperature region, the work fluctuations δw^2 increase while the temperature is increased and vice versa. Now let us consider fixed inverse bath temperatures (β_c and β_h). In this case, the efficiency and work fluctuations behave as a monotonic function of the adiabacity parameter ξ as displayed in Figs. 3(c) and 3(d). We note from Fig. 3(c) that the efficiency is nonpositive if the positive work condition (20) is violated. The increase in the adiabacity parameter ξ yields a decrease (an increase) in the machine efficiency (work fluctuations) as it should.

As no specific form of the driving Hamiltonian is employed throughout our engine model, we do not obtain the adiabacticity parameter ξ in terms of the expansion and compression Hamiltonian driving times τ_{ch} and τ_{hc}. We do not determine the power and power fluctuations; instead we analyze the relative power fluctuations that are independent of the specific driving Hamiltonians. Since the stochastic power is the work divided by the cycle period τ_{cyc}, namely, $\dot{w}[|n(0)\rangle; |n(\tau_{ch} + \tau_h)\rangle] = w[|n(0)\rangle; |n(\tau_{ch} + \tau_h)\rangle]/\tau_{cyc}$, the relative fluctuations of the power $f_{\dot{w}}$ are equivalent to the corresponding ones of work f_w, namely, $f_{\dot{w}} = f_w = \sqrt{\delta w^2}/\langle w\rangle$. For nonadiabatic driving branches (with constant ξ), the relative power fluctuations are increasing with decreasing effective time durations x_c and x_h; see Figs. 4(a) and 4(b). That is, when speeding up the isochoric branches, both the average power and relative power fluctuations are increasing. Figure 5 shows the relative fluctuations $f_{\dot{w}}$ as a function of the adiabaticity parameter ξ. Our calculation shows that the relative fluctuations always take positive values if the positive work condition (20) (i.e., $\xi < \xi^+$) is satisfied. In contrast, if this condition is violated, these relative fluctuations are negative (or divergent) due to nonpositive power. In the physical regime in which the model

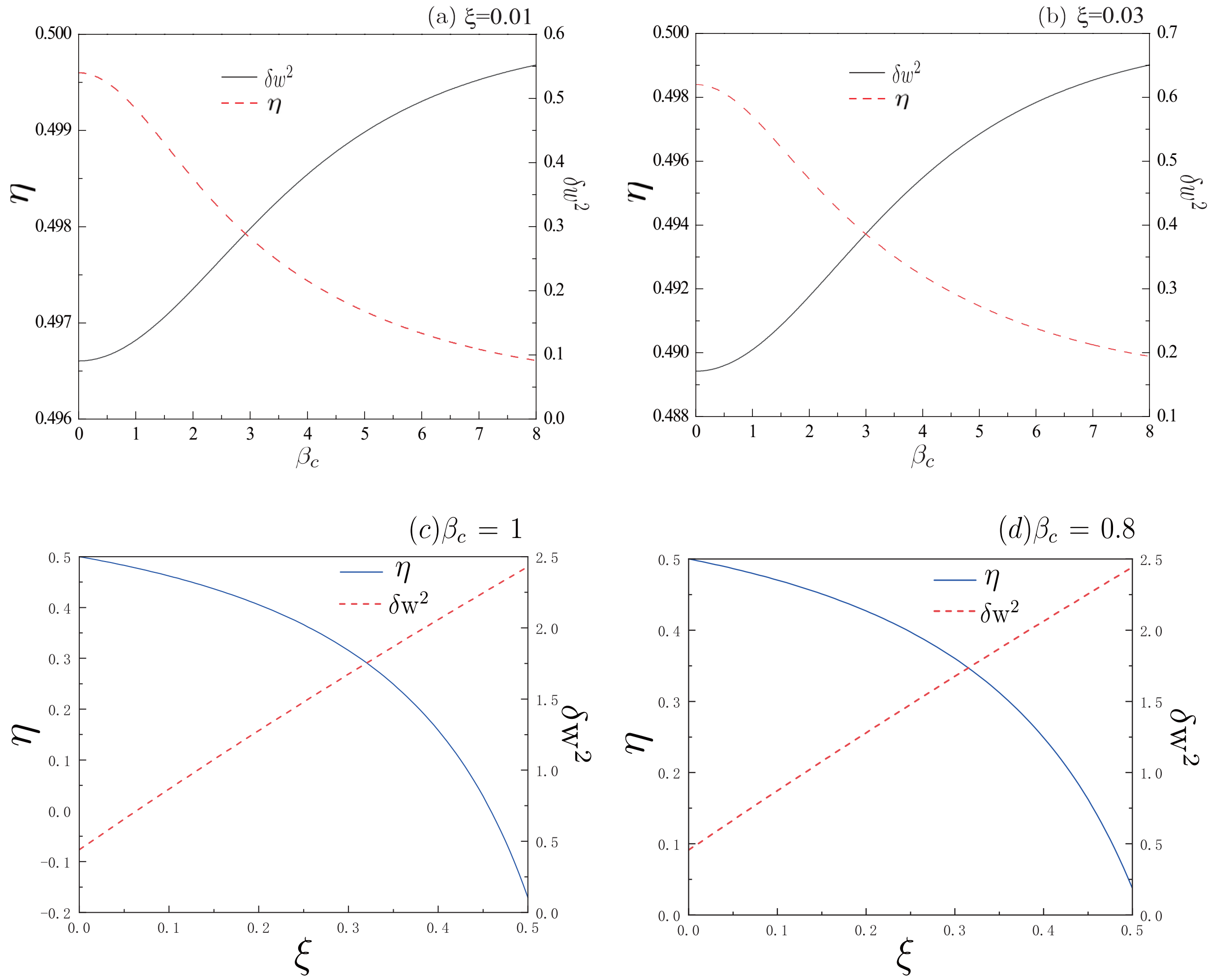

FIG. 3. Efficiency and work fluctuations. (a), (b) The efficiency η and the work fluctuations δw^2 as a function of the inverse temperature of the cold bath, $\beta_c = 5\beta_h$, for $\xi = 0.01$ and $\xi = 0.03$, respectively. (c), (d) The efficiency η and the work fluctuations δw^2 as a function of the adiabaticity parameter ξ for $\beta_c = 1$ and $\beta_c = 0.8$ (with $\beta_h = 0.2\beta_c$), respectively. In all figures, the efficiency and work fluctuations are indicated by blue solid lines and red dashed lines, respectively. The parameters are $x_c = x_h = 8$, $\omega_h = 2\omega_c = 1$, and $\hbar = 2$.

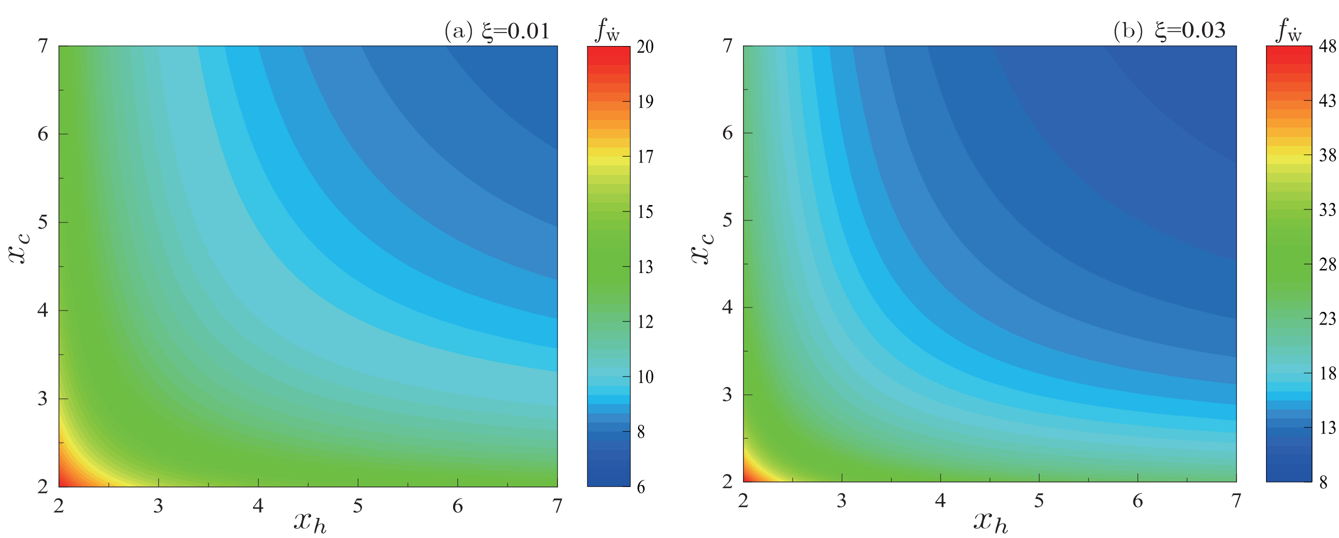

FIG. 4. Contour plot of the relative fluctuations of the power $f_{\dot{w}}$ in the effective time duration (x_h, x_c) plane for a nonadiabatic driving, with $\xi = 0.01$ (a) and $\xi = 0.03$ (b). The parameters are are $\beta_h = 0.2\beta_c = 0.2$, $\omega_h = 2\omega_c = 1$, and $\hbar = 2$.

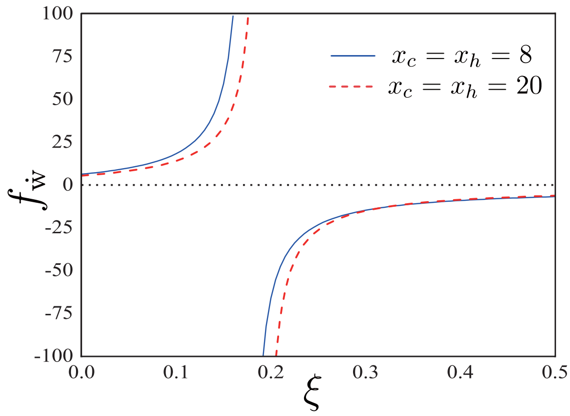

FIG. 5. Relative power fluctuations $f_{\dot{w}}$ as a function of the adiabaticity parameter ξ for $x_h = x_c = 8$ (blue solid lines) and $x_h = x_c = 20$ (red dashed lines). The parameters are $\beta_c = 10\beta_h = 1$, $\omega_h = 2.5\omega_c = 1$, and $\hbar = 2$.

operates as a heat engine by producing positive work, the relative power fluctuations are increasing with increasing ξ. We note that the upper limit of the adiabaticity parameter, ξ^+, increases as time durations (τ_c and τ_h) of isochoric strokes increase. Figure 5, together with Figs. 4(a) and 4(b), shows that the irreversibility either induced by finite heat flux between the system and the bath or caused by internal irreversible dissipation yields larger power fluctuations than those in the reversible cycle operation.

IV. DISCUSSIONS AND CONCLUSIONS

Previous studies [14,18,40] assumed complete thermalization that excludes the irreversibility occurring in the finite-time heat-transfer process studied in our work. Quantum heat engines in which both thermodynamic adiabatic branches and heat-transfer branches proceed in finite time are an interesting issue that requires further work. We made the simplifying assumption that the system would thermalize to a temperature different from one of the heat reservoir due to finite-time operation. When removing this assumption, the independent state approximation for the two driving strokes must break down. A natural extension of our work would be inclusion of correlation between probability distributions of work in two driving processes due to finite-time incomplete thermalization. Moreover, for systems far way from equilibrium, the fluctuations originating from isochoric branches through quantum coherence due to incomplete thermalization must be included, and a generalization of our calculation to such kind of fluctuations makes this an interesting line of future study.

In summary, we derived the probability distribution of stochastic work of quantum Otto engines working within a cycle period of finite time that leads to irreversibility in both the two isochoric and two driving processes. Employing a two-level system as the working substance of these engines, we find that, although the average work is positive, the quantum work may be negative due to quantum fluctuations. Afterwards, we derived the analytical expressions for work and efficiency, all of which are dependent on the time allocations to the four thermodynamic processes. We finally determined statistics of work as well as power at any finite temperatures, and revealed the effects of irreversibility induced by finite-time cycle operation on these work and power statistics.

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APPENDIX A: TIME EVOLUTION OF POPULATION ALONG AN ISOCHORIC PROCESS

The dynamics of the system with energy quantization along the system-bath interaction interval can be described by changes in the occupation probabilities p_n at states $n = 0, 1, 2, \ldots$. In this reduced description, the dynamical response of the heat reservoir is cast in kinetic terms. The master equation is given by [35,53]

$$\dot{p}_n = \sum_{n'} W_{n,n'} p_{n'}, \tag{A1}$$

where the transition rate matrix $W_{n,n'}$ must satisfy $\sum_n W_{n,n'} = 0$. For the system in contact with a heat bath of constant temperature β, we assume that the transition rates from state n' to n, $W_{nn'}$, fulfill the detailed balance $W_{nn'} e^{-\beta E_{n'}} = W_{n'n} e^{-\beta E_n}$, ensuring that the system can achieve asymptotically the thermal state after an infinitely long system-bath interaction duration. At thermal state, the occupation probabilities p_n achieve their asymptotic stationary values p_n^{eq}. These p_n^{eq} can be determined from the steady-state solution of Eq. (A1) and given by the Boltzmann distribution: $p_n^{\text{eq}}(\beta) = e^{-\beta E_n}/Z$, where $Z = \sum_n e^{-\beta E_n}$ is the canonical partition function.

For the two-level system where the energy spectrum reads $E_+ = \frac{1}{2}\hbar\omega$ and $E_- = -\frac{1}{2}\hbar\omega$, the master equation Eq. (A1) becomes

$$\begin{pmatrix} \dot{p}_+ \\ \dot{p}_- \end{pmatrix} = \begin{pmatrix} -W_{-+} & W_{+-} \\ W_{-+} & -W_{+-} \end{pmatrix} \begin{pmatrix} p_+ \\ p_- \end{pmatrix}, \tag{A2}$$

where these two transition rates W_{-+} and W_{+-} obey the detailed balance, $W_{-+}/W_{+-} = e^{-\beta\hbar\omega}$. From Eq. (A2), the motion for the average population can be obtained as

$$\langle \dot{n} \rangle = -\gamma(\langle n \rangle - \langle n \rangle^{\text{eq}}), \tag{A3}$$

where $\gamma = W_{-+} + W_{+-}$ and

$$\langle n \rangle^{\text{eq}} = -\frac{1}{2}\frac{W_{-+} - W_{+-}}{W_{-+} + W_{+-}} = -\frac{1}{2}\tanh\left(\frac{1}{2}\beta\hbar\omega\right). \tag{A4}$$

From Eq. (A3), we find that instantaneous population $\langle n(t) \rangle$ along the thermalization process (staring at initial time $t = 0$) can be written in terms of the population $\langle n(0) \rangle$,

$$\langle n(t) \rangle = \langle n \rangle^{\text{eq}} + [\langle n(0) \rangle - \langle n \rangle^{\text{eq}}] e^{-\gamma t}. \tag{A5}$$

APPENDIX B: RELATION BETWEEN POPULATIONS AT THE BEGINNING AND THE END OF A UNITARY DRIVING PROCESS

We consider the unitary time evolution along the driving compression $A \to B$ from $t = 0$ to $t = \tau_{ch}$ to determine $\langle n^2(0)\rangle$ and $\langle n^2(\tau_{ch})\rangle$ at A and B, respectively. Using $p_n^0 = e^{-\beta_A n\hbar\omega_c}/Z_A$ with $Z_A = e^{-\beta_A\hbar\omega_c/2} + e^{\beta_A\hbar\omega_c/2}$, it follows that $\langle n^2(0)\rangle = \sum_n n^2 p_n^0 = 1/4$. Meanwhile, $\langle n^2(\tau_{ch})\rangle$ can be calculated as

$$\begin{aligned}\langle n^2(\tau_{ch})\rangle &= \sum_{n,m} n^2 p_{m\to n}^{\tau_{ch}} p_m^0(\beta_A) = \sum_{n,m} n^2 |\langle m|U_{\rm com}|n\rangle|^2 p_m^0(\beta_A)\\ &= \frac{1}{4Z_A}[e^{-\beta_A\hbar\omega_c/2}(|\langle +|U_{\rm com}|+\rangle|^2 + \langle +|U_{\rm com}|-\rangle|^2) + e^{\beta_A\hbar\omega_c/2}(|\langle -|U_{\rm com}|+\rangle|^2 + \langle -|U_{\rm com}|-\rangle|^2)]\\ &= \langle n^2(0)\rangle = \frac{1}{4},\end{aligned} \tag{B1}$$

and $\langle n(0)n(\tau_{ch})\rangle$ reads

$$\begin{aligned}\langle n(0)n(\tau_{ch})\rangle &= \sum_{n,m} nm p_{\rm narrowm}^{\tau_{ch}} p_n^0(\beta_A) = \sum_{n,m} nm|\langle n|U_{\rm com}|m\rangle|^2 p_n^0(\beta_A)\\ &= \frac{1}{4Z_A}[e^{-\beta_A\hbar\omega_c/2}(|\langle +|U_{\rm com}|+\rangle|^2 - \langle +|U_{\rm com}|-\rangle|^2) + e^{\beta_A\hbar\omega_c/2}(-|\langle -|U_{\rm com}|+\rangle|^2 + \langle -|U_{\rm com}|-\rangle|^2)] = \frac{1}{4}(1-2\xi),\end{aligned} \tag{B2}$$

where $\xi = |\langle\pm|U_{\rm exp}|\mp\rangle|^2 = |\langle\pm|U_{\rm com}|\mp\rangle|^2$ and $1-\xi = |\langle\pm|U_{\rm exp}|\pm\rangle|^2 = |\langle\pm|U_{\rm com}|\pm\rangle|^2$ have been used. For the unitary expansion $C \to D$ of the two-level engine, we therefore have

$$\langle n^2(\tau_{cyc}-\tau_c)\rangle = \langle n^2(\tau_{ch}+\tau_h)\rangle = \tfrac{1}{4} \tag{B3}$$

and

$$\langle n(\tau_{ch}+\tau_h)n(\tau_{cyc}-\tau_c)\rangle = \tfrac{1}{4}(1-2\xi). \tag{B4}$$

Integrating over the probability distribution function Eq. (15) in the main text, the first two central moments of quantum work can be calculated as

$$\begin{aligned}\langle w\rangle &= \int wp(w)dw\\ &= \int w\,dw \sum_{m,n,i,j} p_{n\to m}^{\tau_{ch}} p_n^0 p_{i\to j}^{\tau_{hc}} p_i^{\tau_{ch}+\tau_h}\delta\{w - w[n(0)\to m(\tau_{ch}); i(\tau_{ch}+\tau_h)\to j(\tau_{cyc}-\tau_c)]\}\\ &= \int w\,dw \sum_{m,n,i,j} p_{n\to m}^{\tau_{ch}} p_n^0 p_{i\to j}^{\tau_{hc}} p_i^{\tau_{ch}+\tau_h}\delta\{w - [(E_i^h - E_j^c) - (E_m^h - E_n^c)]\}\\ &= \int w\,dw \sum_{m,n,i,j} p_{n\to m}^{\tau_{ch}} p_n^0 p_{i\to j}^{\tau_{hc}} p_i^{\tau_{ch}+\tau_h}\delta\{w - \hbar[(i\omega_h - j\omega_c) - (m\omega_h - n\omega_c)]\}\\ &= \hbar[(\omega_c - (1-2\xi)\omega_h]\langle n(0)\rangle + \hbar[(\omega_h - (1-2\xi)\omega_c]\langle n(\tau_{ch}+\tau_h)\rangle\end{aligned} \tag{B5}$$

and

$$\begin{aligned}\langle w^2\rangle &= \int w^2 p(w)dw\\ &= \int w^2 dw \sum_{m,n,i,j} p_{n\to m}^{\tau_{ch}} p_n^0 p_{i\to j}^{\tau_{hc}} p_i^{\tau_{ch}+\tau_h}\delta\{w - w[n(0)\to m(\tau_{ch}); i(\tau_{ch}+\tau_h)\to j(\tau_{cyc}-\tau_c)]\}\\ &= \int w^2 dw \sum_{m,n,i,j} p_{n\to m}^{\tau_{ch}} p_n^0 p_{i\to j}^{\tau_{hc}} p_i^{\tau_{ch}+\tau_h}\delta\{w - [(E_i^h - E_j^c) - (E_m^h - E_n^c)]\}\\ &= \int w^2 dw \sum_{m,n,i,j} p_{n\to m}^{\tau_{ch}} p_n^0 p_{i\to j}^{\tau_{hc}} p_i^{\tau_{ch}+\tau_h}\delta\{w - \hbar[(i\omega_h - j\omega_c) - (m\omega_h - n\omega_c)]\}\\ &= \hbar^2\omega_h^2\langle n^2(\tau_{ch}+\tau_h)\rangle - 2\hbar^2\omega_c\omega_h\langle n(\tau_{ch}+\tau_h)n(\tau_{cyc}-\tau_c)\rangle - 2\hbar^2\omega_c\omega_h\langle n(0)n(\tau_{ch})\rangle\\ &\quad + \hbar^2\omega_c^2\langle n^2(0)\rangle + 2\hbar^2[\omega_c - (1-2\xi)\omega_h]\langle n(0)\rangle[\omega_h - (1-2\xi)\omega_c]\langle n(\tau_{ch}+\tau_h)\rangle\\ &\quad + \hbar^2\omega_c^2\langle n^2(\tau_{cyc}-\tau_c)\rangle + \hbar^2\omega_h^2\langle n^2(\tau_{ch})\rangle.\end{aligned} \tag{B6}$$

With the above results, the second moment of stochastic work can be simplified as

$$\langle w^2\rangle = 2\hbar^2\Big\{\tfrac{1}{4}\big[\omega_h^2+\omega_c^2-2\omega_c\omega_h(1-2\xi)\big]+[\omega_c-(1-2\xi)\omega_h] \times \langle n(0)\rangle[\omega_h-(1-2\xi)\omega_c]\langle n(\tau_{ch}+\tau_h)\rangle\Big\}. \tag{B7}$$

Combining this with Eq. (B5), the work fluctuations, $\delta w^2 = \langle w^2\rangle - \langle w\rangle^2$ can be obtained as Eq. (19) in the main text.

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