Quantum phase transitions mediated by clustered non-Hermitian degeneracies

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(Received 13 November 2020; accepted 24 February 2021; published 15 March 2021)

The phenomenon of degeneracy of an N-plet of bound states is studied in the framework of the quasi-Hermitian (a.k.a. $\mathcal{PT}$-symmetric) formulation of quantum theory of closed systems. For a general non-Hermitian Hamiltonian $H = H(\lambda)$ such a degeneracy may occur at a real Kato's exceptional point $\lambda^{(\mathrm{EPN})}$ of order N and of the geometric multiplicity *alias* clusterization index K. The corresponding unitary process of collapse (loss of observability) can be then interpreted as a generic quantum phase transition. The dedicated literature deals, predominantly, with the non-numerical benchmark models of the simplest processes where $K = 1$. In our present paper it is shown that in the "anomalous" dynamical scenarios with $1 < K \leqslant N/2$ an analogous approach is applicable. A multiparametric anharmonic-oscillator-type exemplification of such systems is constructed as a set of real-matrix N by N Hamiltonians which are exactly solvable, maximally non-Hermitian, and labeled by specific *ad hoc* partitionings $\mathcal{R}(N)$ of N.

DOI: 10.1103/PhysRevE.103.032120

I. INTRODUCTION

The experimentally highly relevant phenomenon of a quantum phase transition during which at least one of the observables loses its observability status is theoretically elusive. In the conventional nonrelativistic Schrödinger picture, for example, the observability of the energy represented by a self-adjoint Hamiltonian $H = H(\lambda)$ appears too robust for the purpose. In phenomenological applications, for this reason, the onset of phase transitions is simulated, typically, by an abrupt, discontinuous change of the operators at $\lambda = \lambda^{(\mathrm{critical})}$ [1,2].

In relativistic quantum mechanics the situation is, paradoxically, under better theoretical control. In the Klein-Gordon equation with Coulomb potential, for example, the energy levels can merge and complexify at a finite, *dynamically* determined critical strength $\lambda^{(\mathrm{critical})}$ of the attraction [3]. The operator representing energy remains unchanged. One must only reconstruct a correct, sophisticated, Hamiltonian-dependent physical Hilbert space of states $\mathcal{H} = \mathcal{H}(\lambda)$ offering a consistent probabilistic interpretation of the system up to $\lambda^{(\mathrm{critical})}$ [4].

The latter result is one of applications of the recent innovative formulation of quantum theory in which a preselected (i.e., relativistic as well as nonrelativistic) Hamiltonian $H = H(\lambda)$ need not be self-adjoint (concerning this theory we shall add more details below; preliminarily, interested readers may consult, say, one of reviews [5–9]). This means that even for non-Hermitian Hamiltonians (with real spectra) and even in the dynamical regime close to $\lambda^{(\mathrm{critical})}$, there exists a feasible strategy of making the evolution of the corresponding quantum system unitary.

In our recent paper [10] we described a few basic aspects of implementation of the latter model-building strategy in the case of quantum systems close to their phase transition. The key technical aspects of the theory were illustrated using the N-by-N–matrix Hamiltonians $H^{(N)}(\lambda)$ for which the critical parameter can be identified with a Kato's [11] exceptional point of order N (EPN; $\lambda^{(\mathrm{critical})} \equiv \lambda^{(\mathrm{EPN})}$). In such an arrangement Hamiltonians $H^{(N)}(\lambda)$ possessed the real and nondegenerate spectra at $\lambda < \lambda^{(\mathrm{EPN})}$ while exhibiting, in the EPN limit, the characteristic phase-transition behavior (a detailed explanation will be given in Sec. II below).

The method used in paper [10] was perturbative so that the specification of the eligible non-Hermitian Hamiltonians $H(\lambda)$ was merely indirect. Thus, the results were complemented by a subsequent technical paper [12] in which several closed-form Hamiltonians were described. Unfortunately, in another, purely numerical study of the phase-transition problem [13] it has been revealed that from the mathematical as well as experiment-oriented points of view the class of models considered in Refs. [10,12] must be declared too narrow. In light of this observation the tone of the overall conclusions was rather discouraging. The existence of "not quite expected technical subtleties" has been emphasized, with the "word of warning ...supported by an explicit ill-behaved illustrative matrix model" [13].

In our present paper we will prolong the latter studies but, first of all, we will strongly oppose the skepticism of their conclusions. We will, first of all, broaden the class of Hamiltonians in a way which will fill the gaps (see Sec. III). As an unexpected mathematical by-product of these efforts, an unusual exhaustive combinatorial classification of our class of models will be formulated and summarized in Appendix.

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In Sec. IV we shall show that the family of our present solvable anharmonic-oscillator-type benchmark models covers all of the mathematically admissible realizations of the EPN-related quantum phase transitions. An explicit sample of our exhaustive classification pattern will be presented, in Sec. V, up to $N = 8$. The overall discussion (emphasizing, e.g., that our classification is non-numerical, circumventing all of the above-mentioned ill-conditioning dangers) and a concise summary will be finally added in Secs. VI and VII.

II. QUANTUM PHASE TRANSITIONS

The main weakness of the models of quantum phase transitions as presented in our preceding papers [10,12] lies in a restriction of their scope to a fairly small subset of the mathematically admissible scenarios. An explanation of this restriction becomes facilitated when one turns attention to the models which are exactly solvable.

A. Exceptional points of order N in solvable non-Hermitian models

In the simplest example taken from [12] the overall discussion of the phase transition processes was based on a detailed analysis of the N-by-N tridiagonal-anharmonic-oscillator (TAO) Hamiltonians

$$H^{(N)}_{(\mathrm{TAO})}(\lambda) = \begin{bmatrix} 1-N & b_1(\lambda) & 0 & \cdots & 0 \\ -b_1(\lambda) & 3-N & \ddots & \ddots & \vdots \\ 0 & -b_2(\lambda) & \ddots & b_2(\lambda) & 0 \\ \vdots & \ddots & \ddots & N-3 & b_1(\lambda) \\ 0 & \cdots & 0 & -b_1(\lambda) & N-1 \end{bmatrix}. \tag{1}$$

Such a model can be interpreted, after an inessential shift of the origin of the energy scale, as an antisymmetric-matrix perturbation of a truncated diagonal matrix form of harmonic oscillator. In the weak-coupling limit the spectrum is known of course. In the opposite, strongly anharmonic dynamical regime the localization of the spectrum becomes, in general, numerical. The Hamiltonian becomes dominated by its off-diagonal part which is chosen real, antisymmetric (i.e., maximally non-Hermitian) and, for the reasons explained in [14], $\mathcal{PT}$ symmetric, i.e., symmetric with respect to the second diagonal.

For the purposes of study of the mechanisms of a unitary passage of quantum systems through their EPN singularities the latter tridiagonal models proved particularly suitable because in a broad range of matrix element functions $b_j(\lambda)$ their spectra remained real. After a more restricted choice of functions $b_j(\lambda)$ these spectra appeared real up to $\lambda = \lambda^{(\mathrm{EPN})}$ (cf. [15]). At the critical value of $\lambda^{(\mathrm{EPN})}$ all of the N energy levels merged while they ceased to be real beyond $\lambda^{(\mathrm{EPN})}$.

From our present point of view the most relevant feature of the model is that at any integer $N \geqslant 2$ the Hamiltonians degenerate, in the phase-transition EPN limit, to the respective nondiagonalizable but elementary, closed-form N-by-N matrices

$$H^{(2)}_{(\mathrm{TAO})}(\lambda^{(\mathrm{EP2})}) = \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix},$$

$$H^{(3)}_{(\mathrm{TAO})}(\lambda^{(\mathrm{EP3})}) = \begin{bmatrix} -2 & \sqrt{2} & 0 \\ -\sqrt{2} & 0 & \sqrt{2} \\ 0 & -\sqrt{2} & 2 \end{bmatrix},$$

$$H^{(4)}_{(\mathrm{TAO})}(\lambda^{(\mathrm{EP4})}) = \begin{bmatrix} -3 & \sqrt{3} & 0 & 0 \\ -\sqrt{3} & -1 & 2 & 0 \\ 0 & -2 & 1 & \sqrt{3} \\ 0 & 0 & -\sqrt{3} & 3 \end{bmatrix},$$

$$H^{(5)}_{(\mathrm{TAO})}(\lambda^{(\mathrm{EP5})}) = \begin{bmatrix} -4 & 2 & 0 & 0 & 0 \\ -2 & -2 & \sqrt{6} & 0 & 0 \\ 0 & -\sqrt{6} & 0 & \sqrt{6} & 0 \\ 0 & 0 & -\sqrt{6} & 2 & 2 \\ 0 & 0 & 0 & -2 & 4 \end{bmatrix}, \tag{2}$$

etc. [12]. The respective spectra degenerate to a single real value,

$$\lim_{\lambda\to\lambda^{(\mathrm{EPN})}} E_n(\lambda) = \eta, \quad n = 1, 2, \ldots, N. \tag{3}$$

This is precisely the mathematical feature of the spectrum which finds its physical interpretation of an instant of quantum phase transition [16].

In our particular models the value of $\eta = 0$ remains N independent. The complete EPN degeneracy of energies (3) proves accompanied by the complete degeneracy of all of the related eigenvectors,

$$\lim_{\lambda\to\lambda^{(\mathrm{EPN})}} |\psi^{(N)}_n(\lambda)\rangle = |\chi^{(N)}(\lambda)\rangle, \quad n = 1, 2, \ldots, N. \tag{4}$$

For our forthcoming analysis of the mechanisms of the limiting loss-of-the-observability transition it will be vital to know that all of the matrix limits (2) can easily be transformed to their unique, "canonical," N-by-N Jordan-matrix respective forms

$$J^{(N)}(\eta) = \begin{bmatrix} \eta & 1 & 0 & \cdots & 0 \\ 0 & \eta & 1 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & 0 & \eta & 1 \\ 0 & \cdots & 0 & 0 & \eta \end{bmatrix}. \tag{5}$$

The transformations are mediated by the Schrödinger-like equation

$$H^{(N)}_{(\mathrm{TAO})}(\lambda^{(\mathrm{EPN})})\, Q^{(N)} = Q^{(N)}\, J^{(N)}(\eta). \tag{6}$$

For models (2), all of the transition matrices $Q^{(N)}$ defined by this equation are available in closed form (see [12]).

B. Numerical models and clustered non-Hermitian degeneracies

The empirical observations published in the strictly numerically oriented paper [13] indicate that the tridiagonal-matrix choice of models (1) may be over-restrictive. Such a suspicion results from the completeness of the degeneracy (4) of the eigenvectors. Indeed, from the point of view of linear algebra or functional analysis such a type of degeneracy represents a special case [11]. In general one can only expect a weaker form of such a degeneracy in which the eigenvectors $|\psi_n^{(N)}(\lambda)\rangle$ of the Hamiltonian would form a K-plet of clusters of the degenerating eigenvectors with the set of subscripts $n = 1, 2, \dots, N$ decomposed into K nonoverlapping subsets S_k,

$$\lim_{\lambda\to\lambda^{(\mathrm{EPN})}} |\psi_{n_k}^{(N)}(\lambda)\rangle = |\chi_k^{(N)}(\lambda)\rangle, \quad n_k \in S_k, \quad k = 1, 2, \dots, K. \tag{7}$$

In light of this observation the results of papers [10,12] must be reinterpreted as covering only a subfamily of all of the possible EPN-related quantum phase transitions. Any complete set of benchmark Hamiltonians $H^{(N)}(\lambda)$ must contain matrices which are more general than tridiagonal. In what follows our attention will be, therefore, redirected to the full real matrices of an analogous, antisymmetrically perturbed general anharmonic oscillator (GAO) form

$$H_{(\mathrm{GAO})}^{(N)}(\lambda) = \begin{bmatrix} 1-N & b_1(\lambda) & c_1(\lambda) & d_1(\lambda) & \cdots & \omega_1(\lambda) \\ -b_1(\lambda) & 3-N & b_2(\lambda) & c_2(\lambda) & \ddots & \vdots \\ -c_1(\lambda) & \ddots & \ddots & \ddots & \ddots & d_1(\lambda) \\ -d_1(\lambda) & \ddots & -b_3(\lambda) & N-5 & b_2(\lambda) & c_1(\lambda) \\ \vdots & \ddots & -c_2(\lambda) & -b_2(\lambda) & N-3 & b_1(\lambda) \\ -\omega_1(\lambda) & \cdots & -d_1(\lambda) & -c_1(\lambda) & -b_1(\lambda) & N-1 \end{bmatrix}. \tag{8}$$

Our present paper could be read, in this light, as a continuation and as an ultimate completion of the project of Ref. [13]. *A priori*, one might be critical towards such a project. Indeed, from the numerical experiments as performed in [13] it can be deduced that at the larger dimensions N the work with tridiagonal Hamiltonians is the only constructively tractable option. In other words, the study of the EPN-supporting toy-model Hamiltonians $H^{(N)}(\lambda)$ should work either with the general tridiagonal matrices as sampled above, or with some of their nontridiagonal generalizations defined at a few smallest matrix dimensions N (thus, e.g., the "easily tractable" maximum was found at $N = 6$ in [13]). In our present paper we will just describe a new, third option showing that an amended and universal EPN-related model-building strategy does exist.

Our results will be based on the use of nontridiagonal matrices which are sparse, rendering the implementation of the strategy feasible in applications. The presentation of the idea may start from the replacement of the special condition of Eq. (6) by the generalized, standard relation

$$H_{(\mathrm{GAO})}^{(N)}(\lambda^{(\mathrm{EPN})})\, Q^{(N)} = Q^{(N)}\, \mathcal{J}^{[\mathcal{R}(N)]}(\eta). \tag{9}$$

The superscript $\mathcal{R}(N)$ denotes here one of the partitions of $N = N_1 + N_2 + \cdots + N_K$ that do not contain 1 as a part (and are such that, say, $N_1 \geqslant N_2 \geqslant \cdots \geqslant N_K \geqslant 2$; see [17] or Table 1 in [18]). The integer K represents the above-mentioned clusterization index *alias* geometric multiplicity of the EPN degeneracy [11].

As long as the first two partitions $\mathcal{R}(2) = 2$ and $\mathcal{R}(3) = 3$ are unique (for both of them the geometric EPN multiplicity K is equal to 1), Eqs. (6) and (9) remain the same at $N = 2$ and $N = 3$. The difference reflecting the existence of nontrivial multiplicities $K > 1$ only emerges at $N = 4$ where we can have $\mathcal{R}_1(1) = 4$ (i.e., $K = 1$) and $\mathcal{R}_2(4) = 2 + 2$ (i.e., $K = 2$). Thus, in our present amended model-building recipe we choose any $N \geqslant 4$, pick up one of the partitionings $\mathcal{R}(N)$, skip the $K = 1$ cases (which are well known), and define the block-diagonal matrix $\mathcal{J}^{[\mathcal{R}(N)]}(\eta)$ in the form of the direct sum of a K-plet of elementary Jordan blocks,

$$\mathcal{J}^{[\mathcal{R}(N)]}(\eta) = J^{(N_1)}(\eta) \oplus J^{(N_2)}(\eta) \oplus \cdots \oplus J^{(N_K)}(\eta). \tag{10}$$

The first alternative options emerge at $N = 4 = 2 + 2$ and at $N = 5 = 3 + 2$, with the following two new, $K = 2$ direct sums of the Jordan blocks,

$$\mathcal{J}^{[2+2]}(\eta) = \left[\begin{array}{cc|cc} \eta & 1 & 0 & 0 \\ 0 & \eta & 0 & 0 \\ \hline 0 & 0 & \eta & 1 \\ 0 & 0 & 0 & \eta \end{array}\right]$$

$$\text{and} \quad \mathcal{J}^{[3+2]}(\eta) = \left[\begin{array}{ccc|cc} \eta & 1 & 0 & 0 & 0 \\ 0 & \eta & 1 & 0 & 0 \\ 0 & 0 & \eta & 0 & 0 \\ \hline 0 & 0 & 0 & \eta & 1 \\ 0 & 0 & 0 & 0 & \eta \end{array}\right]. \tag{11}$$

We are now prepared to turn our attention to the construction of multidiagonal GAO models with the full-matrix structure (8) of their Hamiltonians, and with the general direct-sum structure (10) of their canonical representation in the EPN limit.

III. SYSTEMS WITH PENTADIAGONAL-MATRIX HAMILTONIANS

A. Elementary one-parametric model

A few simulations of dynamics near EPNs using multidiagonal N-by-N matrix Hamiltonians were presented in paper [13]. The scope of the study was restricted, due to the apparently purely numerical nature of the problem, to the smallest matrix dimensions $N \leqslant 6$. Such a restriction helped to keep the necessary evaluations of the spectra non-numerical. Incidentally, the latter decision was a bit unfortunate because, as we will see below, the next option with $N = 7$ would have been perceivably more instructive. Still, the key message of the study remains significant: the search for anomalous $K > 1$ EPN singularities with optional geometric multiplicities should be based on a systematic analysis of nontridiagonal, multidiagonal matrix models.

In light of this experience let us now turn our attention to the following pentadiagonal-matrix example with $N = 7$:

$$H^{(\mathrm{toy})}(\lambda) = \begin{bmatrix} 1 & 0 & \sqrt{3}g & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & \sqrt{2}g & 0 & 0 & 0 \\ -\sqrt{3}g & 0 & 5 & 0 & 2g & 0 & 0 \\ 0 & -\sqrt{2}g & 0 & 7 & 0 & \sqrt{2}g & 0 \\ 0 & 0 & -2g & 0 & 9 & 0 & \sqrt{3}g \\ 0 & 0 & 0 & -\sqrt{2}g & 0 & 11 & 0 \\ 0 & 0 & 0 & 0 & -\sqrt{3}g & 0 & 13 \end{bmatrix}. \tag{12}$$

The unperturbed truncated harmonic-oscillator spectrum is kept unshifted, the anharmonicity is antisymmetric, and the freedom of the λ dependence of the off-diagonal elements of the perturbation is reduced to a single function $g = g(\lambda)$. Such a simplification implies that the related Schrödinger bound-state problem

$$H^{(\mathrm{toy})}(g)\,|\psi_n(g)\rangle = E_n(g)\,|\psi_n(g)\rangle \tag{13}$$

becomes solvable exactly,

$$E_0(g) = 7, \quad E_{\pm 1}(g) = 7 \pm \sqrt{4 - g^2},$$

$$E_{\pm 2}(g) = 7 \pm 2\sqrt{4 - g^2}, \quad E_{\pm 3}(g) = 7 \pm 3\sqrt{4 - g^2}.$$

The model exemplifies the system in which the conventional self-adjoint harmonic-oscillator dynamics is realized at $g = 0$, and in which the small perturbations, in spite of their maximal non-Hermiticity, keep the spectrum real and, hence, observable, in principle at least.

1. Strong-coupling dynamical regime

The above-listed explicit formulas demonstrate that the whole spectrum of our toy model remains real up to the EP7 limit of $g \to g^{(\mathrm{EP7})} = 2$. The bound-state energies remain real and well separated along a path connecting the weakly anharmonic (WA) and the strong-coupling (SC) ends of the open interval of the values of $g \in (0, 2)$. Even when one decides to consider a nontrivial function $g = g(\lambda)$ of parameter λ which would not deviate too much from the linear one, one can still deduce that there exists a fairly broad corridor of unitarity which connects the harmonic-oscillator and the EP7 dynamical extremes.

In the WA regime the optional auxiliary (and, say, monotonously increasing) function $g(\lambda)$ is to be kept small. Then, the anharmonicity will remain easily tractable by the standard Rayleigh-Schrödinger perturbation methods. Near the opposite SC boundary where $g \lessapprox 2$ the EPN degeneracy (3) is reached. The reality of the spectrum becomes mathematically fragile. Whenever the value of the coupling exceeds its critical value of $g^{(\mathrm{EPN})} = 2$, the whole spectrum becomes complex.

The latter form of the EPN-related instability is a real, measurable phenomenon. Its possible detection appeared to be a true challenge during the popular experimental simulations of quantum dynamics via nonquantum systems (cf., e.g., extensive reviews of this point in [19,20]). Fortunately, it has recently been clarified that in the genuine closed quantum systems living in an EPN vicinity an experimental realization of the similar instabilities in the laboratory would be much more complicated if not even impossible [21]. The subtle reasons of the existence of such a paradox were briefly explained in [22]. Their essence lies in the fact that the perturbations which would make the system leave the physical Hilbert space $\mathcal{H}$ would be hard to define in practically any quantum theory of closed systems including not only its conventional textbook forms but, equally well, all of its various quasi-Hermitian [5], pseudo-Hermitian [8], or $\mathcal{PT}$-symmetric [6] versions. In the latter setting, indeed, the Hamiltonian-dependent choice of the physical Hilbert space $\mathcal{H}$ is ambiguous [8]. For this reason, *any* change of the Hamiltonian reopens the ambiguity problem and, in this sense, makes the perturbation theory nonlinear [18,22].

Our present model may be recalled for illustration purposes. In it, we are allowed to introduce a new small parameter $\kappa = \kappa(\lambda) \in (0, 1)$ and to redefine $g = \widetilde{g}(\kappa) = 2\,(1 - \kappa^2)$. This enables us to consider the related ("tilded") modification of our spectral problem (13) with the same exact eigenvalues rewritten in an equivalent but SC-friendlier form

$$\begin{aligned}
\widetilde{E}_0(\kappa) &= 7,\\
\widetilde{E}_{\pm 1}(\kappa) &= 7 \pm 2\sqrt{-\kappa^4 + 2\kappa^2}\\
&\qquad \sim 7 \pm 2\sqrt{2}\,\kappa + O(\kappa^3),\\
\widetilde{E}_{\pm 2}(\kappa) &= 7 \pm 4\sqrt{-\kappa^4 + 2\kappa^2},\\
\widetilde{E}_{\pm 3}(\kappa) &= 7 \pm 6\sqrt{-\kappa^4 + 2\,\kappa^2}.
\end{aligned}$$

Also in this representation these eigenvalues remain all real at small κ, reconfirming the existence of a corridor of unitarity connecting the WA and SC dynamical-regime extremes.

2. Canonical form of the Hamiltonian

In the SC EP7 limit $\kappa \to 0$ the spectrum becomes degenerate and the Hamiltonian itself ceases to be diagonalizable. It may be shown to possess just two eigenvectors so that the EPN value $\eta = 7$ of the energy can be used in the canonical eigenvalue problem (9) where

$$\mathcal{J}^{(4+3)}(\eta) = \left[\begin{array}{cccc|ccc}
\eta & 1 & 0 & 0 & 0 & 0 & 0\\
0 & \eta & 1 & 0 & 0 & 0 & 0\\
0 & 0 & \eta & 1 & 0 & 0 & 0\\
0 & 0 & 0 & \eta & 0 & 0 & 0\\
\hline
0 & 0 & 0 & 0 & \eta & 1 & 0\\
0 & 0 & 0 & 0 & 0 & \eta & 1\\
0 & 0 & 0 & 0 & 0 & 0 & \eta
\end{array}\right]
= \begin{bmatrix}
\eta & 1 & 0 & 0\\
0 & \eta & 1 & 0\\
0 & 0 & \eta & 1\\
0 & 0 & 0 & \eta
\end{bmatrix} \oplus \begin{bmatrix}
\eta & 1 & 0\\
0 & \eta & 1\\
0 & 0 & \eta
\end{bmatrix}. \tag{14}$$

This is a direct sum of two Jordan-block matrices. The partitioning merely guides the eye and emphasizes the fact that in our toy-model Hamiltonian (12) there is no mutual coupling between the even and odd indices. In the EP7 limit, our Hamiltonian $H^{(\mathrm{toy})}(\lambda^{(\mathrm{EP7})})$ may be interpreted as a direct sum of the two independent components,

$$H^{(\mathrm{toy})}(\lambda^{(\mathrm{EP7})}) = H^{[\mathrm{odd}]}_{(\mathrm{EP7})} \oplus H^{[\mathrm{even}]}_{(\mathrm{EP7})}, \tag{15}$$

where

$$H^{[\mathrm{odd}]}_{(\mathrm{EP7})} = \begin{bmatrix}
1 & 2\sqrt{3} & 0 & 0\\
-2\sqrt{3} & 5 & 4 & 0\\
0 & -4 & 9 & 2\sqrt{3}\\
0 & 0 & -2\sqrt{3} & 13
\end{bmatrix},$$

$$H^{[\mathrm{even}]}_{(\mathrm{EP7})} = \begin{bmatrix}
3 & 2\sqrt{2} & 0\\
-2\sqrt{2} & 7 & 2\sqrt{2}\\
0 & -2\sqrt{2} & 11
\end{bmatrix}.$$

Off the EP7 limit, our specific pentadiagonal matrix $H^{(\mathrm{toy})}(\lambda)$ still has the form of a direct sum of the two tridiagonal matrices. Due to our specific choice of the model, they are both exactly solvable so that the whole $N = 7$ model is solvable in closed form as well. Moreover, arbitrary *ad hoc* perturbation terms may be added to couple the components of the direct sum (see, e.g., an application of such an idea to a realistic model of such a type in [23]).

A specific consequence of the simplicity of our present perturbed model lies in the availability of the explicit transition-matrix solution of Eq. (13),

$$Q^{(\mathrm{toy})} = \begin{bmatrix} -48 & 24 & -6 & 1 & 0 & 0 & 0 \\ 0 & 8 & -4 & 1 & 8 & -4 & 1 \\ -48\sqrt{3} & 16\sqrt{3} & -2\sqrt{3} & 0 & 0 & 0 & 0 \\ 0 & 8\sqrt{2} & -2\sqrt{2} & 0 & 8\sqrt{2} & -2\sqrt{2} & 0 \\ -48\sqrt{3} & 8\sqrt{3} & 0 & 0 & 0 & 0 & 0 \\ 0 & 8 & 0 & 0 & 8 & 0 & 0 \\ -48 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Indeed, in the perturbation constructions of the SC states the role of unperturbed basis is relegated, in natural manner, to transition matrices (see more details in [18,23]).

B. Multiparametric Hamiltonians

The tricks used in connection with the one-parametric toy model (12) can immediately be applied to the N-dimensional model

$$H^{(N)}_{(\mathrm{pent.\ special})}(\lambda) = \left[\begin{array}{c|c|c|c|c|c} 1-N & 0 & c_1(\lambda) & 0 & \cdots & 0 \\ \hline 0 & 3-N & 0 & \ddots & \ddots & \vdots \\ \hline -c_1(\lambda) & 0 & \ddots & \ddots & c_2(\lambda) & 0 \\ \hline 0 & \ddots & \ddots & N-5 & 0 & c_1(\lambda) \\ \hline \vdots & \ddots & -c_2(\lambda) & 0 & N-3 & 0 \\ \hline 0 & \cdots & 0 & -c_1(\lambda) & 0 & N-1 \end{array}\right] \tag{16}$$

with an elementary shift of the origin on the energy scale and with the parallels in structure emphasized by the partitioning. Obviously, all of these matrices are equal to direct sums of two tridiagonal matrices, viz.,

$$H^{(N)}_{(\mathrm{component\ one})}(\lambda) = \begin{bmatrix} 1-N & c_1(\lambda) & 0 & \cdots \\ -c_1(\lambda) & 5-N & c_3(\lambda) & \ddots \\ 0 & -c_3(\lambda) & 9-N & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{bmatrix} \tag{17}$$

and

$$H^{(N)}_{(\mathrm{component\ two})}(\lambda) = \begin{bmatrix} 3-N & c_2(\lambda) & 0 & \cdots \\ -c_2(\lambda) & 7-N & c_4(\lambda) & \ddots \\ 0 & -c_4(\lambda) & 11-N & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{bmatrix}. \tag{18}$$

Due to our assumption of maximal non-Hermiticity (i.e., of antisymmetry) of perturbations (i.e., of anharmonicities), their diagonal component will always vanish. Thus, without confusion, we may refer to the matrix of Eq. (16) by its main diagonal put in the box, $\boxed{1-N, 3-N, \dots, N-1}$. Its direct-sum decomposition into components (17) and (18) can be then written in shorthand,

$$\boxed{1-N, 3-N, \dots, N-1} = \boxed{1-N, 5-N, 9-N, \dots} \oplus \boxed{3-N, 7-N, 11-N, \dots}.$$

The last elements of the summands are not displayed because they vary with the parity of N. After the explicit specification of the parity of N we arrive at the following two conclusions.

Lemma 1. At the even matrix dimension $N = 2J$, the decomposition of the pentadiagonal sparse-matrix model (16) into its tridiagonal TAO components (17) and (18) only supports the two $K = 1$ EPJ limits (2), with different respective energies $\eta = \pm 1$. At any one of them, the confluence in Eq. (3) is incomplete, involving just J levels.

Proof. The main diagonal $\boxed{1-N, 3-N, \dots, N-1} = \boxed{1-2J, 3-2J, \dots, 2J-1}$ of matrix (16) does not contain a central zero. The central interval $(-1, 1)$ is "too short." Its two elements -1 and 1 get distributed among both of the components (17) and (18). In the resulting direct sum

$$\boxed{1-N, 3-N, \dots, N-1} = \boxed{1-2J, 5-2J, \dots, 2J-3} \oplus \boxed{3-2J, 7-2J, \dots, 2J-1},$$

both of the components will be centrally asymmetric. ■

The search for an anomalous EPN with $K = 2$ failed. Even after a successful J-by-J realization of the two separate EPJ limits using building blocks (2), requirement (3) will only offer two different values of the eligible limiting EPN energies. The direct-sum decomposition yields the two nonanomalous $K = 1$ EPNs of the same small order $J = N/2$. A better result is the next one.

Lemma 2. At odd $N = 2J + 1$, both of the tridiagonal matrices (17) and (18) admit the respective realizations (2) of their EPN limits. The related energies coincide so that the direct sum (16) admits the anomalous EPN limit with geometric multiplicity 2.

Proof. We have

$$\boxed{1-N, 3-N, \dots, N-1}$$
$$= \boxed{-2J, 2-2J, \dots, 2J} = \boxed{-2J, 4-2J, \dots, 2J}$$
$$\oplus \boxed{2-2J, 6-2J, \dots, 2J-2} \qquad (19)$$

so that out of the central triplet of integers $(-2, 0, 2)$, the doublet $(-2, 2)$ remains long enough to be a component of one of the sub-boxes. Their respective dimensions $J + 1$ and J are now different. This is compensated by the central symmetry of the summands and by the coincidence of the EPN energies, $\eta_\pm = 0$. In the direct sum (16) the respective EPN limits degenerate to a single, anomalous EPN limit. The $K = 2$ clusterization (7) takes place. ■

The highly plausible one-to-one correspondence between the tridiagonality of the Hamiltonian and the $K = 1$ form of its EPN limit as conjectured in [13] is now complemented by our two lemmas which confirm that in a search for models with larger K even the pentadiagonality assumption need not help too much. Nevertheless, the general pentadiagonal models

$$H^{(N)}_{(\text{pentadiagonal})}(\lambda) = \begin{bmatrix} 1-N & b_1(\lambda) & c_1(\lambda) & 0 & \cdots & 0 \\ -b_1(\lambda) & 3-N & b_2(\lambda) & \ddots & \ddots & \vdots \\ -c_1(\lambda) & -b_2(\lambda) & \ddots & \ddots & c_2(\lambda) & 0 \\ 0 & \ddots & \ddots & N-5 & b_2(\lambda) & c_1(\lambda) \\ \vdots & \ddots & -c_2(\lambda) & -b_2(\lambda) & N-3 & b_1(\lambda) \\ 0 & \cdots & 0 & -c_1(\lambda) & -b_1(\lambda) & N-1 \end{bmatrix} \qquad (20)$$

offer another methodical inspiration paving the way towards the full-matrix scenario. The idea is based on an additional assumption that all of the matrix elements $b_n(\lambda)$ are small. Then, the decomposition

$$H^{(N)}_{(\text{pentadiagonal})}(\lambda) = H^{(N)}_{(\text{pent. special})}(\lambda) + \text{small perturbations} \qquad (21)$$

could prove tractable by perturbation techniques [18].

IV. MULTIDIAGONAL SOLVABLE MODELS

The freedom of choice of any positive integer K is desirable, but the task is ill-conditioned [24]. The numerical localization of the EPN degeneracies is difficult at larger N's. This is well sampled, e.g., in Ref. [25]. Therefore, we will only use non-numerical strategies in our model-building project.

A. Clusterization

The core and essence of our forthcoming general non-numerical constructions will lie in the mere generalization of Eq. (21), i.e., in an application of the idea that it makes sense to have some of the "unfriendly" GAO matrix elements reclassified as "small perturbations" (which could be, in the first run, neglected and omitted). Such a reduction should help us to obtain a non-numerically tractable structure [analogous to matrix (16)] which could be factorized into a K-plet of solvable TAO components [sampled, at $K = 2$, by (17) and (18)].

In a way inspired by the pentadiagonal-matrix Lemma 2 and, in particular, by the direct-sum decomposition (19), also the general GAO full-matrix Hamiltonian (8) may still be identified and represented by its left-right antisymmetric main diagonal put in a box,

$$\boxed{1-N, 3-N, \dots, N-3, N-1}. \qquad (22)$$

Formally, such a boxed symbol can be decomposed as follows:

$$\boxed{1-N, 3-N, \dots, N-1}$$
$$= \boxed{1-N, N-1} \oplus \boxed{3-N, 5-N, \dots, N-3}. \qquad (23)$$

Naturally, the first right-hand-side component $\boxed{1-N, N-1}$ of this decomposition could already represent the required TAO type Hamiltonian matrix, provided only that all of the unfriendly elements are omitted from the outer rows and columns of the initial Hamiltonian, yielding

$$H^{(N)}_{(\text{spec. partit.})}(\lambda) = \left[\begin{array}{c|ccccc|c} 1-N & 0 & 0 & \cdots & 0 & \omega_1(\lambda) \\ \hline 0 & 3-N & b_2(\lambda) & \dots & z_2(\lambda) & 0 \\ 0 & -b_2(\lambda) & \ddots & \ddots & \vdots & 0 \\ \vdots & \vdots & \ddots & N-5 & b_2(\lambda) & \vdots \\ 0 & -z_2(\lambda) & \dots & -b_2(\lambda) & N-3 & 0 \\ \hline -\omega_1(\lambda) & 0 & 0 & \cdots & 0 & N-1 \end{array}\right]. \qquad (24)$$

This enables us to decompose

$$H^{(N)}_{(\text{spec. partit.})}(\lambda) = \left[(N-1)\times H^{(2)}_{(\text{toy})}(\lambda)\right] \oplus H^{(N-2)}_{(\text{GAO})}(\lambda),$$

where the dimension of the second full-matrix component is diminished. Thus, the construction of one of the possible direct-sum decompositions could be completed iteratively, with the ultimate result preserving the TAO form of all of its components.

We are now prepared to search for all of the other K-term generalizations of the $K = 2$ direct-sum expansion (19). It is worth emphasizing that for the reasons illustrated in Lemma 1 our fundamental methodical requirement of the non-numerical tractability of the EPN limits of the GAO models is in a one-to-one correspondence with the constraint that all of the components of their K-term direct-sum expansions must keep having the specific TAO form, represented by the centrally antisymmetric boxed symbols. Thus, an exhaustive classification of all of the possible direct-sum decompositions of the initial symbol (22) becomes an interesting combinatorial problem with the solution described in Appendix below.

B. General direct-sum decompositions

The unitary evolution scenarios characterized by an incomplete, anomalous $K > 1$ degeneracy of eigenstates are all equally important [13]. The present continuation of their analysis will be inspired by Eq. (15) where $K = 2$. We will fix the parameter $\lambda = \lambda^{(\text{EPN})}$ and assume an analogous GAO direct-sum decomposition valid, in the EPN limit with specific $\eta = 0$, at any preselected dimension N and multiplicity K,

$$H^{(N)}_{(\text{GAO})}(\lambda^{(\text{EPN})}) = \widetilde{H^{(N_1)}}(\lambda^{(\text{EPN})}) \oplus \widetilde{H^{(N_2)}}(\lambda^{(\text{EPN})}) \oplus \cdots \oplus \widetilde{H^{(N_K)}}(\lambda^{(\text{EPN})}). \tag{25}$$

We should only keep in mind that the value of the geometric multiplicity K cannot exceed $N/2$. We will also insist on the non-numerical tractability of the model. Most easily, this goal will be achieved by the requirement that all of the separate N_j-dimensional tilded matrix components of Hamiltonian (25) are elements of the above-mentioned TAO-Hamiltonian family (2),

$$\widetilde{H^{(N_j)}}(\lambda^{(\text{EPN})}) = c_j\, H^{(N_j)}_{(\text{TAO})}(\lambda^{(\text{EPN})}), \quad j = 1, 2, \ldots, K. \tag{26}$$

The freedom of choice of K different normalization constants c_j will not destroy the non-numerical form and solvability of the model. In a small vicinity of the EPN singularity the exact solvability of the TAO toy models (1) will survive. We may, therefore, extend the definition of the model, accordingly, to parameters $\lambda < \lambda^{(\text{EPN})}$ which do not lie too far from $\lambda^{(\text{EPN})}$,

$$H^{(N)}_{(\text{GAO})}(\lambda) = \widetilde{H^{(N_1)}}(\lambda) \oplus \widetilde{H^{(N_2)}}(\lambda) \oplus \cdots \oplus \widetilde{H^{(N_K)}}(\lambda) + \text{small corrections}. \tag{27}$$

For the sake of simplicity let us ignore the corrections, and let us only consider the components with weights which remain λ independent,

$$\widetilde{H^{(N_j)}}(\lambda) = c_j\, H^{(N_j)}_{(\text{TAO})}(\lambda), \quad j = 1, 2, \ldots, K. \tag{28}$$

TABLE I. The list of all of the alternative TAO-direct-sum decompositions (27) of the GAO Hamiltonian (15) with symbol (label) $\boxed{1-N, 3-N, \ldots, N-1}$ at $N = 6$ ($K > 1$).

		GAO label	$\boxed{-5, -3, -1, 1, 3, 5}$		
K	$\mathcal{R}(6)$	j	N_j	c_j	TAO$_j$ label
2	4+2	1	4	1	$\boxed{-3, -1, 1, 3}$
		2	2	5	$\boxed{-5, 5}$
3	2+2+2	1	2	1	$\boxed{-1, 1}$
		2	2	3	$\boxed{-3, 3}$
		3	2	5	$\boxed{-5, 5}$

The parameter λ may now decrease to zero in a way which parallels the behavior of the tridiagonal TAO models (1) of Refs. [12,14,15].

During the process the direct-sum decomposition of the Hamiltonian [i.e., the exact solvability of the $K > 1$ models (27) where we omitted the "correction" term] will survive. Unfortunately, for a general real K-plet of the normalization constants c_j the spectrum of model (27) would be, in the $\lambda \to 0$ limit, nonequidistant. This means that in such a limit our system would *not* mimic the truncated harmonic oscillator. This would be a truly unpleasant feature of the model, especially in the context of perturbation theory. In our present paper the equidistance of the unperturbed $\lambda = 0$ spectrum of the GAO model will be, therefore, added as an independent WA postulate.

The latter requirement will restrict the freedom of our choice of the normalization constants c_j in Eq. (28), of course. From the practical physical, phase-transition-oriented point of view, such a restriction appears acceptable, being rather severe only at the not too large integers N and K. This is illustrated in Table I. It shows that just two alternative GAO models with $K > 1$ will exist at $N = 6$. Nevertheless, the benefits of the exact solvability of the restricted GAO models will certainly prevail at the larger matrix dimensions because with the growth of N the number of alternative scenarios will grow very quickly. This growth is sampled in Appendix.

V. SYSTEMATICS OF MODELS WITH CLUSTERED EPN LIMITS

Our model-building strategy is based on the partitioning of an N-by-N GAO Hamiltonian labeled by the boxed main diagonal [cf. Eq. (22)] into a K-plet of the TAO components represented by the shorter, centrally antisymmetric boxed equidistant subsets

$$\boxed{(1-N_j)\,c_j, (3-N_j)\,c_j, \ldots, (N_j-3)\,c_j, (N_j-1)\,c_j}. \tag{29}$$

This makes every candidate for a solvable benchmark Hamiltonian equal to a direct sum of TAO building blocks. At the first few dimensions N, the systematic constructive implementation of such a recipe will be made more explicit in what follows.

A. The choice of $N = 2$ and $N = 3$: No anomalous degeneracies

For our present GAO class of λ-dependent Hamiltonians (8) there exists strictly one, unique EP2 limit satisfying our restrictions at $N = 2$, namely, the matrix $H^{(2)}_{(\mathrm{TAO})}(\lambda^{(\mathrm{EPN})})$ as displayed in Eq. (2). In our abbreviated notation such a matrix is characterized by the boxed symbol $\boxed{-1,1}$. In the notation of Appendix the number $a(N)$ of eligible scenarios is one, $a(2) = 1$. In the EPN limit, the geometric multiplicity of the spectrum is $K = 1$.

Similarly, at $N = 3$ we have $a(3) = 1$ and the unique $K = 1$ limit $H^{(3)}_{(\mathrm{TAO})}(\lambda^{(\mathrm{EP3})})$ represented by the boxed symbol $\boxed{-2,0,2}$ and by the matrix displayed in Eq. (2).

B. The simplest anomalous case with $N = 4$ and $K = 2$

Besides the trivial $K = 1$ option with symbol $\boxed{-3,-1,1,3}$, there exists strictly one other possibility of decomposition at $N = 4$, viz.,

$$\boxed{-3,-1,1,3} = \boxed{-1,1} \oplus \boxed{-3,3}, \quad K = 2.$$

In the limit $\lambda \to \lambda^{(\mathrm{EP4})}$ this direct sum represents the seven-diagonal but very sparse GAO matrix

$$H^{(4)}_{(K=2)}(\lambda^{(\mathrm{EP4})}) = \begin{bmatrix} -3 & 0 & 0 & 3 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ -3 & 0 & 0 & 3 \end{bmatrix}. \tag{30}$$

In the unitarity domain where $\lambda \neq \lambda^{(\mathrm{EP4})}$ the number of the eligible dynamical scenarios is two, $a(4) = 2$, one of them representing a nontrivial clusterization with $K = 2$.

C. Two $K = 2$ options at $N = 5$

At $N = 5$ the number of scenarios is three, $a(5) = 3$. Besides the trivial case, we have the two $K = 2$ decompositions $\boxed{-4,-2,0,2,4} = \boxed{-2,0,2} \oplus \boxed{-4,4}$ and $\boxed{-4,-2,0,2,4} = \boxed{-4,0,4} \oplus \boxed{-2,2}$, leading to the two alternative, nonequivalent wave-function clusterizations. These two alternatives are represented by the two respective EP5 limiting matrix Hamiltonians, viz., by the nine-diagonal

$$H^{(5)}_{(K=2,a)}(\lambda^{(\mathrm{EP5})}) = \begin{bmatrix} -4 & 0 & 0 & 0 & 4 \\ 0 & -2 & \sqrt{2} & 0 & 0 \\ 0 & -\sqrt{2} & 0 & \sqrt{2} & 0 \\ 0 & 0 & -\sqrt{2} & 2 & 0 \\ -4 & 0 & 0 & 0 & 4 \end{bmatrix}$$

and/or by the pentadiagonal

$$H^{(5)}_{(K=2,b)}(\lambda^{(\mathrm{EP5})}) = \begin{bmatrix} -4 & 0 & 2\sqrt{2} & 0 & 0 \\ 0 & -2 & 0 & 2 & 0 \\ -2\sqrt{2} & 0 & 0 & 0 & 2\sqrt{2} \\ 0 & -2 & 0 & 2 & 0 \\ 0 & 0 & -2\sqrt{2} & 0 & 4 \end{bmatrix}.$$

The latter matrix fits in the classification pattern as provided by Lemma 2 above.

D. The first occurrence of $K = 3$ at $N = 6$

Besides the trivial, nondegenerate EP6 limit with $K = 1$ we have to consider its anomalous descendants, viz., the unique $K = 2$ decomposition $\boxed{-5,-3,-1,1,3,5} = \boxed{-3,-1,1,3} \oplus \boxed{-5,5}$ and the unique $K = 3$ decomposition $\boxed{-5,-3,-1,1,3,5} = \boxed{-1,1} \oplus \boxed{-3,3} \oplus \boxed{-5,5}$. In both of these cases (listed in illustrative Table I above) the direct-sum components of $H^{(6)}_{(K=2,K=3)}(\lambda^{(\mathrm{EP6})})$ may be found displayed in Eq. (2). In the latter case, for example, the direct sum yields the matrix

$$H^{(6)}_{(K=3)}(\lambda^{(\mathrm{EP6})}) = \begin{bmatrix} -5 & 0 & 0 & 0 & 0 & 5 \\ 0 & -3 & 0 & 0 & 3 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & -3 & 0 & 0 & 3 & 0 \\ -5 & 0 & 0 & 0 & 0 & 5 \end{bmatrix}.$$

The number of scenarios is $a(6) = 3$. The role and consequences of small perturbations of the latter matrix were analyzed in [13].

E. Paradox of decrease of $a(N)$ between $N = 7$ and $N = 8$

At $N = 7$ the number of eligible EP7 scenarios is $a(7) = 6$ because the usual trivial $K = 1$ option can be accompanied by the following quintuplet of anomalous EP7 direct sums,

$$\boxed{-6,-4,-2,0,2,4,6} = \boxed{-4,-2,0,2,4} \oplus \boxed{-6,6}, \quad K = 2,$$

$$\boxed{-6,-4,-2,0,2,4,6} = \boxed{-2,0,2} \oplus \boxed{-4,4} \oplus \boxed{-6,6}, \quad K = 3,$$

$$\boxed{-6,-4,-2,0,2,4,6} = \boxed{-4,0,4} \oplus \boxed{-2,2} \oplus \boxed{-6,6}, \quad K = 3,$$

$$\boxed{-6,-4,-2,0,2,4,6} = \boxed{-4,0,4} \oplus \boxed{-6,-2,2,6}, \quad K = 2,$$

$$\boxed{-6,-4,-2,0,2,4,6} = \boxed{-6,0,6} \oplus \boxed{-2,2} \oplus \boxed{-4,4}, \quad K = 3.$$

In contrast, at $N = 8$ we have $a(8) = 4$, i.e., only the triplet of the anomalous, $K > 1$ direct sums becomes available, viz.,

$$\boxed{-7,-5,-3,-1,1,3,5,7} = \boxed{-5,-3,-1,1,3,5} \oplus \boxed{-7,7}, \quad K = 2,$$

$$\boxed{-7,-5,-3,-1,1,3,5,7} = \boxed{-3,-1,1,3} \oplus \boxed{-5,5} \oplus \boxed{-7,7}, \qquad K=3,$$

$$\boxed{-7,-5,-3,-1,1,3,5,7} = \boxed{-1,1} \oplus \boxed{-3,3} \oplus \boxed{-5,5}, \oplus \boxed{-7,7}, \qquad K=4\,.$$

The latter item is our first four-term direct-sum decomposition example representing a 15-diagonal but very sparse matrix $H^{(8)}_{(K=4)}(\lambda^{(EP8)})$ with bidiagonal structure.

VI. DISCUSSION

The field of study of quantum phase transitions and of the role played by the Kato's exceptional points may be characterized by a lasting conflict between ambition and reality. The enthusiasm accompanying the production of ideas on the side of theory (see two older compact reviews of physics of non-Hermitian degeneracies [26,27]) seems counterbalanced by the difficulties of the search for EPN-related phase transitions in the laboratory [28].

The concept of the exceptional-point value $\lambda^{(EPN)}$ of a real parameter λ in a linear operator $H(\lambda)$ proved, originally, useful in mathematics [9,11]. Physics behind the EPNs remained obscure. The situation has changed after several authors discovered that the concept admits applicability in multiple branches of quantum as well as nonquantum physics [6,19,20]. *Pars pro toto* let us mention that in the subdomain of quantum physics the values of $\lambda^{(EPN)}$ acquired the status of instants of an experimentally realizable quantum phase transition [5,29,30]. The boundary-of-stability role played by the values of $\lambda^{(EPN)}$ attracted, therefore, the attention of experimentalists [27,31] as well as of theoreticians [32–34].

During our study of the problem we felt challenged by both of these aspects of the phenomenon. On the side of experiment we were impressed, first of all, by an intimate connection between the mathematics of EPNs and the very concrete physics of quantum phase transitions [35]. On the side of theory we felt motivated by the existence of its two mutually interrelated aspects. The first one was pragmatic: the description of the processes of the loss of the observability seems to be hardly feasible by the conventional numerical means [25]. At the same time, the perturbation-approximation techniques appeared to be applicable after minor amendments [23]. The second attractive aspect of the theory was conceptual and deeper: the non-Hermitian degeneracy of an N-plet of the stable bound states with $N > 2$ only became tractable as an admissible process in the framework of quasi-Hermitian formulation of quantum mechanics [5] (at present, people more often use the term $\mathcal{PT}$-symmetric theory; cf. reviews [7,8,19,20]).

In the latter framework one encounters challenges and apparent contradictions reflecting the unexpected peaceful coexistence of the non-Hermitian EPN-related degeneracy of spectra (which are, by assumption [6,29], real) with the unitarity of evolution. Another puzzle may be seen in a rather vague correspondence between some of the EPN properties and the structure of the operators, etc. In our present study we felt guided by the contrast between the not too surprising numerical ill-conditioning [24] and the real-matrix nature of the exactly solvable TAO models of Refs. [14,15]. The over-restrictive form of these models was reclassified as inessential. We accepted the conjecture of correlation between the tridiagonality of Hamiltonians and a triviality of the geometric multiplicities [13]. Keeping the warning in mind we constructed the universal GAO $K > 1$ models via the TAO-direct-sum ansatz.

A. Unitary vs nonunitary systems

The theoretical background of our present complete menu of non-numerically tractable phase-transition scenarios lies in the consistent theoretical compatibility of the non-Hermiticity of the Hamiltonian with the unitarity of the quantum evolution in the quasi-Hermitian Schrödinger picture [5,6,8,36]. In order to avoid misunderstandings we must immediately add that in many experiment-oriented descriptions, the quantum-system transmutations mediated by the EPNs *alias* non-Hermitian degeneracies [26] are very often nonunitary. In the extensive related literature [37] the scope of the theory is very broad. In the Feshbach's open-system spirit [38], people work with the non-Hermitian effective Hamiltonians H_{eff} with spectra which are complex. Still, the quantum systems in question are realistic and their analysis profits from sharing the mathematical know-how with the $\mathcal{PT}$-symmetric theories of the closed, unitary system.

From the historical perspective, the open-system philosophy was always dominant. For a broader audience this dominance has only been shattered, in 1998, by Bender and Boettcher [29]. These authors proposed that at least some of the processes of the quantum phase transitions might find a more natural description and explanation in the alternative, closed-system theoretical framework (see reviews [6,8,9]).

The Bender- and Boettcher-inspired change of the paradigm had two roots, both of them related to the problem of quantum phase transitions. One is that even in many closed quantum systems the unitary evolution can be controlled by the Hamiltonian (with real spectrum) which is non-Hermitian (i.e., admitting the EPNs). Although such a conjecture might sound like a paradox, Bender and Boettcher's secret trick was that the latter Hamiltonian can be, via an appropriate amendment of the Hilbert space of states, Hermitized, fitting all of the standard postulates of textbooks (see, e.g., the older review paper [5] for some basic mathematical details).

The second root of the change of the paradigm concerns the quantum phase transitions more immediately, opening their innovative treatment via specific examples. In [29] the innovation was sampled by the spontaneous breakdown of parity-time (i.e., $\mathcal{PT}$) symmetry. The authors related the collapse of the system to the coincidence of the parameter in $H(\lambda)$ with its EPN value $\lambda^{(EPN)}$. In the phase-transition limit $\lambda \to \lambda^{(EPN)}$ they indeed encountered a genuine qualitative novelty. In their non-Hermitian local-interaction models the degeneracy was mostly followed by an abrupt complexification, i.e., by a sudden loss of the observability [29]. Their

EPN-related "quantum catastrophic" behavior was realized as a merger of bound-state energies (3). The key challenge was to show, for every preselected non-Hermitian Hamiltonian, that in the real-spectrum regime with $\lambda < \lambda^{(\mathrm{EPN})}$ the system remains unitary.

B. Correct inner products

For a preselected non-Hermitian quantum Hamiltonian with real spectrum a specification of its correct probabilistic closed-system interpretation can be a straightforward linear-algebraic procedure, especially for the most elementary TAO tridiagonal Hamiltonian matrices (1) (cf. the detailed and constructive discussion of this point in paper [39]). A firm theoretical ground of such a procedure (explained, e.g., in review [8]) is to be sought in an amendment of the conventional Hilbert space. For the sake of definiteness, we may denote the amended, correct Hilbert space by dedicated symbol $\mathcal{H}$, with the other, manifestly unphysical but mathematically friendlier Hilbert space denoted by a different symbol, say, $\mathcal{K}$.

In this notation the spectrum of any candidate $H(\lambda)$ for an observable Hamiltonian must be kept real. Although such an operator must be self-adjoint in $\mathcal{H}$ (i.e., in the physical Hilbert space, obedient to the Stone theorem [40]), it will be, in general, non-Hermitian in $\mathcal{K}$. The key purpose of such a duplicity (often attributed to Dyson [41]) is that via transition to a user-friendlier space $\mathcal{K}$, one achieves a decisive simplification of all calculations. In parallel, in the language of physics one transfers the responsibility for the unitarity of the system from the λ dependence of the Hamiltonian to the more flexible λ dependence of the Hilbert space $\mathcal{H} = \mathcal{H}(\lambda)$. Such a "double picture" of evolution offers a highly welcome physical model-building freedom while still guaranteeing that the evolution generated by $H(\lambda)$ remains unitary in $\mathcal{H}(\lambda)$.

In the majority of applications (including the one used in the present paper) the original Dyson's flowchart of the theory is inverted. The model-building process is initiated by the choice of a non-Hermitian $H(\lambda)$ acting in a λ-independent Hilbert space $\mathcal{K}$ endowed with a conventional inner product $\langle\psi_1|\psi_2\rangle_{\mathcal{K}}$ which is unphysical but user friendly. Naturally, what is then needed is a reconstruction of the "missing" Hilbert space $\mathcal{H}(\lambda)$ [8].

In a way described in [5] the construction of $\mathcal{H}(\lambda)$ becomes significantly facilitated when the candidate for the Hamiltonian is finite dimensional, $H(\lambda) = H^{(N)}(\lambda)$. Then, there will exist multiple Hermitian and positive-definite matrices $\Theta = \Theta^{(N)}(\lambda)$ which satisfy the N-by-N matrix equation

$$[H^{(N)}(\lambda)]^\dagger\,\Theta^{(N)}(\lambda) = \Theta^{(N)}(\lambda)\,H^{(N)}(\lambda).$$

In terms of any one of these matrices one can, subsequently, define the space $\mathcal{H}(\lambda)$ via the mere redefinition of the inner product in $\mathcal{K}$,

$$\langle\psi_1|\psi_2\rangle_{\mathcal{H}} = \langle\psi_1|\Theta|\psi_2\rangle_{\mathcal{K}}. \tag{31}$$

The differences between the two alternative representation spaces is in fact reduced to the mere nonequivalence of the respective inner products. Due to such an elementary mathematical correspondence the standard textbook description of the unitary evolution dynamics defined in the difficult Hilbert space $\mathcal{H}$ finds a simplification in which all of the necessary calculations and predictions are assumed to be made in the much simpler representation space $\mathcal{K}$ (i.e., in our present paper, in the most elementary Euclidean real vector space $\mathcal{K} = \mathbb{R}^N$).

C. The corridors of unitarity

Whenever one keeps the evolution unitary, the values of $\lambda^{(\mathrm{EPN})}$ mark the points of the loss of the observability of the system [42,43]. Our present hierarchy of specific GAO models may be treated as certain exactly solvable quantum analog of the Thom's typology of classical catastrophes [44], with potential applicability to closed as well as open systems.

In our present paper we were exclusively interested in the former type of applications. We knew that the dynamics of any unitary quantum system (i.e., typically, its stability with respect to small perturbations) is strongly influenced by the EPNs. We should only add that extreme care must be paid to the Stone theorem [40] requiring the Hermiticity of $H(\lambda)$ in the related physical Hilbert space $\mathcal{H}$. A Hermitization of the Hamiltonian is needed [5]. Such a process involves a reconstruction of an appropriate amended inner product in the conventional but unphysical Hilbert space $\mathcal{K}$. Interested readers may find one of the rare samples of such a reconstruction of the whole menu of $\mathcal{H}$'s in [45].

In the realistic models one often encounters a paradox that the construction of the correct, amended inner product may happen to be prohibitively complicated, i.e., from the pragmatic point of view, inaccessible. This is the reason why people often postpone the problem and use, temporarily, a simplified inner product. For our present, user-friendly, matrix-represented GAO Hamiltonians of closed systems with $N < \infty$ such a purely technical obstacle does not occur. The reconstruction of $\mathcal{H}$ would be a routine application of linear algebra; reclassifying the real GAO matrices which are non-Hermitian in our auxiliary space $\mathcal{K} = \mathbb{R}^N$ (that is why we write $H \neq H^\dagger$) into operators which are, by construction, Hermitian in $\mathcal{H}$ (in [36], e.g., we wrote $H = H^\ddagger$).

An unusual property of the physical, amended Hilbert space is that it is Hamiltonian- and λ-dependent, $\mathcal{H} = \mathcal{H}(\lambda)$. In the present phase-transition context the Hamiltonian $H(\lambda)$ itself may be interpreted as Hermitian just for "admissible," unitarity-compatible λ's forming a domain $\mathcal{D}$. The system is able to reach, via unitary evolution, the instant of the EPN-related quantum phase transition if and only if the overlaps of $\mathcal{D}$ with the arbitrarily small vicinities of $\lambda^{(\mathrm{EPN})}$ remain all nonempty forming a "corridor of access" $\mathcal{D}^{(\mathrm{EPN})}$ [10].

In paper [10] it has been shown that the corridors $\mathcal{D}^{(\mathrm{EPN})}$ connecting, in the space of matrix elements, the EPN extremes with the points in a deep interior of $\mathcal{D}$ do always exist. For the TAO matrices (2), in particular, the shape of these corridors has been found, in [15], sharply spiked. For the present broader class of the $K > 1$ models the existence of the analogous corridors of the unitary access to EPNs has been conjectured in [18].

VII. SUMMARY

The phenomenological usefulness of the TAO models of Refs. [12,14,15] reappears, unexpectedly, also in the $K > 1$

GAO-based phase-transition context. We showed that in the amended theory the TAO matrices may be assigned the role of building blocks, and that such a trick implies, as one of its by-products, the exact solvability of the resulting benchmark GAO quantum systems at any $K > 1$ and $\lambda \in (0, \lambda^{(EPN)})$. Along these lines, a universal mathematical classification as well as a richer structure of predictions of the measurable phenomena is achieved.

A menu of eligible benchmark EPN-supporting specific models is proposed and described as controlled by alternative multidiagonal N-by-N matrix realizations of the direct-sum anharmonic-oscillator-type Hamiltonians. On methodical level the phenomenological nonequivalence of these partially K-related decompositions of the Hamiltonians is to be stressed.

Via our systematic explicit enumeration of the most elementary special cases we emphasized, first of all, the importance of the proper treatment of the geometric multiplicity K of the N-plets of energy levels in the EPN-related phase-transition dynamical regime. We showed that in a precritical stage of the transition the nontrivial multiplicities $K > 1$ just reflect the existence of the phenomenon of a K-tuple clusterization of the wave functions of bound states. This may be of interest in experiments in which the processes of the loss of observability are currently being studied, mostly due to the existing technical limitations, just at the small N and trivial $K = 1$. Our present results may offer a motivation for performing some extended and subtler analyses or simulations of the clusterized $K > 1$ processes in the laboratory.

ACKNOWLEDGMENTS

The author acknowledges financial support from the Excellence project PřF UHK 2020.

APPENDIX: DIRECT-SUM-DECOMPOSITION PARTITIONINGS $\mathcal{R}(N)$

In the present GAO-based phenomenological models numbered by the Hamiltonian-matrix dimensions $N = (1), 2, 3, \ldots$, the counts of the eligible nonequivalent EPN-related dynamical scenarios [i.e., direct-sum decompositions (25) and (27) as sampled in Table I at $N = 6$) form a sequence

$$a(N) = (0), 1, 1, 2, 3, 3, 6, 4, 11, 6, 17, 7, 32, 8, 47, 13, 66, \ldots, . \tag{A1}$$

The evaluation of the sequence is important and useful for at least two reasons. First, beyond the smallest N, it enables us to check the completeness of the EPN-related dynamical alternatives. Second, the asymptotically exponential growth of the sequence indicates that at the larger N's, the menus of the EPN-supporting toy models numbered by partitionings $\mathcal{R}(N)$ will be dominated by the anomalous, multidiagonal $K > 1$ Hamiltonians.

Besides that, the properties of the sequence are of an independent mathematical interest. First of all we notice that our sequence seems composed of the two apparently simpler, monotonously increasing integer subsequences. They have to be discussed separately.

1. Subsequence of $a(N)$ with even $N = 2J, J = 1, 2, \ldots$

The values

$$b(J) = a(2J) = 1, 2, 3, 4, 6, 7, 8, 13, 14, 15, 25, 26, 33, 50, \ldots \tag{A2}$$

of the even-dimension (sub)sequence may be generated by the algorithm described in [46] and carrying the identification code number A336739. Recalling this source let us summarize a few key mathematical features of the sequence.

Definition 1. The quantity b(n) is the number of decompositions of B(n,1) into disjoint unions of B(j,k) where B(j,k) is the set of numbers {(2 i-1) (2 k-1), 1 ⩽ i ⩽ j }.

It may be instructive to display a few examples:

$B(n, 1)$ are the sets {1}, {1,3}, {1,3,5}, {1,3,5,7},. . .,

$B(n, 2)$ are the sets {3}, {3,9}, {3,9,15}, {3,9,15,21},. . .,

$B(n, 3)$ are the sets {5}, {5,15}, {5,15,25}, {5,15,25, 35},. . .,

etc. There are two decompositions of $B(2, 1) = \{1, 3\}$, viz., trivial $B(2, 1)$ and nontrivial $B(1, 1) + B(1, 2) = \{1\} + \{3\}$. Similarly, the complete list of the $a(5) = 6$ decompositions of {1,3,5,7,9} is as follows:

{{1,3,5,7,9}},
{{1,3,5,7}, {9}},
{{1,3,5}, {7}, {9}},
{{1,3}, {5}, {7}, {9}},
{{1}, {3}, {5}, {7}, {9}},
{{3,9}, {1}, {5}, {7}}.

We should add that the notation used in Definition 1 of quantities $B(j, k)$ is mathematically optimal. For the purposes of our present paper, nevertheless, it is necessary to recall the equivalence of every $B(j, k)$ to one of the present boxed symbols. For example, in place of $B(3, 1) = \{1, 3, 5\}$ we should write $\mathcal{B}[3,1] = \boxed{-5, -3, -1, 1, 3, 5}$, etc. Definition 1 can be modified as follows.

Definition 2. The quantity b(n) is the number of different decompositions of $\mathcal{B}$[n,1] into unions of $\mathcal{B}$[j,k] where $\mathcal{B}$[J,K] is defined as the boxed symbol

$$\boxed{(2K-1)(1-2J), (2K-1)(3-2J), (2K-1)(5-2J), \ldots, (2K-1)(2J-1)}.$$

2. Subsequence of $a(N)$ with odd $N = 2J + 1, J = 1, 2, \ldots$

The values of the subsequence

$$c(J) = a(2J+1) = 1, 3, 6, 11, 17, 32, 47, 66, 105, 162, 198, 376, \ldots \tag{A3}$$

may be found discussed in [47]. Using this source let us summarize a few key aspects of this sequence which carries the identification number A335631.

Definition 3. The quantity c(n) is the number of decompositions of C(n,1) into disjoint unions of C(j,k) and G(q,r) where C(j,k) is the set of numbers {i k, 0 ⩽ i ⩽j} and where G(q,r) is the set of numbers $\{(2p-1)r,\ 1 \leqslant p \leqslant q\}$.

In a more explicit manner let us point out that

$C(n,1)$ are the sets {0,1}, {0,1,2}, {0,1,2,3}, {0,1,2,3,4},...,

$C(n,2)$ are the sets {0,2}, {0,2,4}, {0,2,4,6}, {0,2,4,6,8},...,

$C(n,3)$ are the sets {0,3}, {0,3,6}, {0,3,6,9}, {0,3,6,9,12},...,

etc., and that

$G(n,1)$ are the sets {1}, {1,3}, {1,3,5}, {1,3,5,7},...,

$G(n,2)$ are the sets {2}, {2,6}, {2,6,10}, {2,6,10,14},...,

$G(n,3)$ are the sets {3}, {3,9}, {3,9,15}, {3,9,15,21},...,

etc. We can say that $a(2) = 3$ because the decompositions of $C(2,1) = \{0,1,2\}$ involve not only the trivial copy $C(2,1)$ but also the nontrivial formulas $C(1,2) + G(1,1) = \{0,2\} + \{1\}$ and $C(1,1) + G(1,2) = \{0,1\} + \{2\}$. Similarly: why $a(3) = 6$? Because the decompositions of {0,1,2,3} are as follows:

{{0,1,2,3}},
{{0,1,2}, {3}},
{{0,1}, {2}, {3}},
{{0,2}, {1,3}},
{{0,2}, {1}, {3}},
{{0,3}, {1}, {2}}.

The one-to-one correspondence and the translation of this notation to our present boxed-symbol language is again obvious, fully analogous to the one described in the preceding paragraph.

[1] A. Messiah, *Quantum Mechanics* (North Holland, Amsterdam, 1961).
[2] M. Znojil, Phys. Rev. A **97**, 042117 (2018).
[3] W. Greiner, *Relativistic Quantum Mechanics: Wave Equations* (Springer, Berlin, 2012).
[4] A. Mostafazadeh, Class. Quantum Grav. **20**, 155 (2003).
[5] F. G. Scholtz, H. B. Geyer, and F. J. W. Hahne, Ann. Phys. (NY) **213**, 74 (1992).
[6] C. M. Bender, Rep. Prog. Phys. **70**, 947 (2007).
[7] M. Znojil, Symm. Integ. Geom. Methods Appl. **5**, 001 (2009).
[8] A. Mostafazadeh, Int. J. Geom. Meth. Mod. Phys. **7**, 1191 (2010).
[9] *Non-Selfadjoint Operators in Quantum Physics: Mathematical Aspects*, edited by F. Bagarello, J.-P. Gazeau, F. Szafraniec, and M. Znojil (Wiley, Hoboken, 2015).
[10] M. Znojil, Phys. Rev. A **100**, 032124 (2019).
[11] T. Kato, *Perturbation Theory for Linear Operators* (Springer-Verlag, Berlin, 1966).
[12] M. Znojil, Proc. R. Soc. A. **476**, 20190831 (2020).
[13] M. Znojil and D. I. Borisov, Nucl. Phys. B **957**, 115064 (2020).
[14] M. Znojil, J. Phys. A: Math. Theor. **40**, 4863 (2007).
[15] M. Znojil, J. Phys. A: Math. Theor. **40**, 13131 (2007).
[16] C. M. Bender and K. A. Milton, Phys. Rev. D **55**, R3255 (1997); T. Goldzak, A. A. Mailybaev, and N. Moiseyev, Phys. Rev. Lett. **120**, 013901 (2018).
[17] The On-Line Encyclopedia of Integer Sequences (OEIS), item A002865, http://oeis.org/A002865/ (accessed January 31, 2021).
[18] M. Znojil, Symmetry **12**, 1309 (2020).
[19] *PT Symmetry in Quantum and Classical Physics*, edited by C. M. Bender (World Scientific, Singapore, 2018).
[20] *Parity-Time Symmetry and Its Applications*, edited by D. Christodoulides and J.-K. Yang (Springer-Verlag, Singapore, 2018).
[21] M. Znojil, Phys. Rev. A **97**, 032114 (2018).
[22] M. Znojil and F. Růžička, J. Phys.: Conf. Ser. **1194**, 012120 (2019).
[23] M. Znojil, Proc. R. Soc. A **476**, 20200292 (2020).
[24] J. H. Wilkinson, *The Algebraic Eigenvalue Problem* (Oxford University Press, Oxford, 1965).
[25] M. Znojil, Ann. Phys. (NY) **405**, 325 (2019).
[26] M. V. Berry, Czech. J. Phys. **54**, 1039 (2004).
[27] W. D. Heiss, J. Phys. A: Math. Theor. **45**, 444016 (2012).
[28] R. El-Ganainy, K. G. Makris, M. Khajavikhan, Z. H. Musslimani, S. Rotter, and D. N. Christodoulides, Nat. Phys. **14**, 11 (2018); M.-A. Miri and A. Alu, Science **363**, eaar7709 (2019).
[29] C. M. Bender and S. Boettcher, Phys. Rev. Lett. **80**, 5243 (1998).
[30] D. I. Borisov, Acta Polytech. **54**, 93 (2014); M. Znojil and D. I. Borisov, Ann. Phys. (NY) **394**, 40 (2018); M. Znojil, Phys. Rev. A **98**, 032109 (2018); S.-B. Wang, B. Hou, W.-X. Lu, Y.-T. Chen, Z. Q. Zhang, and C. T. Chan, Nat. Commun. **10**, 832 (2019).
[31] Y. N. Joglekar, D. Scott, M. Babbey, and A. Saxena, Phys. Rev. A **82**, 030103(R) (2010); J.-M. Li, A. K. Harter, J. Liu, L. de Melo, Y. N. Joglekar, and L. Luo, Nat. Commun. **10**, 855 (2019); Z. Bian, L. Xiao, K. Wang, X. Zhan, F. A. Onanga, F. Růžička, W. Yi, Y. N. Joglekar, and P. Xue, Phys. Rev. Res. **2**, 022039 (2020).
[32] L. N. Trefethen and M. Embree, *Spectra and Pseudospectra* (Princeton University Press, Princeton, NJ, 2005).
[33] P. Dorey, C. Dunning, and R. Tateo, J. Phys. A: Math. Gen. **34**, L391 (2001); C. M. Bender, D. C. Brody, and H. F. Jones, Phys. Rev. Lett. **89**, 270401 (2002).
[34] A. V. Smilga, J. Phys. A: Math. Theor. **41**, 244026 (2008); F. Bagarello and M. Znojil, *ibid.* **45**, 115311 (2012); F. Bagarello, A. Inoue and C. Trapani, J. Math. Phys. **55**, 033501 (2014).
[35] K. Ding, Z.-Q. Zhang, and C.-T. Chan, Phys. Rev. B **92**, 235310 (2015).
[36] M. Znojil, *Non-Selfadjoint Operators in Quantum Physics: Mathematical Aspects* (Ref. [9]), pp. 7–58.
[37] N. Moiseyev, *Non-Hermitian Quantum Mechanics* (Cambridge University Press, Cambridge, 2011).
[38] H. Feshbach, Ann. Phys. (NY) **5**, 357 (1958).
[39] M. Znojil, J. Phys. A: Math. Theor. **45**, 085302 (2012).
[40] M. H. Stone, Ann. Math. **33**, 643 (1932).
[41] F. J. Dyson, Phys. Rev. **102**, 1230 (1956).

[42] M. Znojil, J. Phys. A: Math. Theor. **45**, 444036 (2012); V. M. Martinez Alvarez, J. E. Barrios Vargas, and L. E. F. Foa Torres, Phys. Rev. B **97**, 121401(R) (2018).

[43] G. Lévai, F. Růžička, and M. Znojil, Int. J. Theor. Phys. **53**, 2875 (2014); K. Ding, G. Ma, M. Xiao, Z.-Q. Zhang, and C.-T. Chan, Phys. Rev. X **6**, 021007 (2016); D. I. Borisov and M. Znojil, in *Non-Hermitian Hamiltonians in Quantum Physics*, edited by F. Bagarello, R. Passante and C. Trapani (Springer, Basel, 2016), p. 201; H. Jing, Ş. K. Özdemir, H. Lü, and F. Nori, Sci. Rep. **7**, 3386 (2017).

[44] E. C. Zeeman, *Catastrophe Theory-Selected Papers 1972–1977* (Addison-Wesley, Reading, 1977); V. I. Arnold, *Catastrophe Theory* (Springer-Verlag, Berlin, 1992).

[45] M. Znojil, Sci. Rep. **10**, 18523 (2020).

[46] The On-Line Encyclopedia of Integer Sequences (OEIS), item A336739, https://oeis.org/A336739 (accessed Oct. 27, 2020).

[47] The On-Line Encyclopedia of Integer Sequences (OEIS), item A335631, https://oeis.org/A335631 (accessed Oct. 27, 2020).