Distribution of the span of one-dimensional confined random processes before hitting a target

J. Klinger and R. Voituriez
Laboratoire de Physique Théorique de la Matière Condensée, UMR 7600 CNRS/UPMC, 4 Place Jussieu, 75255 Paris Cedex, France, and Laboratoire Jean Perrin, UMR 8237 CNRS/UPMC, 4 Place Jussieu, 75255 Paris Cedex, France

O. Bénichou
Laboratoire de Physique Théorique de la Matière Condensée, UMR 7600 CNRS/UPMC, 4 Place Jussieu, 75255 Paris Cedex, France,

(Received 18 December 2020; accepted 16 February 2021; published 9 March 2021)

We derive the distribution of the number of distinct sites visited by a random walker before hitting a target site of a finite one-dimensional (1D) domain. Our approach holds for the general class of Markovian processes with connected span—i.e., whose trajectories have no "holes." We show that the distribution can be simply expressed in terms of splitting probabilities only. We provide explicit results for classical examples of random processes with relevance to target search problems, such as simple symmetric random walks, biased random walks, persistent random walks, and resetting random walks. As a by-product, explicit expressions for the splitting probabilities of all these processes are given. Extensions to reflecting boundary conditions, continuous processes, and an example of a random process with a nonconnected span are discussed.

DOI: 10.1103/PhysRevE.103.032107

I. INTRODUCTION

Quantifying the exploration of a domain by a random walker is a longstanding question, which has applications in a variety of problems, ranging from living organisms exploring their environment to find resources to robots designed to accomplish specific tasks such as cleaning or demining [1–3]. In this context, the determination of first-passage time statistics to target sites of interest in a domain has played an important role in the literature [4–7]. The case of confined geometries has proved to be particularly relevant in the field of target search problems and has been the focus of many theoretical works [8–13]. Beyond first-passage times, other observables, such as cover times—time needed to explore exhaustively a domain of interest [14–16]—or occupation times—cumulative time spent in a subdomain of interest—have also been studied [17–21].

Another classical quantifier of random exploration is given, in a discrete setting, by the number of distinct sites visited by a random walker after n steps [1,22,23]. In this paper, we focus on a related observable S that is defined as the number of distinct sites visited by a random walker before hitting a target site of a finite domain, where it is assumed to be either trapped or removed. The observable S therefore quantifies the fraction of the domain that has been explored before a target is found, which provides a further characterization of random search processes. Note that this observable does not seem to have received much attention in the literature, with the notable exception of Ref. [24], where it was studied for a continuous one-dimensional unconfined system.

Here we focus on one-dimensional geometries with periodic boundary conditions and derive the full distribution of S for the general class of Markovian processes with connected span, i.e., whose trajectories have no "holes." We show that the explicit determination of this distribution amounts to calculating splitting probabilities [4], i.e., probabilities that the walker reaches a given site x_1 before a second site x_2. We next apply these results to classical examples of random processes with relevance to target search problems, such as simple symmetric random walks, biased random walks, persistent random walks, and resetting random walks [18,25,26]. As a by-product of our study, we provide explicit expressions for the splitting probabilities of all these processes. Finally, we extend our results to the case of reflecting boundary conditions, continuous processes, and an example of a random process with a nonconnected span.

II. GENERAL EXPRESSION OF THE DISTRIBUTION OF S

In what follows, unless specified otherwise, we consider a Markovian random walker on a 1D periodic lattice of N sites, labeled from 0 to $N-1$. We denote by 0 the target (or exit) site and by s_0 the starting position. Let S be the number of distinct sites visited by a trajectory that ends at 0, where by convention 0 and s_0 are included (see Fig. 1).

Importantly, we consider only processes with connected span, for which unvisited sites can only be reached by nearest-neighbor jumps, so that qualitatively trajectories have no holes.

We first consider the case of a semi-infinite lattice (equivalent to the limit $N \to \infty$). The probability for the walker to have visited exactly n sites when it escapes the domain is then exactly given by the probability that it reaches the site $n-1$ and then escapes without going any further than $n-1$. (see Fig. 2).

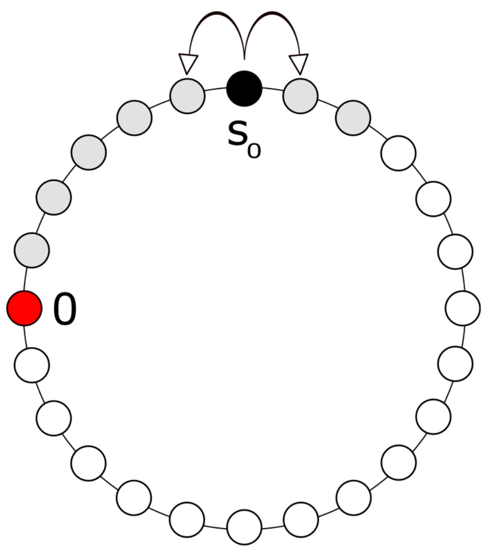

FIG. 1. Number of distinct sites visited by a random walker before hitting a target site of a finite domain. Starting from s_0, the random walker visits all the light gray sites before reaching the red target site 0. In this case, $N = 24$ and $S = 9$.

In other words, one has

$$P(S = n) = \pi_{\underline{n-1},0}(s_0)\ \pi_{\underline{0},n}(n-1), \tag{1}$$

where the Markov property has been used. Here $\pi_{\underline{i},j}(k)$ is the splitting probability that the walker, starting from k, reaches the site i before the site j. The exact distribution of S for the walker in a finite 1D periodic geometry is obtained by using the same idea. Taking into account the fact that S is bounded by N and that the trajectories to the target can be sorted as either "clockwise" or "counterclockwise," we finally obtain

$$\begin{aligned} P(S = n) = {} & 1_{s_0+1\leqslant n}\pi_{\underline{n-1},0}(s_0)\pi_{\underline{0},n}(n-1) \\ & + 1_{n\geqslant N-s_0+1}\pi_{\underline{N-n+1},N}(s_0)\pi_{\underline{N},N-n}(N-n+1), \end{aligned} \tag{2}$$

where $1_{s_0+1\leqslant n} = 1$ if $s_0 + 1 \leqslant n$ and $1_{s_0+1\leqslant n} = 0$ otherwise; $1_{n\geqslant N-s_0+1}$ is defined accordingly. Importantly, the expression (2) is exact for all Markov processes with connected span and can be easily extended to one-step non-Markovian processes, such as the persistent walk considered below. In addition it is fully explicit, provided that the splitting probabilities of the process can be determined. We provide explicit examples in what follows.

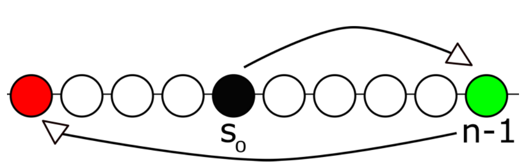

FIG. 2. Example of trajectory. Starting from s_0, the walker first reaches site $n-1$ and then goes back on its tracks to exit at the red site.

III. APPLICATIONS: EXPLICIT EXAMPLES

A. Symmetric nearest-neighbor random walk

Let us first consider the example of the classical symmetric nearest-neighbor random walk. As mentioned above, this case has been studied in the limit $N \to \infty$ (unconfined system) in the continuous space approximation in Ref. [24]. The splitting probability is well known [1,27] and given by

$$0 \leqslant s_0 \leqslant s_1,\ \pi_{\underline{s_1},0}(s_0) = \frac{s_0}{s_1}, \tag{3}$$

$$s_1 \leqslant s_0 \leqslant N,\ \pi_{\underline{s_1},N}(s_0) = \frac{N-s_0}{N-s_1}. \tag{4}$$

The distribution of S is then explicitly determined by Eq. (2). Figure 3 shows the exact distribution $P(S)$ (confirmed by numerical simulations) for examples of parameters s_0 and N. Note the two sharp jumps in the distribution located at $n = s_0 + 1$ and $n = N - s_0 + 1$. These jumps are found to be local maxima and reflect the fact that $P(S)$ can be decomposed as the sum of two contributions [see Eq. (2)]: clockwise and counterclockwise trajectories. For each set of trajectories the most probable span is given by the distance to the exit (therefore s_0 or $N - s_0$). As expected, these sharp jumps are observed for $s_0 = O(N)$ and $N - s_0 = O(N)$, and the distribution smoothens for $N \gg s_0$ (or equivalently $N \gg N - s_0$), where the semi-infinite case is recovered.

B. Biased nearest-neighbor random walk

In many physical situations, external force fields can give rise to biased diffusion. The corresponding model is that of the biased random walk, defined in a discrete setting as follows: at each step the walker steps to the right with probability p and to the left with probability $1 - p$ [1]. The splitting probability can be obtained from the classical backward equation [1,4], which yields

$$0 \leqslant s_0 \leqslant s_1,\ \pi_{\underline{s_1},0}(s_0) = \frac{\alpha^{s_0}-1}{\alpha^{s_1}-1}, \tag{5}$$

$$s_1 \leqslant s_0 \leqslant N,\ \pi_{\underline{s_1},N}(s_0) = \frac{\alpha^{s_0}-\alpha^N}{\alpha^{s_1}-\alpha^N}, \tag{6}$$

where $\alpha = \frac{1-p}{p}$. The distribution of S is then explicitly determined by Eq. (2). Similarly to the unbiased case above, the distribution shows two sharp jumps for $s_0 = O(N)$ and $N - s_0 = O(N)$, whose interpretation is unchanged; their relative weight is now controlled by the bias (see Fig. 4).

C. Persistent nearest-neighbor random walk

Another important example of random walk involved in the context of search processes is the persistent random walk [22,28], defined as follows. At each time step the walker performs a step identical to the previous one with probability p, and opposite with probability $1 - p$. We also introduce the probability a that the first step is clockwise. Even though its derivation relies on rather standard tools, the expression of the splitting probability for the persistent random walk does not seem to be present in the literature; we therefore provide the main steps of the derivation below.

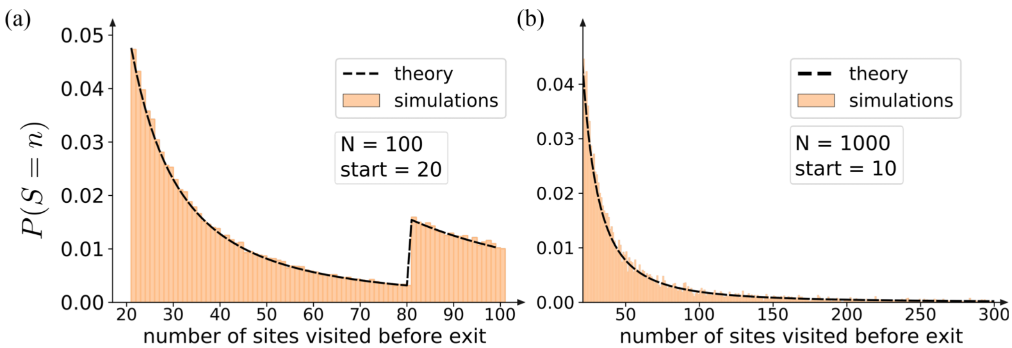

FIG. 3. Distribution $P(S)$ for a symmetric nearest-neighbor random walker. (a) $s_0 = 20$, $N = 100$. (b) $s_0 = 10$, $N = 1000$. Exact results are compared to 10^6 (left panel) and 10^5 (right panel) numerical simulations.

We consider the case $0 < s_0 < s$ and aim at determining $\pi_{\underline{s},0}(s_0, a)$; the case $s < s_0 < N$ will then be deduced by taking $s \to N - s$, $s_0 \to N - s_0$ and $a \to 1 - a$. Boundary conditions yield $\pi_{\underline{s},0}(s, a) = 1$ and $\pi_{\underline{s},0}(0, a) = 0$. It is useful to define $u_{\underline{s},0}(s_0)$ as the probability to reach s before 0, knowing that the step that led to s_0 was to the right, with the boundary condition $u_{\underline{s},0}(s) = 1$. Similarly, we introduce $v_{\underline{s},0}(s_0)$ as the probability to reach s before 0, knowing that the step that led to s_0 was to the left, with the boundary condition $v_{\underline{s},0}(0) = 0$. Partitioning over the first step of the walk, we obtain the following equation satisfied by $\pi_{\underline{s},0}(s_0, a)$ for $0 < s_0 < s$:

$$\pi_{\underline{s},0}(s_0, a) = a u_{\underline{s},0}(s_0 + 1) + (1 - a) v_{\underline{s},0}(s_0 - 1). \quad (7)$$

Similarly, a set of equations for u and v is given by

$$\begin{aligned} u_{\underline{s},0}(s_0) &= p u_{\underline{s},0}(s_0 + 1) + (1 - p) v_{\underline{s},0}(s_0 - 1), \\ v_{\underline{s},0}(s_0) &= p v_{\underline{s},0}(s_0 - 1) + (1 - p) u_{\underline{s},0}(s_0 + 1), \end{aligned} \quad (8)$$

which can be rewritten as

$$\begin{aligned} u_{\underline{s},0}(s_0 + 1) - 2u_{\underline{s},0}(s_0) + u_{\underline{s},0}(s_0 - 1) &= 0, \\ v_{\underline{s},0}(s_0) &= \frac{1}{1 - p}[u_{\underline{s},0}(s_0 + 1) - p u_{\underline{s},0}(s_0 + 2)] \end{aligned} \quad (9)$$

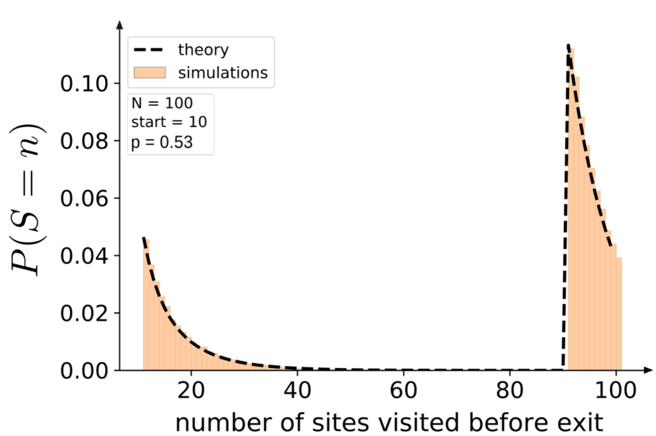

FIG. 4. Distribution $P(S)$ for a biased random walker starting at $s_0 = 10$ with rightward drift $p = 0.53$. The rightmost peak has much higher weight because the walker has a strong tendency to escape clockwise, even though the counterclockwise exit is closer. Exact results are compared to 10^5 numerical simulations.

Enforcing the boundary conditions leads to the following determination of $u_{\underline{s},0}(s_0)$, $v_{\underline{s},0}(s_0)$:

$$\begin{aligned} u_{\underline{s},0}(s_0) &= 1 + B(s_0 - s), \\ v_{\underline{s},0}(s_0) &= B s_0, \end{aligned} \quad (10)$$

where

$$B = \frac{p - 1}{(1 - s)(1 - p) - p}. \quad (11)$$

Finally, the splitting probability reads for $0 < s_0 < s$ as

$$\pi_{\underline{s},0}(s_0, a) = B s_0 + a(1 + 2B - Bs) - B \quad (12)$$

or, equivalently,

$$\pi_{\underline{s},0}(s_0, a) = \frac{p - 1}{(1 - p)(1 - s) - p} s_0 + \frac{1 - p - a}{1 + p(s - 2) - s}. \quad (13)$$

Note that the splitting probability is an affine function of s_0, as in the case of the normal random walk. However, the slope for the persistent random walk is controlled by both p and s. As expected, the case of the normal walk is recovered by taking $p = a = 1/2$. The distribution of S is then explicitly determined by Eq. (2) and reads as follows:

$$\begin{aligned} P(S = n) &= 1_{s_0+1 \leqslant n} \pi_{\underline{n-1},0}(s_0, a) \pi_{\underline{0},n}(n - 1, p) \\ &\quad + 1_{n \geqslant N - s_0 + 1} \pi_{\underline{N-n+1},N}(s_0, a) \pi_{\underline{N},N-n}(N - n + 1, 1 - p). \end{aligned} \quad (14)$$

Figure 5 shows $P(S)$ for $a = 0.8$ and $p = 0.8$. Again, the distribution shows two sharp jumps for $s_0 = O(N)$ and $N - s_0 = O(N)$. Here, the corresponding peaks are sharpened as the persistence time of the random walk (controlled by the parameter p) is increased.

D. Resetting random walk

We now turn to the resetting walk, which is another example of the search process that has recently been given much attention [25,26]. This process has been mostly studied in continuous space and time, with the exception of Refs. [29,30]; we consider here a discrete version. At each time step, the walker either performs a nearest-neighbor jump (drawn symmetrically) with probability $1 - \lambda$ or resets to its

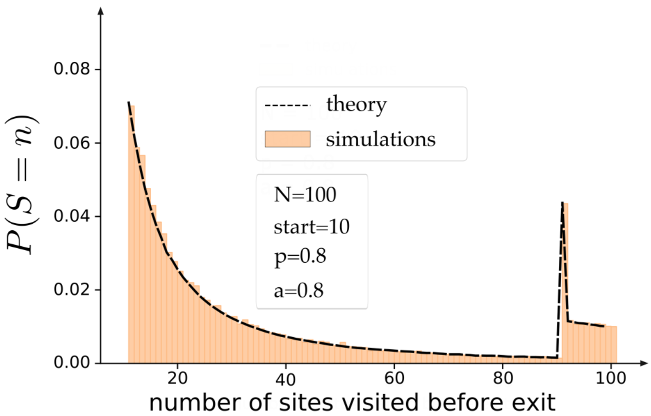

FIG. 5. Distribution $P(S)$ for a persistent random walker with $s_0 = 10, N = 100, p = 0.8$, and $a = 0.8$. Exact results are compared to 5×10^5 numerical simulations.

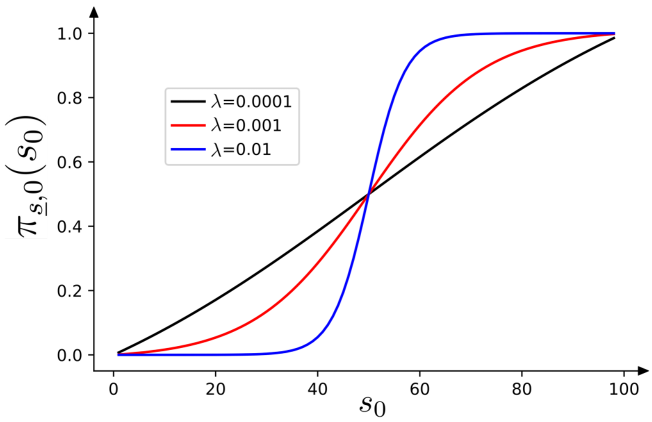

FIG. 6. Splitting probability of a resetting random walker on the interval [0,100], as a function of the initial position for different values of the resetting probability λ. Exact result of Eq. (25).

initial position s_0 with probability λ. Note that despite the resetting jumps whose range can cover many sites, the span of the process remains connected, because unvisited sites can only be reached by nearest-neighbor jumps (indeed resetting jumps all lead to the initial site).

We first determine the splitting probability $\pi_{\underline{s},0}(s_0)$, which has not been given explicitly in the literature for discrete processes to the best of our knowledge, even if related quantities have been studied for continuous resetting processes [31,32]. It is convenient to introduce an auxiliary site s_p (not necessarily equal to s_0), where resetting jumps end. The backward equation satisfied by $\pi_{\underline{s},0}(s_0)$ can then be written as

$$\pi_{\underline{s},0}(s_0) = \frac{1-\lambda}{2}[\pi_{\underline{s},0}(s_0+1) + \pi_{\underline{s},0}(s_0-1)] + \lambda\pi_{\underline{s},0}(s_p) \tag{15}$$

or, equivalently,

$$L_{s_0}\pi_{\underline{s},0}(s_0) = -\frac{2\lambda}{1-\lambda}\pi_{\underline{s},0}(s_p), \tag{16}$$

with

$$L_{s_0}\pi_{\underline{s},0}(s_0) = \pi_{\underline{s},0}(s_0+1) - \frac{2}{1-\lambda}\pi_{\underline{s},0}(s_0) + \pi_{\underline{s},0}(s_0-1). \tag{17}$$

The backward equation is completed by the boundary conditions $\pi_{\underline{s},0}(0) = 0$ and $\pi_{\underline{s},0}(s) = 1$. The solution of the linear equation (16) is then given by the sum of the homogeneous solution $h_s(s_0)$ and a particular solution $p(s_0)$. The homogeneous solution, defined by

$$L_{s_0}h_s(s_0) = 0 \tag{18}$$

and $h_s(0) = 0$ and $h_s(s) = 1$, can be written as

$$h_s(s_0) = \frac{r_+^{s_0} - r_-^{s_0}}{r_+^{s} - r_-^{s}}, \tag{19}$$

where $r_\pm = \frac{1}{1-\lambda} \pm \sqrt{\frac{1}{(1-\lambda)^2} - 1}$. In turn, the particular solution can be constructed using the Green function of the problem defined by

$$L_{s_1}G(s_1, s_2) = \delta_{s_1,s_2}, \tag{20}$$

with vanishing boundary conditions

$$G(0, s_2) = G(s, s_2) = 0. \tag{21}$$

One obtains

$$G(s_1, s_2) = 1_{s_1 \leqslant s_2}G_-(s_1, s_2) + 1_{s_1 > s_2}G_+(s_1, s_2), \tag{22}$$

where

$$G_-(s_1, s_2) = A(s_2)^{-1}\left(r_+^{s_1} - r_-^{s_1}\right),$$

$$G_+(s_1, s_2) = A(s_2)^{-1}\frac{r_+^{s_2} - r_-^{s_2}}{r_+^{s_2} - r_-^{s_2}\frac{r_+^s}{r_-^s}}\left(r_+^{s_1} - r_-^{s_1}\frac{r_+^s}{r_-^s}\right),$$

and

$$A(s_2) = \left(r_+^{s_2} - r_-^{s_2}\right)\left(r_+^{s_2} - r_-^{s_2}\frac{r_+^s}{r_-^s}\right)^{-1}\left(r_+^{s_2+1} - r_-^{s_2+1}\frac{r_+^s}{r_-^s}\right) - \frac{2}{1-\lambda}\left(r_+^{s_2} - r_-^{s_2}\right) + \left(r_+^{s_2-1} - r_-^{s_2-1}\right).$$

Finally, the particular solution can be written as

$$p(s_0) = -\frac{2}{1-\lambda}\pi_{\underline{s},0}(s_p)\sum_{s_2}G(s_0, s_2). \tag{23}$$

Taking now $s_0 = s_p$ and writing $\pi_{\underline{s},0}(s_0) = h_s(s_0) + p(s_0)$ provides a self-consistent equation for $\pi_{\underline{s},0}(s_p)$, which finally yields

$$\pi_{\underline{s},0}(s_p) = \frac{h_s(s_p)}{1 + \frac{2\lambda}{1-\lambda}\sum_{s_2}G(s_p, s_2)}. \tag{24}$$

The splitting probability for arbitrary s_p and s_0 can then be obtained as

$$\pi_{\underline{s},0}(s_0) = h_s(s_0) - \frac{h_s(s_p)2\lambda\sum_{s_2}G(s_0, s_2)}{1 - \lambda + 2\lambda\sum_{s_2}G(s_p, s_2)}, \tag{25}$$

which covers in particular the case of resetting to the initial position $s_p = s_0$ that we consider in this paper. Interestingly, as λ increases, the splitting probability takes a steplike shape (see Fig. 6), which has important consequences on the distribution of the covered territory S, obtained as before from

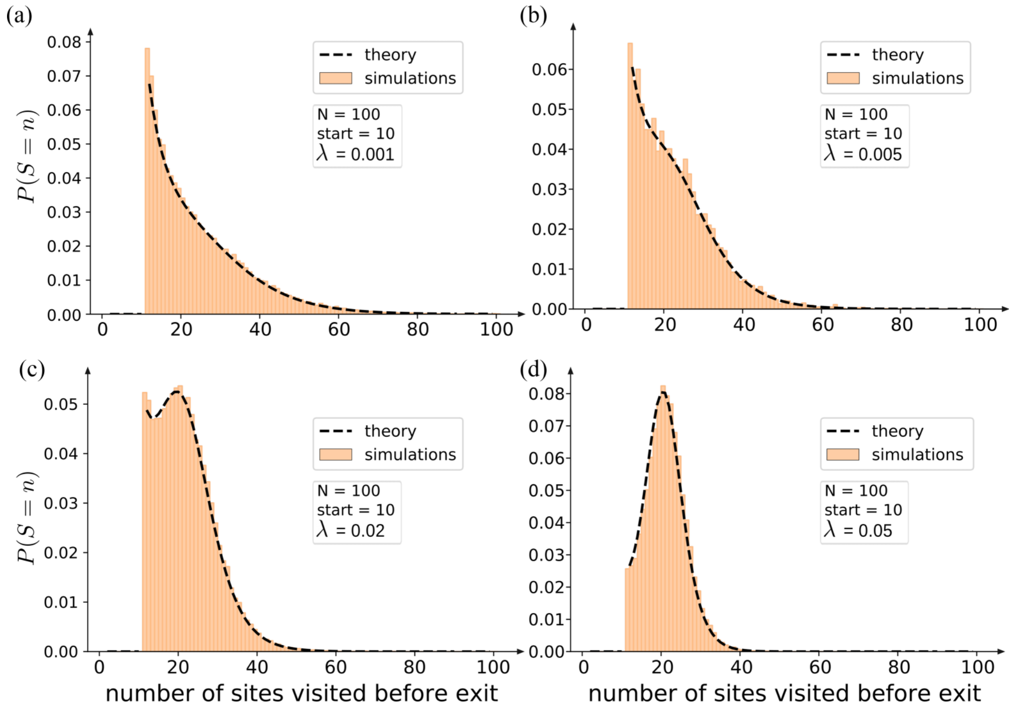

FIG. 7. Distribution $P(S)$ for a resetting random walker, with $s_0 = 10$ and $N = 100$: (a) $\lambda = 0.001$, (b) $\lambda = 0.005$, (c) $\lambda = 0.02$, and (d) $\lambda = 0.05$. Exact results are compared to 5×10^5 numerical simulations.

Eq. (2). Figure 7 reveals an interesting behavior: as λ increases, a spike in the distribution grows at $2s_0$. This can be interpreted as follows. Let us assume $s_0 < N/2$ and define $D = 1/2$ as the discrete diffusion coefficient of the walk. As expected, for $\lambda \ll D/s_0^2$, resetting jumps can be neglected and one recovers the case of the symmetric nearest-neighbor random walk. In turn, for $\lambda \gg D/s_0^2$, many resetting events, which lead to a symmetric exploration of the domain around s_0 occur before the target is reached. The explored territory before exit is thus approximately $2s_0$, which yields the observed peak in the distribution.

IV. EXTENSIONS

Let us summarize our results. So far, we have provided an analytical expression for the probability distribution of the territory explored before exit [Eq. (2)]. This expression holds for discrete Markovian random processes with a connected span on one-dimensional periodic lattices. Below, we extend this result in several directions.

A. Reflecting boundary condition

We first consider the case of reflecting boundary conditions. We thus consider a discrete Markovian random process with a connected span on a 1D lattice of N sites. The target site is still denoted by 0, and we assume that there is a reflecting boundary at site $N-1$: this effectively means that all nearest-neighbor jumps from site $N-1$ lead to site $N-2$. The distribution of S is then readily obtained and can be written as follows:

$$\begin{aligned} &P(S=n) = 0 \text{ if } n \leqslant s_0, \\ &P(S=n) = \pi_{\underline{n-1},0}(s_0)\pi_{\underline{0},n}(n-1) \text{ if } s_0 < n < N, \\ &P(S=n) = \pi_{\underline{N-1},0}(s_0) \text{ if } n = N. \end{aligned} \quad (26)$$

As in the case of periodic boundary conditions, this expression only involves splitting probabilities.

B. Connection to the distribution of the maximum

In this paragraph, we propose an alternative expression of the distribution of the territory explored before exit $P(S)$ for discrete random processes with a connected span, in terms of the distribution $\sigma(s_0, s)$ [respectively, $\mu(s_0, s)$] of the maximum (respectively, minimum) s reached by the random walker starting from s_0 positive (respectively, negative) before exiting the domain at site 0. In the case of periodic boundary conditions, it is clear that

$$\begin{aligned} P(S=n) = {} & 1_{s_0+1\leqslant n}\sigma(s_0, n-1) \\ & + 1_{N-s_0+1\leqslant n}\mu(s_0-N, 1-n), \end{aligned} \quad (27)$$

which takes the following simpler form in the case of a symmetric random walk:

$$\begin{aligned} P(S=n) = {} & 1_{s_0+1\leqslant n}\sigma(s_0, n-1) \\ & + 1_{N-s_0+1\leqslant n}\sigma(N-s_0, n-1), \end{aligned} \quad (28)$$

where $\sigma(s_0, s)$ can be simply expressed in terms of splitting probabilities according to

$$\pi_{\underline{0},s}(s_0) = \sum_{k=s_0}^{s-1} \sigma(s_0, k). \quad (29)$$

Equivalently, one has

$$\sigma(s_0, s) = \pi_{\underline{0},s+1}(s_0) - \pi_{\underline{0},s}(s_0). \tag{30}$$

Note that the Markov property has not been used to obtain Eq. (28). For Markov processes with a connected span, an alternative expression is given by

$$\sigma(s_0, s) = \pi_{\underline{s},0}(s_0)\pi_{\underline{0},s+1}(s), \tag{31}$$

which, together with Eq. (28), immediately yields Eq. (2). Note also that Eqs. (31) and (30) yield the following equality that is valid for Markov processes:

$$\pi_{\underline{0},s+1}(s_0) - \pi_{\underline{0},s}(s_0) = \pi_{\underline{s},0}(s_0)\pi_{\underline{0},s+1}(s), \tag{32}$$

which can be obtained by a direct probabilistic argument.

C. Moments $\langle S^p \rangle$

As a by-product, the expression (28) of the distribution $P(S)$ gives access to all moments of S according to

$$\begin{aligned}\langle S^p(s_0)\rangle &= N^p\pi_{\underline{0},N}(s_0) + N^p\pi_{\underline{0},N}(N-s_0)\\ &\quad - \sum_{n=s_0}^{N-1}[(n+1)^p - n^p]\pi_{\underline{0},n}(s_0)\\ &\quad - \sum_{n=N-s_0}^{N-1}[(n+1)^p - n^p]\pi_{\underline{0},n}(N-s_0).\end{aligned} \tag{33}$$

D. Continuous limit

We now consider the continuous-space limit of the problem; we are interested in continuous-space random processes with a connected span taking place on the $[0, L]$ ring with periodic boundary conditions. This can be readily done by making use of Eq. (27), which can be rewritten as

$$P(S = x) = 1_{x_0\leqslant x}\,\sigma(x_0, x) + 1_{L-x_0\leqslant x}\,\mu(x_0 - L, -x) \tag{34}$$

and in the symmetric case as

$$P(S = x) = 1_{x_0\leqslant x}\,\sigma(x_0, x) + 1_{L-x_0\leqslant x}\,\sigma(L - x_0, x), \tag{35}$$

where

$$\sigma(x_0, x) = \frac{d}{dx}\pi_{\underline{0},x}(x_0). \tag{36}$$

Similarly to Eq. (33), any moment of S can be written as follows:

$$\begin{aligned}\langle S^p(x_0)\rangle &= \pi_{\underline{0},L}(x_0)L^p + \pi_{\underline{0},L}(L-x_0)L^p\\ &\quad - p\int_{x_0}^{L} x^{p-1}\pi_{\underline{0},x}(x_0)dx\\ &\quad - p\int_{L-x_0}^{L} x^{p-1}\pi_{\underline{0},x}(L-x_0)dx.\end{aligned} \tag{37}$$

As an example, we consider the resetting random walk in continuous time and space with the constant resetting rate λ and the diffusion constant D. The splitting probability obeys the following backward equation:

$$\frac{d^2}{dx^2}\pi_{\underline{x},0}(x_0) - r^2\pi_{\underline{x},0}(x_0) = r^2\pi_{\underline{x},0}(x_p), \tag{38}$$

where x_p denotes the resetting site, x_0 denotes the starting position, and $r = \sqrt{\frac{\lambda}{D}}$. The complete solution reads (see also Ref. [32])

$$\begin{aligned}\pi_{\underline{x},0}(x_0) &= \frac{\sinh(rx_0)}{\sinh(rx)} + \frac{\pi_{\underline{x},0}(x_p)}{\sinh(rx)}\\ &\quad\times\{\sinh(rx) - \sinh[r(x-x_0)] - \sinh(rx_0)\}.\end{aligned} \tag{39}$$

Taking $x_0 = x_p$ and using $\pi_{\underline{0},x}(x_0) = 1 - \pi_{\underline{x},0}(x_0)$, one obtains

$$\pi_{\underline{0},x}(x_p) = \frac{\sinh[r(x-x_p)]}{\sinh[r(x-x_p)] + \sinh(rx_p)}, \tag{40}$$

which finally yields

$$\sigma(x_p, x) = \frac{r\sinh(rx_p)\cosh[r(x-x_p)]}{\{\sinh(rx_p) + \sinh[r(x_p - x)]\}^2}. \tag{41}$$

The distribution of S is then obtained from Eq. (28).

E. An example of a process with a nonconnected span: The golden coupon problem

So far, all the results that we have discussed apply to random walks with a connected span. Here we consider a typical example of a random walk with a nonconnected span, which belongs to the family of intermittent random walks [2,33]. We assume that at each time step, the random walker jumps to a site drawn uniformly from the set of N sites. For this process, determining the distribution of the territory explored before exit amounts to solving the following simple coupon problem. Assume that a coupon is drawn randomly out of N different coupons labeled from 1 to N. The experiment is repeated until the coupon N (the golden coupon) is drawn. Recall that the random walk defined above starts from a given site $s_0 \neq 0$. Denoting $p(n)$ the probability that n distinct coupons have been drawn before the golden one, one has

$$P(S = n+1) = p(n|n > 0) = \frac{N}{N-1}p(n). \tag{42}$$

This problem can be solved as follows. We denote by $S_{k,n}$ the number of surjections from $[\![0, k-1]\!]$ to $[\![1, n]\!]$. The probability $p(n)$ of receiving n distinct coupons before N now reads

$$\begin{aligned}p(n) &= \binom{N-1}{n}\sum_{k=0}^{\infty}\frac{1}{N^k}S_{k,n}\frac{1}{N}\\ &= \binom{N-1}{n}\frac{1}{N}\sum_{k=0}^{\infty}\frac{1}{N^k}\sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j}j^k\\ &= \binom{N-1}{n}\frac{1}{N}(-1)^n\underbrace{\sum_{j=0}^{n}(-1)^j\binom{n}{j}\frac{1}{1-\frac{j}{N}}}_{A},\end{aligned} \tag{43}$$

where A can be shown to be given by

$$\begin{aligned}A &= (-1)^n\frac{n!}{(N-1)\cdots(N-n)},\\ A &= \frac{(-1)^n}{\binom{N-1}{n}},\end{aligned} \tag{44}$$

yielding

$$p(n) = \frac{1}{N}. \tag{45}$$

Finally, using Eq. (42) we obtain the strikingly simple exact result:

$$\forall n \geqslant 2,\ P(S = n) = \frac{1}{N-1}. \tag{46}$$

Note that the result (45) can be interpreted as follows. Let us consider that the experiment is repeated until all different coupons are drawn at least once. This event eventually occurs with probability 1. For symmetry reasons, the order of appearance of the golden coupon N is uniformly distributed, which directly yields (45).

V. CONCLUSION

To conclude, we have derived the full distribution of the territory S explored by a one-dimensional random walker before it exits a finite domain. This result applies to the general class of Markovian processes with a connected span. We have demonstrated that this distribution can be expressed in terms of splitting probabilities only, which can in general be derived from backward equations. We have applied our approach to various examples of random processes, which have appeared in the literature in the context of target search problems. These include simple symmetric random walks, biased random walks, persistent random walks, and resetting random walks. As a by-product, we have provided explicit expressions for the splitting probabilities of discrete persistent random walks and discrete resetting random walks. We have finally discussed several extensions of our approach, namely, to the case of reflecting boundary conditions, continuous processes, and an example of a random process with a nonconnected span. We now wish to inquire about higher dimensions and study S thoroughly in different setups. Of note, our approach is by nature limited to one-dimensional processes with a connected span, for which the explored territory has a simple geometry parametrized by a single scalar parameter; the cases of higher-dimensional problems or of general processes with nonconnected spans call for alternative methods.

[1] B. Hughes, *Random Walks and Random Environments* (Oxford University, New York, 1995).

[2] O. Bénichou, C. Loverdo, M. Moreau, and R. Voituriez, Rev. Mod. Phys. **83**, 81 (2011).

[3] G. M. Viswanathan, E. P. Raposo, and M. G. E. da Luz, Phys. Life Rev. **5**, 133 (2008).

[4] S. Redner, *A Guide to First-Passage Processes* (Cambridge University, Cambridge, England, 2001).

[5] R. Metzler, G. Oshanin, and S. Redner, *First Passage Problems: Recent Advances* (World Scientific, Singapore, 2014).

[6] A. J. Bray, S. N. Majumdar, and G. Schehr, Adv. Phys. **62**, 225 (2013).

[7] O. Bénichou and R. Voituriez, Phys. Rep. **539**, 225 (2014).

[8] S. Condamin, O. Bénichou, V. Tejedor, R. Voituriez, and J. Klafter, Nature (London) **450**, 77 (2007).

[9] A. F. Cheviakov, M. J. Ward, and R. Straube, Multiscale Model. Simul. **8**, 836 (2010).

[10] Z. Schuss, A. Singer, and D. Holcman, Proc. Natl. Acad. Sci. USA **104**, 16098 (2007).

[11] O. Bénichou, C. Chevalier, J. Klafter, B. Meyer, and R. Voituriez, Nat. Chem. **2**, 472 (2010).

[12] T. Guérin, O. Bénichou, and R. Voituriez, Nat. Chem. **4**, 568 (2012).

[13] T. Guérin, N. Levernier, O. Bénichou, and R. Voituriez, Nature (London) **534**, 356 (2016).

[14] M. J. A. M. Brummelhuis and H. J. Hilhorst, Phys. A (Amesterdam, Neth.) **176**, 387 (1991).

[15] M. J. A. M. Brummelhuis and H. J. Hilhorst, Phys. A (Amesterdam, Neth.) **185**, 35 (1992).

[16] M. Chupeau, O. Benichou, and R. Voituriez, Nat. Phys. **11**, 844 (2015).

[17] O. Bénichou, M. Coppey, M. Moreau, P. H. Suet, and R. Voituriez, Europhys. Lett. **70**, 42 (2005).

[18] G. H. Weiss and P. P. Calabrese, Phys. A: Stat. Theor. Phys. **234**, 443 (1996).

[19] S. Condamin, V. Tejedor, and O. Benichou, Phys. Rev. E **76**, 050102(R) (2007).

[20] S. Burov and E. Barkai, Phys. Rev. Lett. **98**, 250601 (2007).

[21] S. Condamin, V. Tejedor, R. Voituriez, O. Benichou, and J. Klafter, Proc. Natl. Acad. Sci. USA **105**, 5675 (2008).

[22] G. Weiss, *Aspects and Applications of the Random Walk* (North-Holland, Amsterdam, 1994).

[23] H. Larralde, P. Trunfio, S. Havlin, H. E. Stanley, and G. H. Weiss, Nature (London) **355**, 423 (1992).

[24] I. Dayan and S. Havlin, J. Phys. A: Math. Gen. **25**, L549 (1992).

[25] M. R. Evans and S. N. Majumdar, Phys. Rev. Lett. **106**, 160601 (2011).

[26] M. R. Evans, S. N. Majumdar, and G. Schehr, J. Phys. A: Math. Theor. **53**, 193001 (2020).

[27] N. Van Kampen, *Stochastic Processes in Physics and Chemistry*, 3rd ed. (North-Holland, Amsterdam, 1992).

[28] V. Tejedor, R. Voituriez, and O. Bénichou, Phys. Rev. Lett. **108**, 088103 (2012).

[29] A. P. Riascos, D. Boyer, P. Herringer, and J. L. Mateos, Phys. Rev. E **101**, 062147 (2020).

[30] L. N. Christophorov, J. Phys. A: Math. Theor. **54**, 015001 (2020).

[31] A. Chechkin and I. M. Sokolov, Phys. Rev. Lett. **121**, 050601 (2018).

[32] A. Pal and V. V. Prasad, Phys. Rev. E **99**, 032123 (2019).

[33] O. Benichou, C. Loverdo, M. Moreau, and R. Voituriez, Phys. Chem. Chem. Phys. **10**, 7059 (2008).