Bounding generalized relative entropies: Nonasymptotic quantum speed limits

Diego Paiva Pires [●],[1] Kavan Modi,[2] and Lucas Chibebe Céleri[3,4]

[1]*International Institute of Physics and Departamento de Física Teórica e Experimental,*
Universidade Federal do Rio Grande do Norte, Natal, RN, 59078-970, Brazil
[2]*School of Physics & Astronomy, Monash University, Clayton, Victoria 3800, Australia*
[3]*Department of Physical Chemistry, University of the Basque Country UPV/EHU, Apartado 644, E-48080 Bilbao, Spain*
[4]*Institute of Physics, Federal University of Goiás, 74.690-900 Goiânia, Goiás, Brazil*

Information theory has become an increasingly important research field to better understand quantum mechanics. Noteworthy, it covers both foundational and applied perspectives, also offering a common technical language to study a variety of research areas. Remarkably, one of the key information-theoretic quantities is given by the relative entropy, which quantifies how difficult is to tell apart two probability distributions, or even two quantum states. Such a quantity rests at the core of fields like metrology, quantum thermodynamics, quantum communication, and quantum information. Given this broadness of applications, it is desirable to understand how this quantity changes under a quantum process. By considering a general unitary channel, we establish a bound on the generalized relative entropies (Rényi and Tsallis) between the output and the input of the channel. As an application of our bounds, we derive a family of quantum speed limits based on relative entropies. Possible connections between this family with thermodynamics, quantum coherence, asymmetry, and single-shot information theory are briefly discussed.

I. INTRODUCTION

Since its formulation decades ago by Shannon [1], information theory has played a major role in both applied and fundamental science, ranging from neuroscience [2] to quantum gravity [3,4], and along the way has impacted thermodynamics [5], finance [6], and evolutionary biology [7]. A central element in this theory is the Shannon entropy, which measure how much information is contained in a probability distribution. Shannon entropy also plays a role on the speed of evolution of classical stochastic processes [8]. However, when the basic assumptions of the theory do not hold, e.g., extensivity or very large data sets (nonasymptotic regime), other information measures appear as generalizations of the Shannon entropy. Indeed, such a family of information-theoretic measures include the paradigmatic cases of Tsallis [9] and Rényi [10] entropies.

Each of these developments is based on the idea that a physical process could be understood as an information processing protocol. In such tasks, distinguishing classical probability distributions or quantum states plays a fundamental role. The relative entropy (RE) [11], also called divergence [12], stand as a remarkable information-theoretic distinguishability metric, thus exhibiting distinct operational meanings in several fields. For instance, relative entropy quantifies the dissipated work in a driven evolution [13], the amount of entanglement [14], and quantum coherence in a given state [15,16]. Moreover, it unveils the role of entropy production in thermal relaxation processes [17–19] and also the asymmetry of a state or process [20].

Rényi relative entropy (RRE) determines an entire family of second laws of thermodynamics in the quantum regime [21], which also applies to black hole physics [22], and cutoff rate in the hypothesis testing theory [23] and quantum Gaussian states [24], to name only a few. Furthermore, RRE is linked to an entropic energy-time uncertainty relation for time-independent systems [25], also being related to the concept of multiple quantum coherences [26].

Tsallis entropy is mainly considered in the field of nonextensive statistical mechanics [27]. However, important applications of this theory also appear in several other areas [28]. Interestingly, it has been shown that Tsallis relative entropies (TRE) define a *bona fide* quantum coherence quantifier [29]. Furthermore, TRE satisfies a class of bounds derived from Pinsker and Fannes–type inequalities [30].

Here we consider the fundamental problem of bounding the change in the generalized relative entropies under an arbitrary unitary process. Specifically, we derive an upper bound on both asymmetric and symmetric versions of RRE and TRE, between the initial and final states. As an application of this result, we show that this upper bound implies an entirely family of quantum speed limits (QSLs).

The importance of our results is twofold. First, it establishes a bound on entropic quantifiers that are employed in distinct fields from quantum communication to biology [31,32]. In general, since the computation of relative entropies are usually difficult, our main result can directly be applied in all of these fields by providing bounds on central quantities. Second, our family of QSLs provide nonasymptotic bounds on the time evolution of quantum systems in the sense of the

so-called single-shot information theory [33]. Furthermore, due to the broadness application of RRE, it provides a bridge among the speed of quantum evolution, thermodynamics [34], and quantum resources, e.g., entanglement, coherence, and asymmetry [35]. Importantly, since our results apply to TRE, it also provides a nonextensive version of the QSL, which can found several applications, both on fundamental and practical aspects [36].

The paper is organized as follows. In Sec. II we briefly review the main properties of Rényi and Tsallis relative entropies, which in turn can be recasted in terms of a generalized entropy. In Sec. III, we introduce the physical setting and present an upper bound on those generalized entropies. Next, in Sec. IV we discuss an entire family of QSLs derived from the referred upper bound on Rényi and Tsallis relative entropies. In Sec. V we illustrate our findings via the prototypical case of the single-qubit state, thus presenting analytical results for the family of QSLs, and also discussing the tightness of the main bound on generalized entropies. Finally, in Sec. VI we close the paper discussing these results and comment on possible applications.

II. GENERALIZED RELATIVE ENTROPIES

Let us start by defining our physical system, which is described by a finite-dimensional Hilbert space $\mathcal{H}$, with $d = \dim(\mathcal{H})$. In general, the state of the system will be given by a density matrix $\rho \in \Omega$, where $\Omega = \{\rho \in \mathcal{H} \mid \rho^\dagger = \rho, \rho \geqslant 0, \mathrm{Tr}(\rho) = 1\}$ defines the convex space of density operators. In this setting, given two states $\rho, \omega \in \Omega$, the Rényi (RRE) and Tsallis (TRE) relative entropies are defined, respectively, as [37]

$$R_\alpha(\rho \| \omega) = \frac{1}{\alpha - 1} \ln [g_\alpha(\rho, \omega)], \tag{1}$$

and

$$H_\alpha(\rho \| \omega) = \frac{1}{1 - \alpha} [1 - g_\alpha(\rho, \omega)], \tag{2}$$

where $g_\alpha(\rho, \omega) = \mathrm{Tr}(\rho^\alpha \omega^{1-\alpha})$ is the α-relative purity, also called Petz-Rényi relative quasi-entropy [24], with the parameter $\alpha \in (0, 1) \cup (1, +\infty)$ labeling the family of quantum relative entropies [38]. Equation (1) is also called Petz-Rényi relative entropy [39], standing as the first quantum extension of the classical RRE. Indeed, due to the noncommutativity of quantum states, the nonuniqueness of quantum information-theoretic quantifiers has triggered the search for a plethora of quantum entropies, e.g., sandwiched Rényi relative entropy [37,40–42], and α-z-relative Rényi entropy [43].

Importantly, relative purity satisfies the property $g_\alpha(\omega, \rho) = g_{1-\alpha}(\rho, \omega)$, i.e., it is skew symmetric with respect to α. In particular, when $\rho = \omega$ we have $g_\alpha(\rho, \rho) = 1$ for all α, and thus one gets $R_\alpha(\rho \| \rho) = H_\alpha(\rho \| \rho) = 0$. Noteworthy, for $\alpha = 1/2$ one recovers the so-called quantum affinity, which is related to Hellinger angle [44]. In turn, Hellinger angle is associated to Wigner-Yanase skew information metric and characterizes the length of the geodesic path connecting states $\rho, \omega \in \Omega$ [45,46].

In the following we summarize the main properties of RRE and TRE. A more complete presentation can be found in

Ref. [47]. Starting with RRE, the limit $\alpha \to 1$ recovers the well-known quantum relative entropy $R_1(\rho \| \omega) = S(\rho \| \omega) := \mathrm{Tr}(\rho \ln \rho - \rho \ln \omega)$. For $\alpha = 0$, RRE reduces to the min-relative entropy $R_0(\rho \| \omega) = -\ln \mathrm{Tr}(\Pi_\rho \omega)$, with Π_ρ being the projector onto the support of the state ρ [48]. Noteworthy, for $0 \leqslant \alpha \leqslant 2$, RRE satisfies the data-processing inequality, i.e., $R_\alpha(\Lambda(\rho) \| \Lambda(\omega)) \leqslant R_\alpha(\rho \| \omega)$, thus being monotonic under any completely positive and trace preserving map $\Lambda(\bullet)$ [49]. This is a fundamental inequality not only within information theory but also for physics (see, for instance, Ref. [50], where the second law of thermodynamics is obtained from such inequality). Moving to Tsallis relative entropy, it has been shown that, for $0 \leqslant \alpha < 1$, TRE is (i) nonnegative, i.e., $H_\alpha(\rho \| \omega) \geqslant 0$ for all $\rho, \omega \in \Omega$, with the equality holding if and only if $\rho = \omega$; (ii) jointly convex; (iii) nonaditive; and (iv) contractive under completely positive and trace preserving maps [28,51,52]. Importantly, TRE also recovers the standard quantum relative entropy in the limit $\alpha \to 1$, i.e., $H_1(\rho \| \omega) = S(\rho \| \omega)$.

We shall stress that RRE and TRE are asymmetric with respect to states $\rho, \sigma \in \Omega$. However, a *bona fide* distance measure within the information geometry theory is usually symmetric. For instance, the so-called quantum Jensen-Shannon divergence, i.e., the square root of symmetrized quantum relative entropy, was proved to be a metric on the space of density matrices [53]. For the case at hand, the aforementioned entropies can be symmetrized as

$$\mathcal{O}_\alpha(\rho : \omega) := \mathcal{O}_\alpha(\rho \| \omega) + \mathcal{O}_\alpha(\omega \| \rho), \tag{3}$$

where index $\mathcal{O} \equiv \{R, H\}$ labels RRE and TRE, respectively. We are now ready to present our main result.

III. BOUNDS ON GENERALIZED RELATIVE ENTROPIES

The dynamics of our system is governed by a time-dependent Hamiltonian $H_t \in \mathcal{B}(\mathcal{H})$, with $\mathcal{B}(\mathcal{H})$ being the set of bounded operators acting on $\mathcal{H}$. In general, the Hamiltonian H_t is not self-commuting at different times, i.e., $[H_s, H_t] \neq 0$ for $s \neq t$. The initial state $\rho_0 \in \Omega$ undergoes the unitary evolution $\rho_t = U_t \rho_0 U_t^\dagger$, for $t \in [0, \tau]$, where $U_t = \mathcal{T} e^{-i \int_0^t ds H_s}$ is the time-ordered unitary evolution operator satisfying the equation $-i(dU_t/dt) = H_t U_t$. From now on, we will work in natural units, $\hbar = k_B = 1$.

Based on this physical setting, our goal is to provide a class of nontrivial upper bounds for RRE and TRE. In Appendix A 2 we have proved that, for $\alpha \in (0, 1)$, RRE and TRE satisfy the inequality

$$\mathcal{O}_\alpha(\rho_\tau \| \rho_0) \leqslant \frac{\tau \left\langle\!\left\langle \mathcal{G}_\alpha^\mathcal{O}(t) \right\rangle\!\right\rangle_\tau}{|1 - \alpha|}, \tag{4}$$

where $\langle\!\langle \bullet \rangle\!\rangle_\tau = \tau^{-1} \int_0^\tau \bullet \, dt$ stands for the time average, and

$$\mathcal{G}_\alpha^\mathcal{O}(t) := \Phi_\alpha^\mathcal{O} \left\| \rho_0^{1-\alpha} \right\|_2 \left\| \left[H_t, \rho_0^\alpha \right] \right\|_2, \tag{5}$$

with $\|A\|_2 = \sqrt{\mathrm{Tr}(A^\dagger A)}$ being the Schatten 2-norm. Here $\Phi_\alpha^\mathcal{O}$ is an auxiliary function which reads

$$\Phi_\alpha^\mathcal{O} = \begin{cases} |1 + (1 - \alpha) \ln(\lambda_{\min}(\rho_0))|^{-1}, & \text{for } \mathcal{O} \equiv R \\ 1, & \text{for } \mathcal{O} \equiv H, \end{cases} \tag{6}$$

where $\lambda_{\min}(\rho_0)$ sets the smallest eigenvalue of the input state ρ_0. Noteworthy, Eq. (4) is the first main result of this article. Remarkably, the bound mostly depends on ρ_0 and H_t. Naturally, a similar bound can be obtained for the case in which the arrangement of states ρ_0 and ρ_τ in Eq. (4) is swapped, which is given by (see Appendix A 2)

$$\mathcal{O}_\alpha(\rho_0\|\rho_\tau) \leqslant \frac{\tau \left\langle\!\!\left\langle \mathcal{G}^{\mathcal{O}}_{1-\alpha}(t) \right\rangle\!\!\right\rangle_\tau}{|1-\alpha|}. \tag{7}$$

Furthermore, the corresponding inequality for symmetrized forms of RRE and TRE is then obtained, roughly speaking, by combining the two nonsymmetric upper bounds, and reads (see details in Appendix A 3)

$$\mathcal{O}_\alpha(\rho_\tau : \rho_0) \leqslant \frac{\tau \left\langle\!\!\left\langle \mathcal{G}^{\mathcal{O}}_{\alpha}(t) + \mathcal{G}^{\mathcal{O}}_{1-\alpha}(t) \right\rangle\!\!\right\rangle_\tau}{|1-\alpha|}. \tag{8}$$

Remarkably, such bounds do not depend on the time-ordered evolution operator U_t, neither on the evolved state of the system. Thus, for states ρ_0 and ρ_τ of a given closed quantum system, our results provide a route to estimate both RRE and TRE entropies which will depend mostly on the spectral properties of the initial state ρ_0 and the driving Hamiltonian.

Importantly, Eq. (4) can be recast in terms of the Fröbenius norm of the initial state of the system, while being a function of the time average of the Schatten 2-norm of the Hamiltonian. To see this, we first point out that for two arbitrary complex matrices X and Y, it has been proved the Schatten 2-norm fulfills the inequality $\|[X, Y]\|_2 \leqslant \sqrt{2}\,\|X\|_2\|Y\|_2$ [54–56]. Hence, from Eq. (5) one readily obtains the upper bound $\mathcal{G}^{\mathcal{O}}_\alpha(t) \leqslant \sqrt{2}\,\|\rho_0^{1-\alpha}\|_2 \,\|\rho_0^\alpha\|_2 \,\|H_t\|_2$. Therefore, the bound in Eq. (4) can be recast as

$$\mathcal{O}_\alpha(\rho_\tau\|\rho_0) \leqslant \frac{\sqrt{2}\,\tau\,\Phi^{\mathcal{O}}_\alpha \,\|\rho_0^{1-\alpha}\|_2 \,\|\rho_0^\alpha\|_2 \,\langle\!\langle\,\|H_t\|_2\,\rangle\!\rangle_\tau}{|1-\alpha|}. \tag{9}$$

In particular, note that $\langle\!\langle\,\|H_t\|_2\,\rangle\!\rangle_\tau = \|H\|_2$ for the case in which the Hamiltonian is time independent, i.e., $H_t \equiv H$. Overall, the bound in Eq. (9) does requires minimal information about the system, e.g., its initial state ρ_0 and the energy levels of the Hamiltonian H_t. Indeed, the latter comes from the fact that, given the spectral decomposition $H_t = \sum_{j=1}^{d} \epsilon_j(t)|\phi_j(t)\rangle\langle\phi_j(t)|$, with $d = \dim \mathcal{H}$ the dimension of the Hilbert space, where the eigenvalues $\{\epsilon_j(t)\}_{j=1,\dots,d}$ and eigenstates $\{|\phi_j(t)\rangle\}_{j=1,\dots,d}$ are time dependent, and thus one obtains $\|H_t\|_2^2 = \sum_{j=1}^{d} \epsilon_j(t)^2$.

Finally, regardless of the simplicity and usefulness of the bound in Eq. (9), we shall point out the original bound in Eq. (4) might stand as the general and tighter one. Before discussing the physical significance of this bound, we will make use of it to obtain a family of QSLs.

IV. QUANTUM SPEED LIMITS

The QSL signals the minimum time of evolution between two quantum states undergoing an arbitrary dynamics. Indeed, Mandelstam and Tamm addressed this question around 75 years ago for closed quantum systems, thus showing the QSL time for orthogonal states is given by $\tau_{\rm QSL} = \hbar\pi/(2\Delta E)$, where $(\Delta E)^2 = \langle\psi_0|H^2|\psi_0\rangle - \langle\psi_0|H|\psi_0\rangle^2$ stands for the variance of the Hamiltonian with respect to the initial state

$|\psi_0\rangle$ of the system [57]. Later, Margolus and Levitin derived the QSL time $\tau_{\rm QSL} = \hbar\pi/(2\langle\psi_0|H|\psi_0\rangle)$ for the same physical setting, i.e., closed quantum systems evolving between orthogonal states [58]. Importantly, both lower bounds can be combined accordingly onto a tighter one as $\tau \geqslant \max\{\hbar\pi/(2\Delta E), \hbar\pi/(2\langle\psi_0|H|\psi_0\rangle)\}$ [59]. In the last decade, several bounds were introduced in the literature covering the QSL time for different physical settings, e.g., addressing pure and mixed states, for closed and open quantum systems [60–72].

Here we will present a family of QSLs by time averaging the right-hand side of Eq. (4), thus followed by a rearrangement of the resulting inequality. Indeed, the time τ required for an arbitrary unitary evolution driving a closed quantum system from ρ_0 to ρ_τ is lower bounded as

$$\tau \geqslant \tau^{\mathcal{O}}_\alpha := \max\left\{\tau^{\mathcal{O}}_\alpha(\rho_\tau\|\rho_0), \tau^{\mathcal{O}}_\alpha(\rho_0\|\rho_\tau), \tau^{\mathcal{O}}_\alpha(\rho_0 : \rho_\tau)\right\}, \tag{10}$$

where

$$\tau^{\mathcal{O}}_\alpha(\rho_\tau\|\rho_0) := \frac{|1-\alpha|\,\mathcal{O}_\alpha(\rho_\tau\|\rho_0)}{\left\langle\!\!\left\langle \mathcal{G}^{\mathcal{O}}_\alpha(t) \right\rangle\!\!\right\rangle_\tau}, \tag{11}$$

and

$$\tau^{\mathcal{O}}_\alpha(\rho_0\|\rho_\tau) := \frac{|1-\alpha|\,\mathcal{O}_\alpha(\rho_0\|\rho_\tau)}{\left\langle\!\!\left\langle \mathcal{G}^{\mathcal{O}}_{1-\alpha}(t) \right\rangle\!\!\right\rangle_\tau}, \tag{12}$$

while the QSL time due to symmetrized relative entropies reads [see Appendix A 3]

$$\tau^{\mathcal{O}}_\alpha(\rho_0 : \rho_\tau) := \frac{|1-\alpha|\,\mathcal{O}_\alpha(\rho_0 : \rho_\tau)}{\left\langle\!\!\left\langle \mathcal{G}^{\mathcal{O}}_\alpha(t) + \mathcal{G}^{\mathcal{O}}_{1-\alpha}(t) \right\rangle\!\!\right\rangle_\tau}. \tag{13}$$

Equation (10) stands as our second main result, thus establishing a family of entropic QSLs, i.e., RRE and TRE provide lower bounds on the time of evolution between the initial and final states of the quantum system. Recently, a related family of QSLs, based on the relative entropy, were derived bounding the time it takes to generate or consume a given quantum resource such as entanglement, asymmetry, and athermality [73]. These bounds, dubbed as *resource speed limits* (RSL) were shown to be tighter than QSLs in several instances. However, as RSLs are constructed using the standard relative entropy ($\alpha \to 1$), thus being only meaningful in the asymptotic limit. RSLs for single shot scenarios requires working with Rényi relative entropies. Here, we have taken the first step in this direction.

We shall stress that the previous discussion is valid for $\alpha \in (0, 1)$. In the following, we will discuss the limiting cases $\alpha \to 1$ and $\alpha \to 0$, which crucially reduce to the standard relative entropy and the so-called min-relative entropy, respectively. Importantly, these results cannot be simply obtained from the results above. While the case $\alpha \to 1$ is clearly delicate from the definition of RRE and TRE given in Eqs. (1) and (2), respectively, the case $\alpha \to 0$ must be carefully considered since simply taking $\alpha = 0$ in Eq. (4) would provide us a trivial bound, independently of the initial state and the dynamics.

A. Limiting case of $\alpha \to 1$

Let us start by considering the limit $\alpha \to 1$, in which both RRE and TRE recover the quantum relative entropy, i.e.,

$\lim_{\alpha \to 1} \mathcal{O}_\alpha(\varrho \| \omega) = S(\varrho \| \omega)$. In this case, it can be proved that the following upper bound applies (see details in Appendix B)

$$S(\rho_\tau \| \rho_0) \leqslant \tau \| \ln \rho_0 \|_2 \langle\!\langle \| [H_t, \rho_t] \|_2 \rangle\!\rangle_\tau. \tag{14}$$

In this case, the corresponding QSL family reads

$$\tau \geqslant \tau_1^{\text{RE}} := \max \left\{ \tau_1^{\text{RE}}(\rho_\tau \| \rho_0), \tau_1^{\text{RE}}(\rho_0 \| \rho_\tau), \tau_1^{\text{RE}}(\rho_0 : \rho_\tau) \right\}, \tag{15}$$

where

$$\tau_1^{\text{RE}}(A \| B) := \frac{S(A \| B)}{\| \ln \rho_0 \|_2 \langle\!\langle \| [H_t, A] \|_2 \rangle\!\rangle_\tau}, \tag{16}$$

and

$$\tau_1^{\text{RE}}(\rho_0 : \rho_\tau) := \frac{S(\rho_\tau \| \rho_0) + S(\rho_0 \| \rho_\tau)}{\| \ln \rho_0 \|_2 \langle\!\langle \| [H_t, \rho_\tau] \|_2 + \| [H_t, \rho_0] \|_2 \rangle\!\rangle_\tau}. \tag{17}$$

In particular, when the Hamiltonian is time independent, i.e., $H_t \equiv H$, one obtains $\| [H, \rho_t] \|_2 = \| [H, \rho_0] \|_2 = 2\sqrt{\mathcal{I}_L(\rho_0, H)}$, where we have used the fact that Schatten 2-norm is unitarily invariant. Here $\mathcal{I}_L(\rho_0, H) = (1/4) \| [H, \rho_0] \|_2^2 = -(1/4) \text{Tr}([\rho_0, H]^2)$ define a time-independent quantum coherence quantifier which sets a lower bound on Wigner-Yanase skew information [74,75]. Now, since $\mathcal{I}_L(\rho_0, H) \leqslant (\Delta H)^2$, where $(\Delta H)^2 = \text{Tr}(\rho_0 H^2) - [\text{Tr}(\rho_0 H)]^2$ is the squared deviation of the Hamiltonian, thus Eq. (16) implies the lower bounds $\tau_1^{\text{RE}}(\rho_0 \| \rho_\tau) \geqslant S(\rho_0 \| \rho_\tau)/(2 \Delta H \| \ln \rho_0 \|_2)$, and $\tau_1^{\text{RE}}(\rho_\tau \| \rho_0) \geqslant S(\rho_\tau \| \rho_0)/(2 \Delta H \| \ln \rho_0 \|_2)$.

B. Limiting case of $\alpha \to 0$

Considering now the case $\alpha \to 0$, in Appendix C we have proved that Rényi min-relative entropy is upper bounded as

$$|R_0(\rho_\tau \| \rho_0)| \leqslant \tau \langle\!\langle \mathcal{Q}_0^t(\rho_0, \Pi_{\rho_0}) \rangle\!\rangle_\tau, \tag{18}$$

with

$$\mathcal{Q}_0^t(A, B) := \frac{\| A \|_2 \| [U_t^\dagger H_t U_t, B] \|_2}{|\text{Tr}(A U_t B U_t^\dagger)|}. \tag{19}$$

Here Π_{ρ_0} is the projector onto the support of the initial state ρ_0. From Eq. (18), we can derive the QSL time as

$$\tau \geqslant \tau_0^R := \max \left\{ \tau_0^R(\rho_\tau \| \rho_0), \tau_0^R(\rho_0 \| \rho_\tau), \tau_0^R(\rho_0 : \rho_\tau) \right\}, \tag{20}$$

where

$$\tau_0^R(\rho_\tau \| \rho_0) := \frac{|R_0(\rho_\tau \| \rho_0)|}{\langle\!\langle \mathcal{Q}_0^t(\rho_0, \Pi_{\rho_0}) \rangle\!\rangle_\tau}, \tag{21}$$

and

$$\tau_0^R(\rho_0 \| \rho_\tau) := \frac{|R_0(\rho_0 \| \rho_\tau)|}{\langle\!\langle \mathcal{Q}_0^t(\Pi_{\rho_0}, \rho_0) \rangle\!\rangle_\tau}, \tag{22}$$

while the QSL related to the symmetric min-entropy is given by

$$\tau_0^R(\rho_0 : \rho_\tau) := \frac{|R_0(\rho_\tau \| \rho_0) + R_0(\rho_0 \| \rho_\tau)|}{\langle\!\langle \mathcal{Q}_0^t(\rho_0, \Pi_{\rho_0}) + \mathcal{Q}_0^t(\Pi_{\rho_0}, \rho_0) \rangle\!\rangle_\tau}. \tag{23}$$

Noteworthy, the "speed" contribution $\mathcal{Q}_0^t$ is closely related to the QSL derived with respect to Euclidean distance in the Bloch sphere [76,77]. Importantly, when the density matrix

ρ_0 has full-rank, i.e., $\dim(\rho_0) = \text{supp}(\rho_0)$, thus Rényi min-relative entropy vanishes and implies that $\tau_R^{(0)} = 0$. To see this, let $\rho_0 = \sum_{\ell=1}^d p_\ell |\psi_\ell\rangle\langle\psi_\ell|$ be the spectral decomposition of the input state, with $d = \dim(\mathcal{H})$. Note the projector Π_{ρ_0} onto the support of the full-rank state ρ_0 is equal to the identity, $\Pi_{\rho_0} = \sum_{\ell: p_\ell \neq 0} |\psi_\ell\rangle\langle\psi_\ell| = \mathbb{I}$. Hence, it is straightforward to verify the symmetric min-entropy is identically zero because $R_0(\rho_\tau \| \rho_0) = -\ln[\text{Tr}(U_\tau^\dagger \rho_0 U_\tau)] = 0$ and $R_0(\rho_0 \| \rho_\tau) = -\ln[\text{Tr}(U_\tau \rho_0 U_\tau^\dagger)] = 0$. Furthermore, from Eq. (19) one readily gets $\mathcal{Q}_0^t(\rho_0, \Pi_{\rho_0}) = 0$, while the functional $\mathcal{Q}_0^t(\Pi_{\rho_0}, \rho_0) = \sqrt{d} \| [U_t^\dagger H_t U_t, \rho_0] \|_2$ remains finite for $t \in [0, \tau]$.

V. EXAMPLE

We now provide an analytical example in order to make clearer the physical implications of our results. Let us consider a single-qubit state, whose Bloch sphere representation is written as $\rho_0 = (1/2)(\mathbb{I} + \vec{r} \cdot \vec{\sigma})$, where $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ is the vector of Pauli matrices, $\vec{r} = r \hat{r}$ is the Bloch vector, with $\hat{r} = \{\sin\theta \cos\phi, \sin\theta \sin\phi, \cos\theta\}$, $0 < r < 1$, $\theta \in [0, \pi]$ and $\phi \in [0, 2\pi]$, while $\mathbb{I}$ is the 2×2 identity matrix. Particularly, for $0 < \alpha < 1$, the operator ρ_0^α respective to the initial single-qubit state is written as

$$\rho_0^\alpha = \frac{1}{2}[\xi_\alpha^+ \mathbb{I} + \xi_\alpha^- (\hat{r} \cdot \vec{\sigma})], \tag{24}$$

where

$$\xi_\alpha^\pm = 2^{-\alpha}[(1+r)^\alpha \pm (1-r)^\alpha]. \tag{25}$$

The dynamics of the system is governed by the time-dependent Hamiltonian $H_t = \varpi \mathbb{I} + \hat{n}_t \cdot \vec{\sigma}$, where $\hat{n}_t = \{n_t^x, n_t^y, n_t^z\}$ is a time-dependent unit vector, $|\hat{n}_t| = 1$, and $\varpi \in \mathbb{R}$. In this case, the time ordered evolution operator becomes $U_t = e^{-it\varpi}[\cos(|\vec{u}_t|) \mathbb{I} - i \sin(|\vec{u}_t|) (\hat{u}_t \cdot \vec{\sigma})]$, where $\hat{u}_t = \vec{u}_t/|\vec{u}_t|$ is a unit vector, with $\vec{u}_t := \int_0^t ds \, \hat{n}_s$. In particular, if $H_t \equiv H$ is time independent, i.e., $\hat{n}_t = \hat{n}$ is a constant unit vector, we directly obtain $\hat{u}_t = \hat{n}$. Next, by performing a lengthy but straightforward calculation, one may verify the evolved single-qubit state becomes

$$\rho_t^\alpha = U_t \rho_0^\alpha U_t^\dagger = \frac{1}{2}[\xi_\alpha^+ \mathbb{I} + \xi_\alpha^- (\hat{v}_t \cdot \vec{\sigma})], \tag{26}$$

with the unit vector $\hat{v}_t := \hat{r} + \sin(2|\vec{u}_t|) (\hat{u}_t \times \hat{r}) + 2 \sin^2(|\vec{u}_t|)[(\hat{u}_t \cdot \hat{r})\hat{u}_t - \hat{r}]$. Hence, by considering the range $0 < \alpha < 1$, from Eqs. (24) and (26) we thus have that α-relative purity reads

$$g_\alpha(\rho_t, \rho_0) = 1 - \xi_\alpha^- \xi_{1-\alpha}^- [1 - (\hat{u}_t \cdot \hat{r})^2] \sin^2(|\vec{u}_t|). \tag{27}$$

Interestingly, since relative purity is skew symmetric over the index α, Eq. (27) implies that $g_\alpha(\rho_t, \rho_0) = g_{1-\alpha}(\rho_t, \rho_0) = g_\alpha(\rho_0, \rho_t)$. In turn, both RRE and TRE will satisfy the constraint $\mathcal{O}_\alpha(\rho_t \| \rho_0) = \mathcal{O}_\alpha(\rho_0 \| \rho_t)$ for single-qubit states. Furthermore, from Eq. (27), note that α-relative purity is equal to 1 for $|\vec{u}_t| = n\pi$, with $n \in \mathbb{Z}$ and $t \in [0, \tau]$. Furthermore, $g_\alpha(\rho_t, \rho_0) = 1$ if vectors $\hat{u}_t$ and $\hat{r}$ are parallel. Conversely, α-relative purity becomes $g_\alpha(\rho_t, \rho_0) = 1 - \xi_\alpha^- \xi_{1-\alpha}^- \sin^2(|\vec{u}_t|)$ if vectors $\hat{u}_t$ and $\hat{r}$ are orthogonal. In Table I we summarize the quantities required to evaluate the QSL bounds $\tau_\alpha^\mathcal{O}$, τ_1^{RE}, and τ_0^R, for the case of a single-qubit state.

For simplicity, let us focus on a Hamiltonian with $\varpi = 0$ and $\hat{n}_t = \gamma^{-1}\{\Delta, 0, vt\}$, with $\gamma := \sqrt{\Delta^2 + (vt)^2}$, where v

TABLE I. Theoretical-information quantifiers related the single qubit state $\rho_0 = (1/2)(\mathbb{I} + \vec{r} \cdot \vec{\sigma})$, evolving under the Hamiltonian $H_t = \varpi \, \mathbb{I} + \hat{n}_t \cdot \vec{\sigma}$. Note that $\hat{\mu}_t := \hat{n}_t - \sin(2|\vec{u}_t|)(\hat{u}_t \times \hat{n}_t) + 2\sin^2(|\vec{u}_t|)[(\hat{u}_t \cdot \hat{n}_t)\hat{u}_t - \hat{n}_t]$, with $\hat{u}_t = \vec{u}_t/|\vec{u}_t|$, and $\vec{u}_t = \int_0^t ds\, \hat{n}_s$. If the Hamiltonian is time independent, i.e., $\hat{n}_t = \hat{n}$, then one must apply the changes $\hat{u}_t \to \hat{n}$, $|\vec{u}_t| \to t$, and $\hat{\mu}_t \to \hat{n}$ into the listed quantities.

Quantifier	Analytical value
$\|[H_t, \rho_0^\alpha]\|_2$	$\xi_\alpha^- \sqrt{2[1 - (\hat{n}_t \cdot \hat{r})^2]}$
$\|[H_t, \rho_0]\|_2$	$r\sqrt{2[1 - (\hat{n}_t \cdot \hat{r})^2]}$
$\|[H_t, \rho_t]\|_2$	$r\sqrt{2[1 - (\hat{\mu}_t \cdot \hat{r})^2]}$
$S(\rho_t \| \rho_0)$	$r \ln\left(\frac{1+r}{1-r}\right)[1 - (\hat{u}_t \cdot \hat{r})^2]\sin^2(|\vec{u}_t|)$
$\|\ln \rho_0\|_2$	$\sqrt{\ln^2(\frac{1-r}{2}) + \ln^2(\frac{1+r}{2})}$
$\|\rho_0^\alpha\|_2$	$\sqrt{\xi_{2\alpha}^+}$

stands as a "level velocity" of the energies of the system, and Δ is the level splitting [78]. Figure 1 shows the QSL τ_α^R and τ_α^H as function of time τ and the parameter α, for the initial single-qubit state with $\{r, \theta, \phi\} = \{1/4, \pi/4, \pi/4\}$, also setting the ratio $\Delta/v = 0.5$. In Appendix D we provide a complementary numerical study, thus exploiting in details the qualitative behavior depicted in Fig. 1 for RRE and TRE.

Finally, we will investigate the tightness of our bounds for nonsymmetric and symmetric relative entropies. Overall, the tightness of the bounds is related with the tightness of the QSL. To see this, we introduce the following figures of merit

for nonsymmetric RRE and TRE

$$\delta_1^{\mathcal{O}}(\alpha, \tau) := \tau \left\langle\!\!\left\langle \mathcal{G}_\alpha^{\mathcal{O}}(t) \right\rangle\!\!\right\rangle_\tau - |1 - \alpha|\mathcal{O}_\alpha(\rho_\tau \| \rho_0), \qquad (28)$$

and

$$\delta_2^{\mathcal{O}}(\alpha, \tau) := \tau \left\langle\!\!\left\langle \mathcal{G}_{1-\alpha}^{\mathcal{O}}(t) \right\rangle\!\!\right\rangle_\tau - |1 - \alpha|\mathcal{O}_\alpha(\rho_0 \| \rho_\tau), \qquad (29)$$

while for the symmetrized relative entropies one gets

$$\delta_3^{\mathcal{O}}(\alpha, \tau) := \delta_1^{\mathcal{O}}(\alpha, \tau) + \delta_2^{\mathcal{O}}(\alpha, \tau), \qquad (30)$$

with $\mathcal{O} \equiv \{R, H\}$. We note that Eqs. (28), (29), and (30) will quantify how much the bounds deviate from the actual value of RRE and TRE entropies, for both nonsymmetric and symmetric cases. Hereafter, we will set the initial single-qubit state parameterized as $\{r, \theta, \phi\} = \{1/4, \pi/4, \pi/4\}$, and also fixing the ratio $\Delta/v = 0.5$. Figure 2 shows the plot of the normalized quantity $\widetilde{\delta}_l^{\mathcal{O}}(\alpha, \tau) = \delta_l^{\mathcal{O}}(\alpha, \tau)/\max[\delta_l^{\mathcal{O}}(\alpha, \tau)]$, with $l = \{1, 2, 3\}$. On the one hand, Figs. 2(a) and 2(b) show that, for $0 \leqslant \alpha \leqslant 1$, both quantities $\widetilde{\delta}_1^R(\alpha, \tau)$ and $\widetilde{\delta}_2^R(\alpha, \tau)$ approaches zero for short times ($0 \lesssim \tau \lesssim 2$). On the other hand, Figs. 2(d) and 2(e) indicate that, as the time increases, the figure of merit $\widetilde{\delta}_1^H(\alpha, \tau)$ will remain close to zero as long as $\alpha \lesssim 0.4$, while $\widetilde{\delta}_2^H(\alpha, \tau)$ approaches a small value for $\alpha \gtrsim 0.4$ during the range $0 \lesssim \tau \lesssim 4$. Finally, Figs. 2(c) and 2(f) suggest that, for the chosen initial parameters, both quantities $\widetilde{\delta}_3^R(\alpha, \tau)$ and $\widetilde{\delta}_3^H(\alpha, \tau)$ behave similarly, being slightly tight in the time window $0 \lesssim \tau \lesssim 2$, for $0 \leqslant \alpha \leqslant 1$. Nonetheless, we stress that a more general analysis requires varying those parameters to include a larger class of initial states and Hamiltonians. Of course, this subject should deserve further investigation, also including a detailed study of Eqs. (28), (29), and (30), as well as the family of QSLs bounds presented in Sec. IV, for higher-dimensional systems.

VI. DISCUSSION

The main contribution of this paper is to provide an upper bound on generalized entropies when the physical system undergoes a unitary transformation. As the first application of our bound, we derived a family of QSLs in Eqs. (11), (12), and (13). From this result, the minimum time required for the state transformation is inversely proportional to a quantity that involves the average energy of the system [see Eq. (5)]. In turn, this determines the evolution speed, while being directly proportional to $\mathcal{O}_\alpha$, and thus implying the entropies play the role of distances. Furthermore, our derivations of QSLs, based on RRE, is first step toward *resource speed limits* quantifying consumption of resource in single shot scenarios [73].

Another interesting connection can be built based on asymmetry monotones (AMs), which in turn characterize conservation laws for general quantum systems, in the sense of Noether's theorem [20]. In short, AM is a function $f : \mathcal{B}(\mathcal{H}) \to \mathbb{R}$ that quantifies how much the state of the system breaks a given symmetry. Mathematically, the action of a symmetry group G on a quantum system is described by the operation $\mathcal{U}_g(\rho) = U_g \, \rho \, U_g^\dagger$, where the variable g labels the group elements. The key idea behind AMs is to recognize the orbit of each quantum state as an encoding process, while the map $\mathcal{U}_g(\rho_0) \to \mathcal{U}_g(\rho_\tau)$ is viewed as data processing. This implies that we can employ any contractive information measure to

FIG. 1. Density plot of QSL (a) τ_α^R, and (b) τ_α^H, as a function of time τ and α, for the time-dependent Hamiltonian $H_t = \hat{n}_t \cdot \vec{\sigma}$, with $\hat{n}_t = \gamma^{-1}\{\Delta, 0, vt\}$, and $\gamma := \sqrt{\Delta^2 + (vt)^2}$. Here the initial state is defined by $\{r, \theta, \phi\} = \{1/4, \pi/4, \pi/4\}$, and the ratio $\Delta/v = 0.5$.
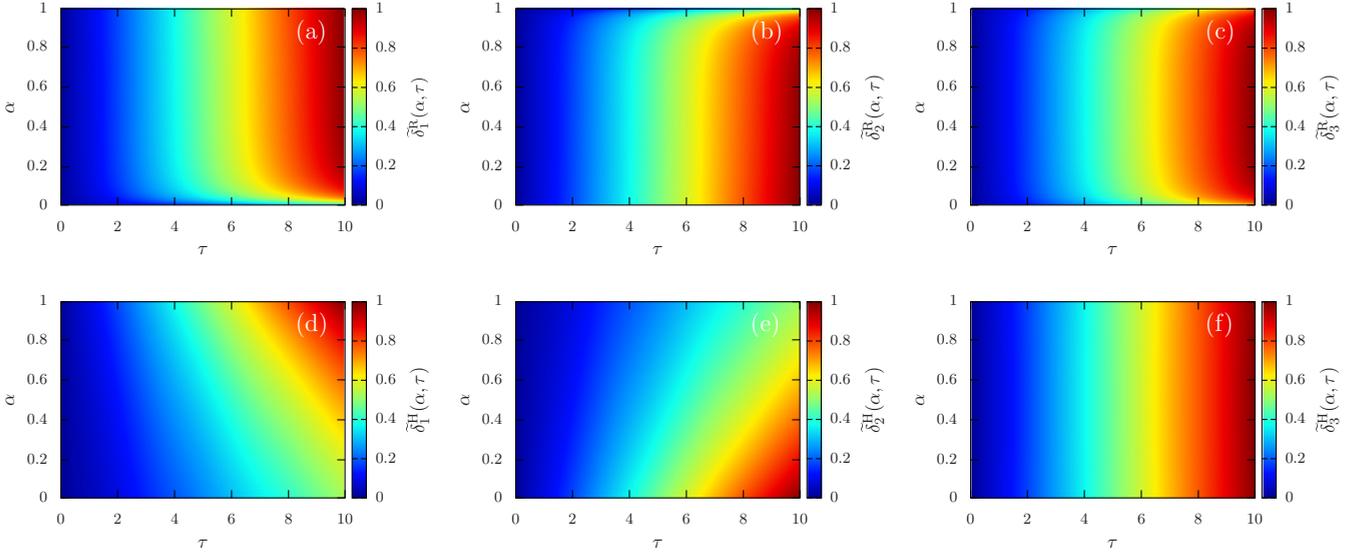
FIG. 2. Density plot of the normalized figure of merit $\widetilde{\delta}_l^{\mathcal{O}}(\alpha, \tau) = \delta_l^{\mathcal{O}}(\alpha, \tau)/\max[\delta_l^{\mathcal{O}}(\alpha, \tau)]$, with $l = \{1, 2, 3\}$ [see Eqs. (28), (29), and (30)], for Rényi relative entropy (a) $\widetilde{\delta}_1^R(\alpha, \tau)$, (b) $\widetilde{\delta}_2^R(\alpha, \tau)$, and (c) $\widetilde{\delta}_3^R(\alpha, \tau)$; and for Tsallis relative entropy (d) $\widetilde{\delta}_1^H(\alpha, \tau)$, (e) $\widetilde{\delta}_2^H(\alpha, \tau)$, and (f) $\widetilde{\delta}_3^H(\alpha, \tau)$. Here we set the initial single-qubit state with $\{r, \theta, \phi\} = \{1/4, \pi/4, \pi/4\}$, and also the ratio $\Delta/v = 0.5$.

characterize the orbit of each state, which leads to an AM satisfying $f(\mathcal{U}_g(\rho)) \leqslant f(\rho)$ [20].

Considering the range of α in which RRE and TRE are contractive under the action of a completely-positive and trace-preserving map, we immediately see the symmetrized relative entropies in Eq. (3) (as well as its nonsymmetric versions) define an entire family of AMs. Indeed, the standard relative entropy ($\alpha \rightarrow 1$) was previously considered as an asymmetry monotone [79].

Next, since Eqs. (4), (7), and (8) are valid for unitary transformations encoding any unknown parameter into the state of the system, i.e., it goes beyond the paradigmatic case of time evolutions, thus we can replace t by the group variable g. Indeed, this is due to the fact the unitary representation of the symmetry group leads to the evolution equation $d\rho_g/dg = -i[K, \rho_g]$, with K being the generator of the transformation. Therefore, our results provide upper bounds on how much the state breaks the symmetry generated by K. This sets upper bounds of how much the conservation of the associated physical quantity can be broken. In the specific case of QSLs, this implies the minimum evolution time is determined by the asymmetry measure respective to nonsymmetric and symmetric RRE and TRE, which in turn stands as a measure of how much the initial state breaks the time-translation symmetry [see Eqs. (11), (12), and (13)].

Finally, we will discuss the relation between our results and the concept of nonequilibrium entropy production. We begin by setting the initial state of the system as a thermal one, $\rho_0 = \rho_\beta = \exp{-\beta H_0/\mathcal{Z}}$, where $\mathcal{Z}$ is the partition function, and H_0 is the "bare" Hamiltonian of the system, i.e., $[H_0, H_t] \neq 0$ for all $t \neq 0$. From Eq. (16), one may verify the lower bound on the time of evolution is proportional to $S(\rho_\tau \| \rho_\beta)$, which in turn stands as the entropy production associated with the process under consideration. Therefore, a natural question arises about the extension of this connection to general entropies and systems. Indeed, such a general picture could be addressed

by exploiting the entire family of second laws of thermodynamics based on RRE [21,34]. This may open an avenue for the comprehension of QSLs [57,58,60–62,80,81], asymmetry monotones [79], and quantum thermodynamics [19,82–84], based on a strictly geometric framework.

The results presented here raise another questions. First, we could consider the extension of our results to open quantum systems. Moreover, given the link between quantum coherence and TRE [85], we could investigate the trade-off among entropy production, QSL and quantum coherence in this scenario. Furthermore, since our results also apply for the min-entropy, i.e., the limit $\alpha \rightarrow 0$ regarding to RRE, they can be employed in the single-shot information theory, where the relations among asymmetry, QSL, and thermodynamics can be further developed into the nonasymptotic regime.

APPENDIX A: BOUND ON RELATIVE ENTROPIES

In this Appendix we will present in details the derivation of the results discussed in Sec. II.

1. Bounding α-relative purity

Here we will derive a nontrivial lower bound on the α-relative purity, which will be useful throughout this supplemental material. Let $\rho_1, \rho_2 \in \Omega$ be two arbitrary density matrices, with $\Omega \subset \mathcal{H}$, where $\Omega = \{\rho \in \mathcal{H} \mid \rho^\dagger = \rho,\ \rho \geqslant 0,\ \mathrm{Tr}(\rho) = 1\}$ sets the convex space of quantum states, while $\mathcal{H}$ is a d-dimensional Hilbert space, with $d = \dim(\mathcal{H})$. The TRE is defined as $H_\alpha(\rho_1\|\rho_2) = (1-\alpha)^{-1}[1 - g_\alpha(\rho_1, \rho_2)]$ where $g_\alpha(\rho_1, \rho_2) = \mathrm{Tr}(\rho_1^\alpha \rho_2^{1-\alpha})$ stands for the α-relative purity [37]. In particular, for $0 \leqslant \alpha \leqslant 1$, it has been proved that TRE is upper bounded as follows [86]:

$$H_\alpha(\rho_1\|\rho_2) \leqslant S(\rho_1\|\rho_2), \tag{A1}$$

where $S(\rho_1\|\rho_2) = \mathrm{Tr}[\rho_1(\ln\rho_1 - \ln\rho_2)]$ is the "standard" quantum relative entropy. Interestingly, from Eq. (A1), we readily derive the lower bound on α-relative purity,

$$g_\alpha(\rho_1, \rho_2) \geqslant 1 - (1-\alpha)S(\rho_1\|\rho_2). \tag{A2}$$

Noteworthy, bound in Eq. (A2) exhibits two important features. On the one hand, for $\alpha = 1$, it trivially saturates since $g_1(\rho_1, \rho_2) = 1$ for all states $\rho_1, \rho_2 \in \Omega$. On the other hand, for $\alpha = 0$ one recovers the Klein's inequality, $S(\rho_1\|\rho_2) \geqslant 0$, due to the fact that $g_0(\rho_1, \rho_2) = 1$ for all $\rho_1, \rho_2 \in \Omega$, thus stating the quantum relative entropy is nonnegative. Next, quantum relative entropy satisfies the following lower bound [87,88]

$$S(\rho_1\|\rho_2) \leqslant -\ln(\lambda_{\min}(\rho_2)), \tag{A3}$$

where $\lambda_{\min}(\bullet)$ sets the minimum eigenvalue of the referred density matrix. Importantly, authors in Refs. [87,88] have derived a plethora of lower and upper bounds on $S(\rho_1\|\rho_2)$, which in turn depend on some distance measures as the operator norm, Schatten 1-norm, and Fröbenius norm. However, we shall stress the bound in Eq. (A3) is more suitable for our purposes, mostly because it stands as one of the simplest nontrivial lower bounds on quantum relative entropy. Therefore, by combining Eqs. (A2) and (A3), we thus have that

$$g_\alpha(\rho_1, \rho_2) \geqslant 1 + (1-\alpha)\ln(\lambda_{\min}(\rho_2)), \tag{A4}$$

which implies the following upper bound on α-relative purity:

$$[g_\alpha(\rho_1, \rho_2)]^{-1} \leqslant [1 + (1-\alpha)\ln(\lambda_{\min}(\rho_2))]^{-1}. \tag{A5}$$

2. Bound on nonsymmetric relative entropies

Here we will address the upper bound for nonsymmetric quantum relative entropies. From now on, we will focus on both Rényi, $R_\alpha(\rho_t\|\rho_0)$, and Tsallis, $H_\alpha(\rho_t\|\rho_0)$, relative entropies, where ρ_0 is the initial state of the system, and $\rho_t = U_t \rho_0 U_t^\dagger$ its respective evolved state, with $U_t = \mathcal{T}e^{-i\int_0^t ds\, H_s}$ being the unitary time-ordered evolution operator. The absolute value of the time derivative of RRE of states ρ_0 and ρ_t is given by

$$\left|\frac{d}{dt}R_\alpha(\rho_t\|\rho_0)\right| = \frac{1}{|1-\alpha|\, g_\alpha(\rho_t, \rho_0)}\left|\frac{d}{dt}g_\alpha(\rho_t, \rho_0)\right|. \tag{A6}$$

In Appendix A 1 we have derived an upper bound on the inverse of the α-relative purity, which in turn will exhbit a logarithmic dependence on the smallest eigenvalue of the initial state of the system. Indeed, by substituting Eq. (A5) into (A6), one obtains the following inequality

$$\left|\frac{d}{dt}R_\alpha(\rho_t\|\rho_0)\right| \leqslant \frac{\left|\frac{d}{dt}g_\alpha(\rho_t, \rho_0)\right|}{|1-\alpha||1 + (1-\alpha)\ln(\lambda_{\min}(\rho_0))|}. \tag{A7}$$

For completeness, the absolute value of the time derivative of TRE of states ρ_0 and ρ_t is given by

$$\left|\frac{d}{dt}H_\alpha(\rho_t\|\rho_0)\right| = \frac{1}{|1-\alpha|}\left|\frac{d}{dt}g_\alpha(\rho_t, \rho_0)\right|. \tag{A8}$$

Next, the time derivative of relative purity $g_\alpha(\rho_t, \rho_0) = \mathrm{Tr}(\rho_t^\alpha \rho_0^{1-\alpha})$ is evaluated as follows. Because ρ_t evolves unitarily, it is possible to verify the operator $\rho_t^\alpha = U_t \rho_0^\alpha U_t^\dagger$ satisfies the von Neumann equation $d\rho_t^\alpha/dt = -i[H_t, \rho_t^\alpha]$, where we used the identity $U_t(dU_t^\dagger/dt) = -(dU_t/dt)U_t^\dagger = -iH_t$. Hence, we thus have that

$$\frac{d}{dt}g_\alpha(\rho_t, \rho_0) = -i\,\mathrm{Tr}\left(\rho_t^\alpha\left[\rho_0^{1-\alpha}, H_t\right]\right), \tag{A9}$$

where we have used the cyclic property of trace. By taking the absolute value of Eq. (A9), and applying the Cauchy-Schwarz inequality, $|\mathrm{Tr}(\mathcal{A}_1\mathcal{A}_2)| \leqslant \|\mathcal{A}_1\|_2\|\mathcal{A}_2\|_2$, with $\|\mathcal{A}\|_2 = \sqrt{\mathrm{Tr}(\mathcal{A}^\dagger\mathcal{A})}$, one readily gets

$$\left|\frac{d}{dt}g_\alpha(\rho_t, \rho_0)\right| \leqslant \|\rho_0^\alpha\|_2 \left\|[H_t, \rho_0^{1-\alpha}]\right\|_2, \tag{A10}$$

where we have used that $\|\rho_t^\alpha\|_2 = \|U_t\rho_0^\alpha U_t^\dagger\|_2 = \|\rho_0^\alpha\|_2$, i.e., Schatten 2-norm is unitarily invariant. Hence, by substituting Eq. (A10) into Eqs. (A7) and (A8), one finds the generalized upper bound

$$\left|\frac{d}{dt}\mathcal{O}_\alpha(\rho_t\|\rho_0)\right| \leqslant |1-\alpha|^{-1}\mathcal{G}_\alpha^{\mathcal{O}}(t), \tag{A11}$$

where we define the functional

$$\mathcal{G}_\alpha^{\mathcal{O}}(t) := \Phi_\alpha^{\mathcal{O}} \left\|\rho_0^\alpha\right\|_2 \left\|[H_t, \rho_0^{1-\alpha}]\right\|_2, \tag{A12}$$

while the auxiliary function reads

$$\Phi_\alpha^{\mathcal{O}} = \begin{cases} |1 + (1-\alpha)\ln(\lambda_{\min}(\rho_0))|^{-1}, & \text{for } \mathcal{O} \equiv R \\ 1, & \text{for } \mathcal{O} \equiv H. \end{cases} \tag{A13}$$

Just to clarify, in the remainder of the paper the index $\mathcal{O} \equiv \{R, H\}$ will label RRE and TRE, respectively. Finally, by integrating Eq. (A11) over the interval $0 \leqslant t \leqslant \tau$, we thus obtain the upper bound

$$\mathcal{O}_\alpha(\rho_\tau\|\rho_0) \leqslant |1-\alpha|^{-1}\int_0^\tau dt\, \mathcal{G}_\alpha^{\mathcal{O}}(t), \tag{A14}$$

where we have invoked the inequality $|\int dx f(x)| \leqslant \int dx |f(x)|$, and used the fact that both RRE and TRE are nonnegative, real-valued, information-theoretic measures.

Importantly, an analogous bound can be derived for the case in which the states ρ_0 and ρ_τ are swapped. We will briefly sketch the proof since the calculations are similar to the previous discussion. The property $g_\alpha(\rho_0, \rho_t) = g_{1-\alpha}(\rho_t, \rho_0)$ for the relative purity implicates the skew symmetry

$$(1-\alpha)\mathcal{O}_\alpha(\rho_0\|\rho_t) = \alpha\,\mathcal{O}_{1-\alpha}(\rho_t\|\rho_0), \tag{A15}$$

which holds for all ρ_0, ρ_t and $0 < \alpha < 1$. In turn, the identity in Eq. (A15) allow us to write down the time derivative

$$\left|\frac{d}{dt}\mathcal{O}_\alpha(\rho_0\|\rho_t)\right| = \frac{\alpha}{|1-\alpha|}\left|\frac{d}{dt}\mathcal{O}_{1-\alpha}(\rho_t\|\rho_0)\right|. \quad \text{(A16)}$$

Invoking Eq. (A11) and mapping $\alpha \rightarrow 1 - \alpha$, one obtains $|d\mathcal{O}_{1-\alpha}(\rho_t\|\rho_0)/dt| \leqslant \alpha^{-1}\mathcal{G}_{1-\alpha}^{\mathcal{O}}(t)$. Hence, by inserting this expression into the right-hand side of Eq. (A16), we thus have that

$$\left|\frac{d}{dt}\mathcal{O}_\alpha(\rho_0\|\rho_t)\right| \leqslant |1-\alpha|^{-1}\mathcal{G}_{1-\alpha}^{\mathcal{O}}(t), \quad \text{(A17)}$$

which implies the upper bound

$$\mathcal{O}_\alpha(\rho_0\|\rho_\tau) \leqslant |1-\alpha|^{-1}\int_0^\tau dt\, \mathcal{G}_{1-\alpha}^{\mathcal{O}}(t). \quad \text{(A18)}$$

3. Bound on symmetric relative entropies

Here we will address the upper bound for symmetric quantum relative entropies. We shall begin with the symmetric relative entropy

$$\mathcal{O}_\alpha(\rho_t : \rho_0) := \mathcal{O}_\alpha(\rho_t\|\rho_0) + \mathcal{O}_\alpha(\rho_0\|\rho_t), \quad \text{(A19)}$$

where ρ_0 is the initial state of the system, while $\rho_t = U_t\rho_0 U_t^\dagger$ is the evolved state. Let $|d\,\mathcal{O}_\alpha(\rho_t : \rho_0)/dt|$ be the absolute value of the time derivative of Eq. (A19). Hence, by applying the triangle inequality $|a_1 + a_2| \leqslant |a_1| + |a_2|$, one obtains

$$\left|\frac{d}{dt}\mathcal{O}_\alpha(\rho_t : \rho_0)\right| \leqslant \left|\frac{d}{dt}\mathcal{O}_\alpha(\rho_t\|\rho_0)\right| + \left|\frac{d}{dt}\mathcal{O}_\alpha(\rho_0\|\rho_t)\right|. \quad \text{(A20)}$$

Based on Eqs. (A11) and (A17), it follows that

$$\left|\frac{d}{dt}\mathcal{O}_\alpha(\rho_t : \rho_0)\right| \leqslant |1-\alpha|^{-1}\big[\mathcal{G}_\alpha^{\mathcal{O}}(t) + \mathcal{G}_{1-\alpha}^{\mathcal{O}}(t)\big]. \quad \text{(A21)}$$

Finally, by integrating Eq. (A21) over the interval $t \in [0, \tau]$, and using the fact that $|\int dx f(x)| \leqslant \int dx |f(x)|$, one finds the upper bound for symmetric relative entropies

$$|\mathcal{O}_\alpha(\rho_\tau : \rho_0)| \leqslant |\alpha - 1|^{-1}\int_0^\tau dt\big[\mathcal{G}_\alpha^{\mathcal{O}}(t) + \mathcal{G}_{1-\alpha}^{\mathcal{O}}(t)\big]. \quad \text{(A22)}$$

APPENDIX B: RECOVERING RELATIVE ENTROPY

In this Appendix we present the details in the calculation of the limit $\alpha \rightarrow 1$. The idea here is going back few steps and pinpoint the main features needed to properly address such nontrivial limit. For the two states ρ_0 and $\rho_t = U_t\rho_0 U_t^\dagger$, we shall begin evaluating the limiting case $\alpha \rightarrow 1$ of the time derivative of symmetric relative entropy defined in Eq. (A19), i.e.,

$$\lim_{\alpha \rightarrow 1}\frac{d}{dt}\mathcal{O}_\alpha(\rho_t : \rho_0)$$

$$= \lim_{\alpha \rightarrow 1}\frac{d}{dt}\mathcal{O}_\alpha(\rho_t\|\rho_0)$$

$$+ \lim_{\alpha \rightarrow 1}\frac{\alpha}{(1-\alpha)}\frac{d}{dt}\mathcal{O}_{1-\alpha}(\rho_t\|\rho_0), \quad \text{(B1)}$$

where we have used the fact that $\mathcal{O}_\alpha(\rho_0\|\rho_t)$ is skew symmetric [see Eq. (A15)]. Next, by taking the absolute

value of Eq. (B1), then applying the triangle inequality $|a_1 + a_2| \leqslant |a_1| + |a_2|$, and finally integrating the resulting expression over interval $t \in [0, \tau]$, we thus have that

$$\int_0^\tau dt\left|\lim_{\alpha \rightarrow 1}\frac{d}{dt}\mathcal{O}_\alpha(\rho_t : \rho_0)\right|$$

$$\leqslant \int_0^\tau dt\left|\lim_{\alpha \rightarrow 1}\frac{d}{dt}\mathcal{O}_\alpha(\rho_t\|\rho_0)\right|$$

$$+ \int_0^\tau dt\left|\lim_{\alpha \rightarrow 1}\frac{\alpha}{(1-\alpha)}\frac{d}{dt}\mathcal{O}_{1-\alpha}(\rho_t\|\rho_0)\right|. \quad \text{(B2)}$$

Based on the definition of the relative entropy, $S(A\|B) = \text{Tr}[A(\ln A - \ln B)]$, note that one may write down

$$|S(\rho_\tau\|\rho_0) + S(\rho_0\|\rho_\tau)|$$

$$= \left|\lim_{\alpha \rightarrow 1}\mathcal{O}_\alpha(\rho_\tau : \rho_0)\right|$$

$$= \left|\int_0^\tau dt \lim_{\alpha \rightarrow 1}\frac{d}{dt}\mathcal{O}_\alpha(\rho_t : \rho_0)\right|$$

$$\leqslant \int_0^\tau dt\left|\lim_{\alpha \rightarrow 1}\frac{d}{dt}\mathcal{O}_\alpha(\rho_t : \rho_0)\right|. \quad \text{(B3)}$$

Just to clarify, here we assume that RRE and TRE are continuous real-valued functions over the set $\alpha \in (0, 1) \cup (1, +\infty)$ and $t \in [0, \tau]$. In this sense, we are formally able to switch the limit on parameter α with the integration sign over variable t. Thus, by combining Eqs. (B2) and (B3), one readily obtains

$$|S(\rho_\tau\|\rho_0) + S(\rho_0\|\rho_\tau)|$$

$$\leqslant \int_0^\tau dt\left|\lim_{\alpha \rightarrow 1}\frac{\mathcal{E}_\alpha^{\mathcal{O}}(\rho_t, \rho_0)}{(1-\alpha)}\frac{d}{dt}g_\alpha(\rho_t, \rho_0)\right|$$

$$+ \int_0^\tau dt\left|\lim_{\alpha \rightarrow 1}\frac{\mathcal{E}_{1-\alpha}^{\mathcal{O}}(\rho_t, \rho_0)}{(1-\alpha)}\frac{d}{dt}g_{1-\alpha}(\rho_t, \rho_0)\right|, \quad \text{(B4)}$$

where we have used that

$$\left|\lim_{\alpha \rightarrow 1}\frac{d}{dt}\mathcal{O}_\alpha(\rho_t\|\rho_0)\right|$$

$$= \left|\lim_{\alpha \rightarrow 1}\frac{\mathcal{E}_\alpha^{\mathcal{O}}(\rho_t, \rho_0)}{(1-\alpha)}\frac{d}{dt}g_\alpha(\rho_t, \rho_0)\right|, \quad \text{(B5)}$$

and

$$\left|\lim_{\alpha \rightarrow 1}\frac{\alpha}{(1-\alpha)}\frac{d}{dt}\mathcal{O}_{1-\alpha}(\rho_t\|\rho_0)\right|$$

$$= \left|\lim_{\alpha \rightarrow 1}\frac{\mathcal{E}_{1-\alpha}^{\mathcal{O}}(\rho_t, \rho_0)}{(1-\alpha)}\frac{d}{dt}g_{1-\alpha}(\rho_t, \rho_0)\right|, \quad \text{(B6)}$$

with the auxiliary functional $\mathcal{E}_s^{\mathcal{O}}(A, B)$ defined as

$$\mathcal{E}_s^{\mathcal{O}}(A, B) = \begin{cases} [g_s(A, B)]^{-1}, & \text{for } \mathcal{O} \equiv R \\ 1, & \text{for } \mathcal{O} \equiv H. \end{cases} \quad \text{(B7)}$$

Interestingly, one may verify the right-hand side of Eq. (B4) exhibits an indeterminacy in the limit $\alpha \rightarrow 1$. Indeed, since $(d\rho_t^s/dt) = -i[H_t, \rho_t^s]$, it follows that

$$\lim_{\alpha \rightarrow 1}[dg_\alpha(\rho_t, \rho_0)/dt]$$

$$= (-i)\lim_{\alpha \rightarrow 1}\text{Tr}\big(\rho_0^{1-\alpha}[H_t, \rho_t^\alpha]\big) = 0, \quad \text{(B8)}$$

and also

$$\lim_{\alpha \to 1} [dg_{1-\alpha}(\rho_t, \rho_0)/dt]$$
$$= (-i) \lim_{\alpha \to 1} \mathrm{Tr}\big(\rho_0^\alpha [H_t, \rho_t^{1-\alpha}]\big) = 0. \qquad (B9)$$

Similarly, one readily verifies that $\lim_{\alpha \to 1} (1 - \alpha)[\mathcal{E}_\alpha^\mathcal{O}(\rho_t, \rho_0)]^{-1} = 0$ and $\lim_{\alpha \to 1} (1 - \alpha)[\mathcal{E}_{1-\alpha}^\mathcal{O}(\rho_t, \rho_0)]^{-1} = 0$, where we used the fact that $\lim_{\alpha \to 1} [\mathcal{E}_\alpha^\mathcal{O}(\rho_t, \rho_0)]^{-1} = 1$, and $\lim_{\alpha \to 1} [\mathcal{E}_{1-\alpha}^\mathcal{O}(\rho_t, \rho_0)]^{-1} = 1$, for all $\mathcal{O} \equiv \{R, H\}$. In this case, one obtains

$$\lim_{\alpha \to 1} \frac{\mathcal{E}_\alpha^\mathcal{O}(\rho_t, \rho_0)}{(1 - \alpha)} \frac{d}{dt} g_\alpha(\rho_t, \rho_0) \longrightarrow \frac{0}{0}, \qquad (B10)$$

and

$$\lim_{\alpha \to 1} \frac{\mathcal{E}_{1-\alpha}^\mathcal{O}(\rho_t, \rho_0)}{(1 - \alpha)} \frac{d}{dt} g_{1-\alpha}(\rho_t, \rho_0) \longrightarrow \frac{0}{0}, \qquad (B11)$$

which in turn suggests the right-hand side of Eq. (B4) is not well behaved. We note, however, this such issue is readily

circumvented by applying the L'Hospital rule, leading us to

$$\lim_{\alpha \to 1} \frac{\mathcal{E}_\alpha^\mathcal{O}(\rho_t, \rho_0)}{(1 - \alpha)} \frac{d}{dt} g_\alpha(\rho_t, \rho_0)$$
$$= \lim_{\alpha \to 1} \frac{d[d\, g_\alpha(\rho_t, \rho_0)/dt]/d\alpha}{d\big((1 - \alpha)\big[\mathcal{E}_\alpha^\mathcal{O}(\rho_t, \rho_0)\big]^{-1}\big)/d\alpha}, \qquad (B12)$$

and

$$\lim_{\alpha \to 1} \frac{\mathcal{E}_{1-\alpha}^\mathcal{O}(\rho_t, \rho_0)}{(1 - \alpha)} \frac{d}{dt} g_{1-\alpha}(\rho_t, \rho_0)$$
$$= \lim_{\alpha \to 1} \frac{d[d\, g_{1-\alpha}(\rho_t, \rho_0)/dt]/d\alpha}{d\big((1 - \alpha)\big[\mathcal{E}_{1-\alpha}^\mathcal{O}(\rho_t, \rho_0)\big]^{-1}\big)/d\alpha}. \qquad (B13)$$

In the following, we will discuss in details each contribution in the right-hand side of Eqs. (B12) and (B13). Let us start by evaluating the following derivatives:

$$\lim_{\alpha \to 1} \frac{d}{d\alpha} \left(\frac{d}{dt} g_\alpha(\rho_t, \rho_0) \right) = -i \lim_{\alpha \to 1} \frac{d}{d\alpha} \big(\mathrm{Tr}\big(\rho_0^{1-\alpha}[H_t, \rho_t^\alpha]\big)\big)$$
$$= -i \lim_{\alpha \to 1} \sum_{j,\ell} \frac{d}{d\alpha} \big(p_j^{1-\alpha} p_\ell^\alpha\big) \langle \psi_j | [H_t, U_t | \psi_\ell \rangle \langle \psi_\ell | U_t^\dagger] | \psi_j \rangle$$
$$= -i \sum_{j,\ell} p_\ell (\ln p_\ell - \ln p_j) \langle \psi_j | [H_t, U_t | \psi_\ell \rangle \langle \psi_\ell | U_t^\dagger] | \psi_j \rangle$$
$$= i \, \mathrm{Tr}(\ln \rho_0 \, [H_t, \rho_t]), \qquad (B14)$$

and

$$\lim_{\alpha \to 1} \frac{d}{d\alpha} \left(\frac{d}{dt} g_{1-\alpha}(\rho_t, \rho_0) \right) = -i \lim_{\alpha \to 1} \frac{d}{d\alpha} \big(\mathrm{Tr}\big(\rho_0^\alpha [H_t, \rho_t^{1-\alpha}]\big)\big)$$
$$= -i \lim_{\alpha \to 1} \sum_{j,\ell} \frac{d}{d\alpha} \big(p_j^\alpha p_\ell^{1-\alpha}\big) \langle \psi_j | [H_t, U_t | \psi_\ell \rangle \langle \psi_\ell | U_t^\dagger] | \psi_j \rangle$$
$$= i \sum_{j,\ell} p_j (\ln p_\ell - \ln p_j) \langle \psi_j | [H_t, U_t | \psi_\ell \rangle \langle \psi_\ell | U_t^\dagger] | \psi_j \rangle$$
$$= -i \, \mathrm{Tr}(U_t \ln \rho_0 U_t^\dagger [H_t, \rho_0]), \qquad (B15)$$

where we have used that $d\,(p_j^{1-\alpha} p_\ell^\alpha)/d\alpha = (\ln p_\ell - \ln p_j)p_j^{1-\alpha} p_\ell^\alpha$. Moving forward, note that

$$\frac{d}{d\alpha} \Big((1 - \alpha) \big[\mathcal{E}_\alpha^\mathcal{O}(\rho_t, \rho_0)\big]^{-1}\Big) = -\big[\mathcal{E}_\alpha^\mathcal{O}(\rho_t, \rho_0)\big]^{-1} + (1 - \alpha) \frac{d}{d\alpha} \big[\mathcal{E}_\alpha^\mathcal{O}(\rho_t, \rho_0)\big]^{-1}, \qquad (B16)$$

and

$$\frac{d}{d\alpha} \Big((1 - \alpha) \big[\mathcal{E}_{1-\alpha}^\mathcal{O}(\rho_t, \rho_0)\big]^{-1}\Big) = -\big[\mathcal{E}_{1-\alpha}^\mathcal{O}(\rho_t, \rho_0)\big]^{-1} + (1 - \alpha) \frac{d}{d\alpha} \big[\mathcal{E}_{1-\alpha}^\mathcal{O}(\rho_t, \rho_0)\big]^{-1}. \qquad (B17)$$

From Eqs. (B16) and (B17), we point out that $\lim_{\alpha \to 1} [\mathcal{E}_\alpha^\mathcal{O}(\rho_t, \rho_0)]^{-1} = 1$, and $\lim_{\alpha \to 1} [\mathcal{E}_{1-\alpha}^\mathcal{O}(\rho_t, \rho_0)]^{-1} = 1$, for all $\mathcal{O} \equiv \{R, H\}$. Hence, from now on it suffices to proceed by showing the derivatives $d\,[\mathcal{E}_\alpha^\mathcal{O}(\rho_t, \rho_0)]^{-1}/d\alpha$, and $d\,[\mathcal{E}_{1-\alpha}^\mathcal{O}(\rho_t, \rho_0)]^{-1}/d\alpha$, are indeed well behaved for $\alpha \to 1$. On the one hand, for Tsallis relative entropy the auxiliary functional becomes $\mathcal{E}_\alpha^H(\rho_t, \rho_0) = \mathcal{E}_{1-\alpha}^H(\rho_t, \rho_0) = 1$, for all α, and the aforementioned derivatives are identically zero. In

this case, from Eqs. (B16) and (B17), one obtains

$$\frac{d}{d\alpha} \Big\{(1 - \alpha) \big[\mathcal{E}_\alpha^H(\rho_t, \rho_0)\big]^{-1}\Big\}$$
$$= \frac{d}{d\alpha} \Big\{(1 - \alpha) \big[\mathcal{E}_{1-\alpha}^H(\rho_t, \rho_0)\big]^{-1}\Big\} = -1. \qquad (B18)$$

On the other hand, for Rényi relative entropy the auxiliary functional behave as $[\mathcal{E}_\alpha^R(\rho_t, \rho_0)]^{-1} = g_\alpha(\rho_t, \rho_0)$ and

$[\mathcal{E}^R_{1-\alpha}(\rho_t, \rho_0)]^{-1} = g_{1-\alpha}(\rho_t, \rho_0)$, and the calculation is more involved. To see this, let $\rho_0 = \sum_\ell p_\ell |\psi_\ell\rangle\langle\psi_\ell|$ be the spectral decomposition of the initial state, with $0 \leqslant p_\ell \leqslant 1$ and $\sum_\ell p_\ell = 1$. In this case, given the evolved state $\rho_t = U_t \rho_0 U_t^\dagger$, we thus have that $\rho_t^\alpha = \sum_\ell p_\ell^\alpha U_t|\psi_\ell\rangle\langle\psi_\ell|U_t^\dagger$, and the relative purity becomes $g_\alpha(\rho_t, \rho_0) = \sum_{j,\ell} p_j^{1-\alpha} p_\ell^\alpha |\langle\psi_j|U_t|\psi_\ell\rangle|^2$. Hence, the derivative with respect to the parameter α is simply given by

$$\lim_{\alpha \to 1} \frac{d}{d\alpha}\big[\mathcal{E}^R_\alpha(\rho_t, \rho_0)\big]^{-1}$$
$$= \lim_{\alpha \to 1} \frac{d}{d\alpha}\mathrm{Tr}\big(\rho_t^\alpha \rho_0^{1-\alpha}\big)$$
$$= \lim_{\alpha \to 1} \sum_{j,\ell} p_\ell^\alpha p_j^{1-\alpha} (\ln p_\ell - \ln p_j)|\langle\psi_j|U_t|\psi_\ell\rangle|^2$$
$$= S(\rho_t\|\rho_0), \tag{B19}$$

and

$$\lim_{\alpha \to 1} \frac{d}{d\alpha}\big[\mathcal{E}^R_{1-\alpha}(\rho_t, \rho_0)\big]^{-1}$$
$$= \lim_{\alpha \to 1} \frac{d}{d\alpha}\mathrm{Tr}\big(\rho_t^{1-\alpha} \rho_0^\alpha\big)$$
$$= -\lim_{\alpha \to 1} \sum_{j,\ell} p_\ell^{1-\alpha} p_j^\alpha (\ln p_\ell - \ln p_j)|\langle\psi_j|U_t|\psi_\ell\rangle|^2$$
$$= S(\rho_0\|\rho_t). \tag{B20}$$

Hence, by combining Eqs. (B16), (B17), (B18), (B19), and (B20), we get the result

$$\lim_{\alpha \to 1} \frac{d}{d\alpha}\big\{(1-\alpha)\big[\mathcal{E}^\mathcal{O}_\alpha(\rho_t, \rho_0)\big]^{-1}\big\}$$
$$= \lim_{\alpha \to 1} \frac{d}{d\alpha}\big\{(1-\alpha)\big[\mathcal{E}^\mathcal{O}_{1-\alpha}(\rho_t, \rho_0))\big]^{-1}\big\} = -1. \tag{B21}$$

Therefore, by inserting Eqs. (B14), (B15), (B18), and (B21) into Eqs. (B12) and (B13), we conclude

$$\lim_{\alpha \to 1} \frac{\mathcal{E}^\mathcal{O}_\alpha(\rho_t, \rho_0)}{(1-\alpha)} \frac{d}{dt} g_\alpha(\rho_t, \rho_0)$$
$$= -i\,\mathrm{Tr}(\ln\rho_0\,[H_t, \rho_t]), \tag{B22}$$

and

$$\lim_{\alpha \to 1} \frac{\mathcal{E}^\mathcal{O}_{1-\alpha}(\rho_t, \rho_0)}{(1-\alpha)} \frac{d}{dt} g_{1-\alpha}(\rho_t, \rho_0)$$
$$= i\,\mathrm{Tr}(U_t \ln\rho_0 U_t^\dagger[H_t, \rho_0]). \tag{B23}$$

Finally, by substituting Eqs. (B22) and (B23) into Eq. (B4), and then applying the Cauchy-Schwarz inequality, $|\mathrm{Tr}(\mathcal{A}_1\mathcal{A}_2)| \leqslant \|\mathcal{A}_1\|_2\|\mathcal{A}_2\|_2$, with $\|\mathcal{A}\|_2 = \sqrt{\mathrm{Tr}(\mathcal{A}^\dagger\mathcal{A})}$, it yields the result

$$|S(\rho_\tau\|\rho_0) + S(\rho_0\|\rho_\tau)|$$
$$\leqslant \|\ln\rho_0\|_2 \int_0^\tau dt\,(\|[H_t, \rho_t]\|_2 + \|[H_t, \rho_0]\|_2). \tag{B24}$$

APPENDIX C: RECOVERING MIN-RELATIVE ENTROPY

In this Appendix we will discuss the case $\alpha = 0$ for symmetric Rényi relative entropy, which is related to the min-

relative entropy

$$R_0(\rho : \omega) := R_0(\rho\|\omega) + R_0(\omega\|\rho), \tag{C1}$$

where $R_0(\rho\|\omega) = -\ln\mathrm{Tr}(\Pi_\rho\,\omega)$, with Π_ρ being the projector onto the support of the state ρ. Here we will focus on the time-independent initial state ρ_0, and its evolved state $\rho_t = U_t\rho_0 U_t^\dagger$. By using the triangle inequality $|a_1 + a_2| \leqslant |a_1| + |a_2|$, the absolute value of the time derivative of Eq. (C1) is written as

$$\left|\frac{d}{dt}R_0(\rho_t : \rho_0)\right| \leqslant \left|\frac{d}{dt}R_0(\rho_t\|\rho_0)\right| + \left|\frac{d}{dt}R_0(\rho_0\|\rho_t)\right|. \tag{C2}$$

From now on we will discuss the evaluation of each contribution in right-hand side of Eq. (C2). In order to do so, let $\rho_0 = \sum_j p_\ell|\psi_\ell\rangle\langle\psi_\ell|$ be the spectral decomposition of the initial state ρ_0 into the basis $\{|\psi_\ell\rangle\}_{\ell=1,\dots,d}$, with $0 \leqslant p_\ell \leqslant 1$ and $\sum_\ell p_\ell = 1$. By hypothesis, the support of ρ_0 has dimension $d_{\rho_0} := \dim[\mathrm{supp}(\rho_0)]$, and is given by $\mathrm{supp}(\rho_0) = \mathrm{span}\{|\psi_\ell\rangle : p_\ell \neq 0\}$. Thus, the projector onto the support of state ρ_0 is defined as $\Pi_{\rho_0} := \sum_{\ell:\, p_\ell \neq 0}|\psi_\ell\rangle\langle\psi_\ell|$. The evolved state is given by $\rho_t = U_t\rho_0 U_t^\dagger = \sum_j p_\ell|\psi_\ell^t\rangle\langle\psi_\ell^t|$, with $|\psi_\ell^t\rangle := U_t|\psi_\ell\rangle$, and its support is defined as $\mathrm{supp}(\rho_t) = \mathrm{span}\{|\psi_\ell^t\rangle : p_\ell \neq 0\}$. Noteworthy, since the unitary evolution does not change the purity of the initial state, i.e., both states ρ_0 and ρ_t share the same set of eigenvalues, we thus have $\dim[\mathrm{supp}(\rho_t)] = \dim[\mathrm{supp}(\rho_0)]$. The projector Π_{ρ_t} onto the support of ρ_t read as

$$\Pi_{\rho_t} = \sum_{\ell:\, p_\ell \neq 0}|\psi_\ell^t\rangle\langle\psi_\ell^t| = \sum_{\ell:\, p_\ell \neq 0} U_t|\psi_\ell\rangle\langle\psi_\ell|U_t^\dagger = U_t\,\Pi_{\rho_0}U_t^\dagger. \tag{C3}$$

Interestingly, starting from Eq. (C3), the projector Π_{ρ_t} fulfills the von Neumann-like equation $(d\,\Pi_{\rho_t}/dt) = -i[H_t, \Pi_{\rho_t}]$, where we applied the identity $U_t(dU_t^\dagger/dt) = -(dU_t/dt)U_t^\dagger = -iH_t$. Thus, the time derivative of min-relative entropy $R_0(\rho_t\|\rho_0) = -\ln\mathrm{Tr}(\Pi_{\rho_t}\rho_0)$ read as

$$\frac{d}{dt}R_0(\rho_t\|\rho_0)$$
$$= \frac{i\,\mathrm{Tr}(\rho_0[H_t, \Pi_{\rho_t}])}{\mathrm{Tr}(\Pi_{\rho_t}\rho_0)} = \frac{i\,\mathrm{Tr}(U_t^\dagger\rho_0 U_t[U_t^\dagger H_t U_t, \Pi_{\rho_0}])}{\mathrm{Tr}(\Pi_{\rho_0}U_t^\dagger\rho_0 U_t)}, \tag{C4}$$

where we have explicitly used the property obtained in Eq. (C3). By taking the absolute value of Eq. (C4), and thus applying the Cauchy-Schwarz inequality, $|\mathrm{Tr}(\mathcal{A}_1\mathcal{A}_2)| \leqslant \|\mathcal{A}_1\|_2\|\mathcal{A}_2\|_2$, with $\|\mathcal{A}\|_2 = \sqrt{\mathrm{Tr}(\mathcal{A}^\dagger\mathcal{A})}$, one obtains

$$\left|\frac{d}{dt}R_0(\rho_t\|\rho_0)\right| \leqslant \frac{\|\rho_0\|_2\,\|[U_t^\dagger H_t U_t, \Pi_{\rho_0}]\|_2}{|\mathrm{Tr}(\Pi_{\rho_0}U_t^\dagger\rho_0 U_t)|}. \tag{C5}$$

Let us now move to the second term in the right-hand side of Eq. (C2), which is related to the time derivative of $R_0(\rho_0\|\rho_t) = -\ln\mathrm{Tr}(\Pi_{\rho_0}\rho_t)$. In this case, one readily obtains

$$\frac{d}{dt}R_0(\rho_0\|\rho_t) = \frac{i\,\mathrm{Tr}(\Pi_{\rho_0}[H_t, \rho_t])}{\mathrm{Tr}(\Pi_{\rho_0}\rho_t)}$$
$$= \frac{i\,\mathrm{Tr}(U_t^\dagger\Pi_{\rho_0}U_t[U_t^\dagger H_t U_t, \rho_0])}{\mathrm{Tr}(\Pi_{\rho_0}U_t\rho_0 U_t^\dagger)}, \tag{C6}$$

where we used the fact that ρ_t fulfills the von Neumann equation $(d\rho_t/dt) = -i[H_t, \rho_t]$. By taking the absolute value of Eq. (C6), and thus applying the aforementioned Cauchy-Schwarz inequality, one obtains

$$\left| \frac{d}{dt} R_0(\rho_0 \| \rho_t) \right| \leqslant \frac{\|\Pi_{\rho_0}\|_2 \|[U_t^\dagger H_t U_t, \rho_0]\|_2}{|\mathrm{Tr}(\Pi_{\rho_0} U_t \rho_0 U_t^\dagger)|}. \quad \text{(C7)}$$

Hence, by substituting Eqs. (C5) and (C7) into Eq. (C2), we thus have that

$$\left| \frac{d}{dt} R_0(\rho_t : \rho_0) \right| \leqslant \mathcal{Q}_0^t(\rho_0, \Pi_{\rho_0}) + \mathcal{Q}_0^t(\Pi_{\rho_0}, \rho_0), \quad \text{(C8)}$$

where the functional $\mathcal{Q}_0(\rho_t, \rho)$ is defined as follows:

$$\mathcal{Q}_0^t(A, B) := \frac{\|A\|_2 \|[U_t^\dagger H_t U_t, B]\|_2}{|\mathrm{Tr}(A U_t B U_t^\dagger)|}. \quad \text{(C9)}$$

Finally, by integrating Eq. (C8) over the interval $t \in [0, \tau]$, and thus applying the inequality $|\int dx f(x)| \leqslant \int dx |f(x)|$, one gets the inequality

$$\begin{aligned} |R_0(\rho_\tau : \rho_0)| & \\ \leqslant \int_0^\tau & dt \left[\mathcal{Q}_0^t(\rho_0, \Pi_{\rho_0}) + \mathcal{Q}_0^t(\Pi_{\rho_0}, \rho_0) \right]. \quad \text{(C10)} \end{aligned}$$

APPENDIX D: EXAMPLE: SINGLE-QUBIT STATE

1. Tsallis relative entropy

We shall begin showing that, for an initial single-qubit state evolving unitarily according the physical setting presented in Sec. V, the Tsallis relative entropy is given by

$$\begin{aligned} H_\alpha(\rho_\tau \| \rho_0) &= H_\alpha(\rho_0 \| \rho_\tau) \\ &= \frac{1 - g_\alpha(\rho_\tau, \rho_0)}{1 - \alpha} \\ &= \frac{\xi_\alpha^- \xi_{1-\alpha}^- |1 - (\hat{u}_\tau \cdot \hat{r})^2| \sin^2(|\vec{u}_\tau|)}{1 - \alpha}, \quad \text{(D1)} \end{aligned}$$

where the first equality follows from the property $g_\alpha(\rho_0, \rho_\tau) = g_{1-\alpha}(\rho_\tau, \rho_0) = g_\alpha(\rho_\tau, \rho_0)$ which holds for single-qubit states ρ_0 and ρ_τ [see Eq. (27)]. From Eqs. (5) and (6), we thus have that (see details in Table I)

$$\begin{aligned} \langle\!\langle \mathcal{G}_\alpha^H(t) \rangle\!\rangle_\tau &= \Phi_\alpha^H \|\rho_0^{1-\alpha}\|_2 \langle\!\langle \|[H_t, \rho_0^\alpha]\|_2 \rangle\!\rangle_\tau \\ &= \sqrt{2\,\xi_{2-2\alpha}^+}\, \xi_\alpha^- \frac{1}{\tau} \int_0^\tau dt \sqrt{1 - (\hat{n}_t \cdot \hat{r})^2}, \quad \text{(D2)} \end{aligned}$$

and

$$\begin{aligned} \langle\!\langle \mathcal{G}_{1-\alpha}^H(t) \rangle\!\rangle_\tau &= \Phi_{1-\alpha}^H \|\rho_0^\alpha\|_2 \langle\!\langle \|[H_t, \rho_0^{1-\alpha}]\|_2 \rangle\!\rangle_\tau \\ &= \sqrt{2\xi_{2\alpha}^+}\, \xi_{1-\alpha}^- \frac{1}{\tau} \int_0^\tau dt \sqrt{1 - (\hat{n}_t \cdot \hat{r})^2}. \quad \text{(D3)} \end{aligned}$$

Based on Eqs. (D1), (D2), and (D3), the QSL time for TRE in Eqs. (11) and (12) can be written, respectively, as

$$\tau_\alpha^H(\rho_\tau \| \rho_0) = \frac{\xi_{1-\alpha}^- |1 - (\hat{u}_\tau \cdot \hat{r})^2| \sin^2(|\vec{u}_\tau|)}{\sqrt{2\,\xi_{2-2\alpha}^+} \left[\frac{1}{\tau} \int_0^\tau dt \sqrt{1 - (\hat{n}_t \cdot \hat{r})^2} \right]}, \quad \text{(D4)}$$

and also

$$\tau_\alpha^H(\rho_0 \| \rho_\tau) = \frac{\xi_\alpha^- |1 - (\hat{u}_\tau \cdot \hat{r})^2| \sin^2(|\vec{u}_\tau|)}{\sqrt{2\,\xi_{2\alpha}^+} \left[\frac{1}{\tau} \int_0^\tau dt \sqrt{1 - (\hat{n}_t \cdot \hat{r})^2} \right]}, \quad \text{(D5)}$$

while the QSL time for symmetrized TRE in Eq. (13) reads

$$\begin{aligned} &\tau_\alpha^H(\rho_0 : \rho_\tau) \\ &= \frac{\sqrt{2}\, \xi_\alpha^- \xi_{1-\alpha}^- |1 - (\hat{u}_\tau \cdot \hat{r})^2| \sin^2(|\vec{u}_\tau|)}{(\sqrt{\xi_{2-2\alpha}^+}\, \xi_\alpha^- + \sqrt{\xi_{2\alpha}^+}\, \xi_{1-\alpha}^-) \left[\frac{1}{\tau} \int_0^\tau dt \sqrt{1 - (\hat{n}_t \cdot \hat{r})^2} \right]}. \end{aligned} \quad \text{(D6)}$$

In Fig. 3 we plot the QSL time $\tau_\alpha^H = \max\{\tau_\alpha^H(\rho_\tau \| \rho_0), \tau_\alpha^H(\rho_0 \| \rho_\tau), \tau_\alpha^H(\rho_0 : \rho_\tau)\}$, as a function of time τ and α, for the initial single qubit state $\rho_0 = (1/2)(\mathbb{I} + \vec{r} \cdot \vec{\sigma})$ with $\{r, \theta, \phi\} = \{1/4, \pi/4, \pi/4\}$, and also varying the ratio (a) $\Delta/v = 0.5$, (b) $\Delta/v = 1$, (c) $\Delta/v = 5$, and (d) $\Delta/v = 10$.

2. Rényi relative entropy

We shall begin showing that, for an initial single-qubit state evolving unitarily according the physical setting presented in Sec. V, the Rényi relative entropy is given by

$$\begin{aligned} R_\alpha(\rho_\tau \| \rho_0) &= R_\alpha(\rho_0 \| \rho_\tau) \\ &= \frac{\ln[g_\alpha(\rho_\tau, \rho_0)]}{\alpha - 1} \\ &= \frac{\ln\{1 - \xi_\alpha^- \xi_{1-\alpha}^- [1 - (\hat{u}_\tau \cdot \hat{r})^2] \sin^2(|\vec{u}_\tau|)\}}{\alpha - 1}, \end{aligned} \quad \text{(D7)}$$

where the first equality follows from the identity $g_\alpha(\rho_0, \rho_\tau) = g_{1-\alpha}(\rho_\tau, \rho_0) = g_\alpha(\rho_\tau, \rho_0)$, which holds for single-qubit states ρ_0 and ρ_τ [see Eq. (27)]. From Eqs. (5) and (6), we thus have that (see details in Table I)

$$\begin{aligned} \langle\!\langle \mathcal{G}_\alpha^R(t) \rangle\!\rangle_\tau &= \frac{\|\rho_0^{1-\alpha}\|_2 \langle\!\langle \|[H_t, \rho_0^\alpha]\|_2 \rangle\!\rangle_\tau}{|1 + (1-\alpha)\ln(\lambda_{\min}(\rho_0))|} \\ &= \frac{\tau^{-1} \sqrt{2\,\xi_{2-2\alpha}^+}\, \xi_\alpha^- \int_0^\tau dt \sqrt{1 - (\hat{n}_t \cdot \hat{r})^2}}{|1 + (1-\alpha)\ln\left(\frac{1-r}{2}\right)|}, \quad \text{(D8)} \end{aligned}$$

and

$$\begin{aligned} \langle\!\langle \mathcal{G}_{1-\alpha}^R(t) \rangle\!\rangle_\tau &= \frac{\|\rho_0^\alpha\|_2 \langle\!\langle \|[H_t, \rho_0^{1-\alpha}]\|_2 \rangle\!\rangle_\tau}{|1 + \alpha\ln(\lambda_{\min}(\rho_0))|} \\ &= \frac{\tau^{-1} \sqrt{2\xi_{2\alpha}^+}\, \xi_{1-\alpha}^- \int_0^\tau dt \sqrt{1 - (\hat{n}_t \cdot \hat{r})^2}}{|1 + \alpha\ln\left(\frac{1-r}{2}\right)|}. \quad \text{(D9)} \end{aligned}$$

Based on Eqs. (D7), (D8), and (D9), the QSL time for RRE in Eqs. (11) and (12) can be written, respectively, as

$$\begin{aligned} &\tau_\alpha^R(\rho_\tau \| \rho_0) \\ &= \frac{|\ln\{1 - \xi_\alpha^- \xi_{1-\alpha}^- [1 - (\hat{u}_\tau \cdot \hat{r})^2] \sin^2(|\vec{u}_\tau|)\}|}{\frac{\sqrt{2\xi_{2-2\alpha}^+}\, \xi_\alpha^- \left[\frac{1}{\tau} \int_0^\tau dt \sqrt{1 - (\hat{n}_t \cdot \hat{r})^2}\right]}{|1 + (1-\alpha)\ln\left(\frac{1-r}{2}\right)|}}, \quad \text{(D10)} \end{aligned}$$
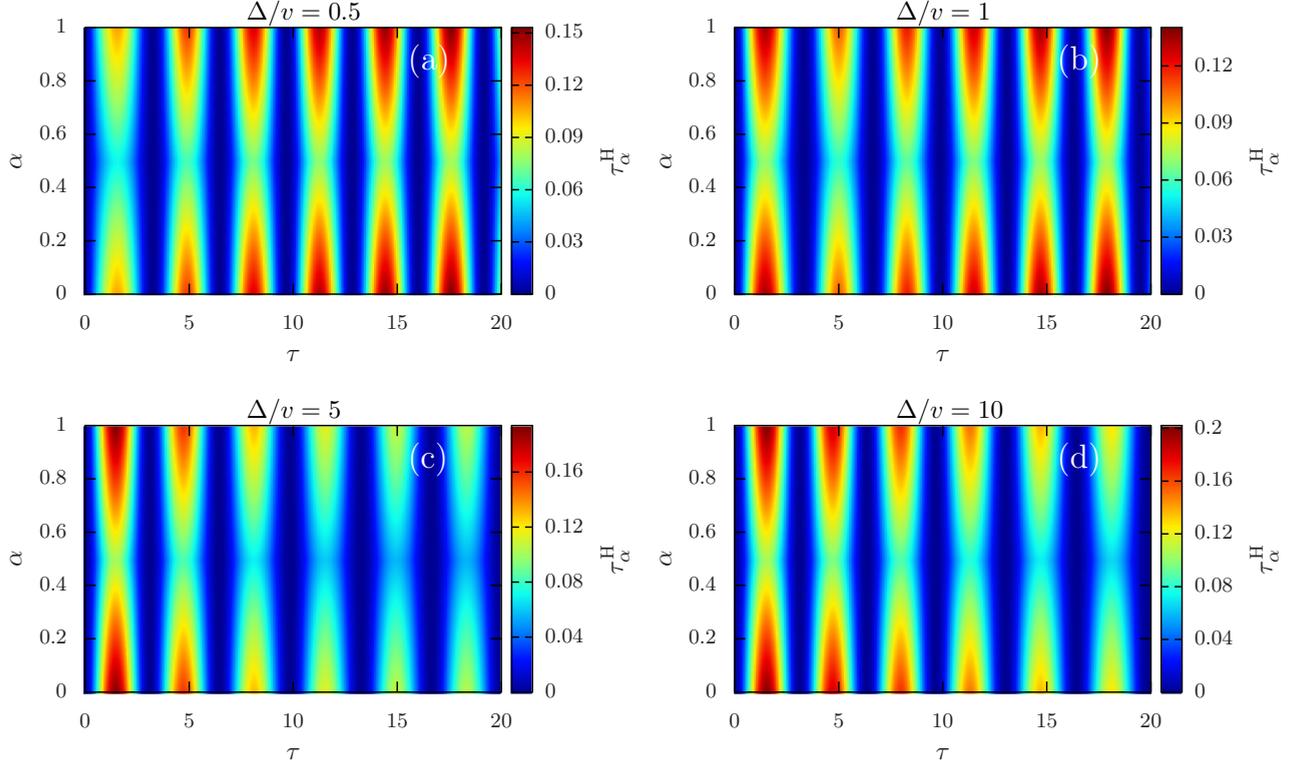
FIG. 3. Density plot of QSL time τ_α^H, as a function of time τ and α, respective to the unitary evolution generated by the time-dependent Hamiltonian $H_t = \hat{n}_t \cdot \vec{\sigma}$, with $\hat{n}_t = N^{-1}\{\Delta, 0, vt\}$ and $N := \sqrt{\Delta^2 + (vt)^2}$. Here we choose the initial single qubit state $\rho_0 = (1/2)(\mathbb{I} + \vec{r} \cdot \vec{\sigma})$ with $\{r, \theta, \phi\} = \{1/4, \pi/4, \pi/4\}$, and also setting the ratio (a) $\Delta/v = 0.5$, (b) $\Delta/v = 1$, (c) $\Delta/v = 5$, and (d) $\Delta/v = 10$.

and

$$
\begin{aligned}
&\tau_\alpha^R(\rho_0 \| \rho_\tau) \\
&= \frac{|\ln\{1 - \xi_\alpha^- \xi_{1-\alpha}^-[1 - (\hat{u}_\tau \cdot \hat{r})^2]\sin^2(|\vec{u}_\tau|)\}|}{\frac{\sqrt{2\xi_{2\alpha}^+ \xi_{1-\alpha}^-}\left[\frac{1}{\tau}\int_0^\tau dt \sqrt{1-(\hat{n}_t \cdot \hat{r})^2}\right]}{|1+\alpha\ln\left(\frac{1-r}{2}\right)|}},
\end{aligned} \quad \text{(D11)}
$$

while the QSL time for symmetrized RRE in Eq. (13) reads

$$
\begin{aligned}
&\tau_\alpha^R(\rho_0 : \rho_\tau) \\
&= \frac{\sqrt{2}\,|\ln\{1 - \xi_\alpha^- \xi_{1-\alpha}^-[1 - (\hat{u}_\tau \cdot \hat{r})^2]\sin^2(|\vec{u}_\tau|)\}|}{\left[\frac{\sqrt{\xi_{2-2\alpha}^+ \xi_\alpha^-}}{|1+(1-\alpha)\ln\left(\frac{1-r}{2}\right)|} + \frac{\sqrt{\xi_{2\alpha}^+ \xi_{1-\alpha}^-}}{|1+\alpha\ln\left(\frac{1-r}{2}\right)|}\right]\left[\frac{1}{\tau}\int_0^\tau dt \sqrt{1-(\hat{n}_t \cdot \hat{r})^2}\right]}.
\end{aligned}
$$
(D12)

In Fig. 4 we plot the QSL time $\tau_\alpha^R = \max\{\tau_\alpha^R(\rho_\tau \| \rho_0), \tau_\alpha^R(\rho_0 \| \rho_\tau), \tau_\alpha^R(\rho_0 : \rho_\tau)\}$, as a function of time τ and α, for the initial single qubit state $\rho_0 = (1/2)(\mathbb{I} + \vec{r} \cdot \vec{\sigma})$ with $\{r, \theta, \phi\} = \{1/4, \pi/4, \pi/4\}$, and also varying the ratio [Fig. 4(a)] $\Delta/v = 0.5$, [Fig. 4(b)] $\Delta/v = 1$, [Fig. 4(c)] $\Delta/v = 5$, and [Fig. 4(d)] $\Delta/v = 10$.

3. Relative entropy

For the case of single-qubit states ρ_0 and ρ_τ discussed in Sec. V, it is possible to show that relative entropy fulfills the identity $S(\rho_\tau \| \rho_0) = S(\rho_0 \| \rho_\tau)$, with (see Table I)

$$
S(\rho_\tau \| \rho_0) = r \ln\left(\frac{1+r}{1-r}\right)[1 - (\hat{u}_\tau \cdot \hat{r})^2]\sin^2(|\vec{u}_\tau|). \quad \text{(D13)}
$$

Next, by using the results in Table I, one readily obtains the time averages

$$
\langle\!\langle \|[H_t, \rho_t]\|_2 \rangle\!\rangle_\tau = \sqrt{2}\,r\,\frac{1}{\tau}\int_0^\tau dt \sqrt{1 - (\hat{\mu}_t \cdot \hat{r})^2}, \quad \text{(D14)}
$$

and

$$
\langle\!\langle \|[H_t, \rho_0]\|_2 \rangle\!\rangle_\tau = \sqrt{2}\,r\,\frac{1}{\tau}\int_0^\tau dt \sqrt{1 - (\hat{n}_t \cdot \hat{r})^2}. \quad \text{(D15)}
$$

Based on Eqs. (D13), (D14), and (D15), the QSL time related to the relative entropy in Eq. (16) implies that

$$
\tau_1^{\mathrm{RE}}(\rho_\tau \| \rho_0) = \frac{\ln\left(\frac{1+r}{1-r}\right)[1 - (\hat{u}_\tau \cdot \hat{r})^2]\sin^2(|\vec{u}_\tau|)}{\sqrt{2}\sqrt{\ln^2\left(\frac{1-r}{2}\right) + \ln^2\left(\frac{1+r}{2}\right)}\left[\frac{1}{\tau}\int_0^\tau dt \sqrt{1 - (\hat{\mu}_t \cdot \hat{r})^2}\right]}, \quad \text{(D16)}
$$
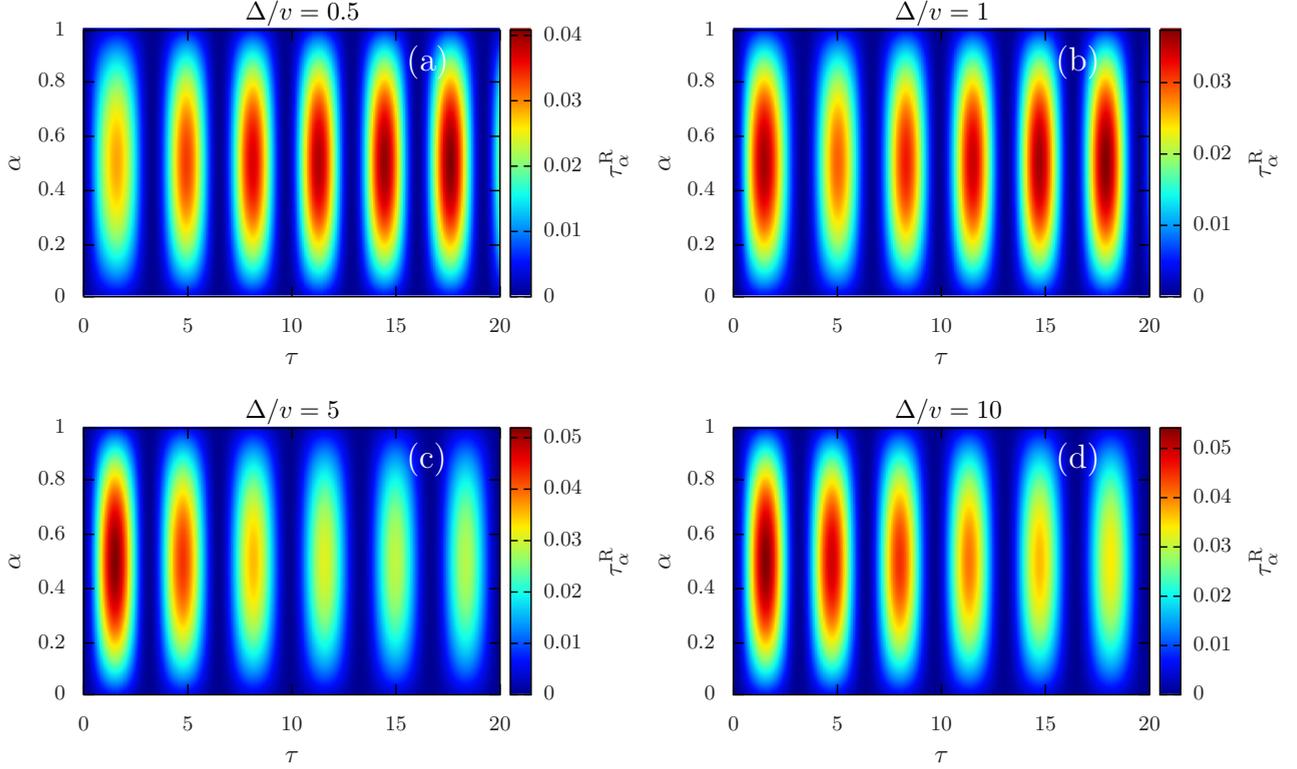
FIG. 4. Density plot of QSL time τ_α^R, as a function of time τ and α, respective to the unitary evolution generated by the time-dependent Hamiltonian $H_t = \hat{n}_t \cdot \vec{\sigma}$, with $\hat{n}_t = N^{-1}\{\Delta, 0, vt\}$ and $N := \sqrt{\Delta^2 + (vt)^2}$. Here we choose the initial single qubit state $\rho_0 = (1/2)(\mathbb{I} + \vec{r} \cdot \vec{\sigma})$ with $\{r, \theta, \phi\} = \{1/4, \pi/4, \pi/4\}$, and also setting the ratio (a) $\Delta/v = 0.5$, (b) $\Delta/v = 1$, (c) $\Delta/v = 5$, and (d) $\Delta/v = 10$.

and by swapping the arrangement of states ρ_0 and ρ_τ, it follows

$$\tau_1^{\text{RE}}(\rho_0\|\rho_\tau) = \frac{\ln\left(\frac{1+r}{1-r}\right)[1 - (\hat{u}_\tau \cdot \hat{r})^2]\sin^2(|\vec{u}_\tau|)}{\sqrt{2}\,\sqrt{\ln^2\left(\frac{1-r}{2}\right) + \ln^2\left(\frac{1+r}{2}\right)}\left[\frac{1}{\tau}\int_0^\tau dt\,\sqrt{1 - (\hat{n}_t \cdot \hat{r})^2}\right]}, \tag{D17}$$

while the QSL time for the symmetrized relative entropy in Eq. (17) reads

$$\tau_1^{\text{RE}}(\rho_0 : \rho_\tau) = \frac{\sqrt{2}\,\ln\left(\frac{1+r}{1-r}\right)[1 - (\hat{u}_\tau \cdot \hat{r})^2]\sin^2(|\vec{u}_\tau|)}{\sqrt{\ln^2\left(\frac{1-r}{2}\right) + \ln^2\left(\frac{1+r}{2}\right)}\left\{\frac{1}{\tau}\int_0^\tau dt\,\left[\sqrt{1 - (\hat{\mu}_t \cdot \hat{r})^2} + \sqrt{1 - (\hat{n}_t \cdot \hat{r})^2}\,\right]\right\}}. \tag{D18}$$

In Fig. 5 we plot the QSL time $\tau_1^{\text{RE}} = \max\{\tau_1^{\text{RE}}(\rho_\tau\|\rho_0), \tau_1^{\text{RE}}(\rho_0\|\rho_\tau), \tau_1^{\text{RE}}(\rho_0 : \rho_\tau)\}$, as a function of time τ and Δ/v, for the initial single qubit state $\rho_0 = (1/2)(\mathbb{I} + \vec{r} \cdot \vec{\sigma})$ with [Fig. 5(a)] $\{r, \theta, \phi\} = \{1/4, \pi/4, \pi/4\}$, [Fig. 5(b)] $\{r, \theta, \phi\} = \{1/4, \pi/3, \pi/4\}$, [Fig. 5(c)] $\{r, \theta, \phi\} = \{1/2, \pi/4, \pi/4\}$, and [Fig. 5(d)] $\{r, \theta, \phi\} = \{1/2, \pi/3, \pi/4\}$.

4. Min-relative entropy

Here we will present the details of the QSL time related to the min-relative entropy, with the latter defined as $R_0(\rho\|\omega) = -\ln \text{Tr}(\Pi_\rho\,\omega)$, with Π_ρ being the projector onto the support of the state ρ. From now on, we will choose the initial state ρ_0 being a pure one, i.e., a non-full-rank density matrix. In particular, for a single-qubit state such a

condition is equivalent to imposing the purity value $r = 1$, i.e., $\rho_0 = (1/2)(\mathbb{I} + \hat{r} \cdot \vec{\sigma})$, which in turn implies the spectral decomposition $\rho_0 = \sum_{\ell=\pm} p_\ell|\psi_\ell\rangle\langle\psi_\ell|$, with eigenvalues $p_+ = 1$ and $p_- = 0$, and eigenstates $|\psi_+\rangle = |\theta, \phi\rangle$ and $|\psi_-\rangle = |\theta - \pi, \phi\rangle$, with $|\theta, \phi\rangle := \cos(\theta/2)|0\rangle + e^{-i\phi}\sin(\theta/2)|1\rangle$. Just to clarify, here $|0\rangle = (1 \quad 0)^{\mathsf{T}}$ and $|1\rangle = (0 \quad 1)^{\mathsf{T}}$ define the standard states of the computational basis. In this case, the projector onto the support of ρ_0 read as $\Pi_{\rho_0} = |\psi_+\rangle\langle\psi_+|$. Moving forward, one may proceed the calculation as follows:

$$\begin{aligned} &\text{Tr}(\Pi_{\rho_0}U_\tau\rho_0 U_\tau^\dagger) \\ &= \text{Tr}(\rho_0 U_\tau \Pi_{\rho_0} U_\tau^\dagger) \\ &= |\langle\psi_+|U_\tau|\psi_+\rangle|^2 = 1 - [1 - (\hat{u}_t \cdot \hat{r})^2]\sin^2(|\vec{u}_t|). \end{aligned} \tag{D19}$$
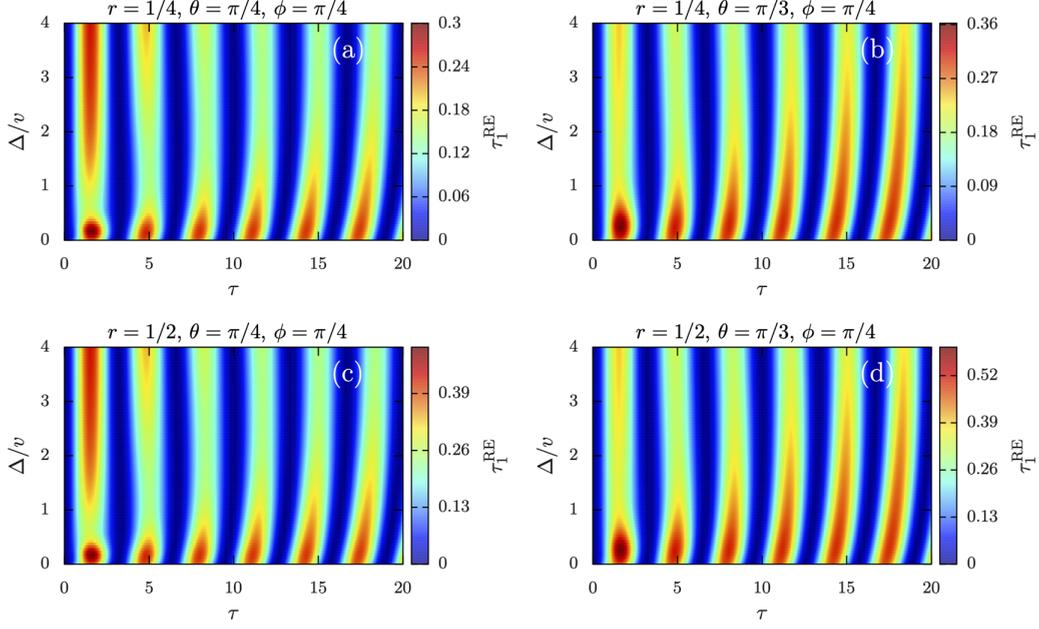
FIG. 5. Density plot of QSL time τ_1^{RE}, as a function of time τ and the ratio Δ/v, respective to the unitary evolution generated by the time-dependent Hamiltonian $H_t = \hat{n}_t \cdot \vec{\sigma}$, with $\hat{n}_t = N^{-1}\{\Delta, 0, vt\}$ and $N := \sqrt{\Delta^2 + (vt)^2}$. Here we choose the initial single qubit state $\rho_0 = (1/2)(\mathbb{I} + \vec{r} \cdot \vec{\sigma})$ with (a) $\{r, \theta, \phi\} = \{1/4, \pi/4, \pi/4\}$, (b) $\{r, \theta, \phi\} = \{1/4, \pi/3, \pi/4\}$, (c) $\{r, \theta, \phi\} = \{1/2, \pi/4, \pi/4\}$, and (d) $\{r, \theta, \phi\} = \{1/2, \pi/3, \pi/4\}$.

Next, by using the result of Eq. (D19), the min-relative entropy becomes

$$R_0(\rho_\tau \| \rho_0) = R_0(\rho_0 \| \rho_\tau)$$
$$= -\ln\{1 - [1 - (\hat{u}_t \cdot \hat{r})^2]\sin^2(|\vec{u}_t|)\}. \quad \text{(D20)}$$

Furthermore, given that $\langle \psi_+|(U_t^\dagger H_t U_t)^2|\psi_+\rangle = 1 + \varpi^2 + 2\varpi(\hat{\mu}_t \cdot \hat{r})$ and also $\langle \psi_+|U_t^\dagger H_t U_t|\psi_+\rangle = \varpi + \hat{\mu}_t \cdot \hat{r}$, one obtains

$$\|[U_t^\dagger H_t U_t, \rho_0]\|_2^2$$
$$= \|[U_t^\dagger H_t U_t, \Pi_{\rho_0}]\|_2^2$$
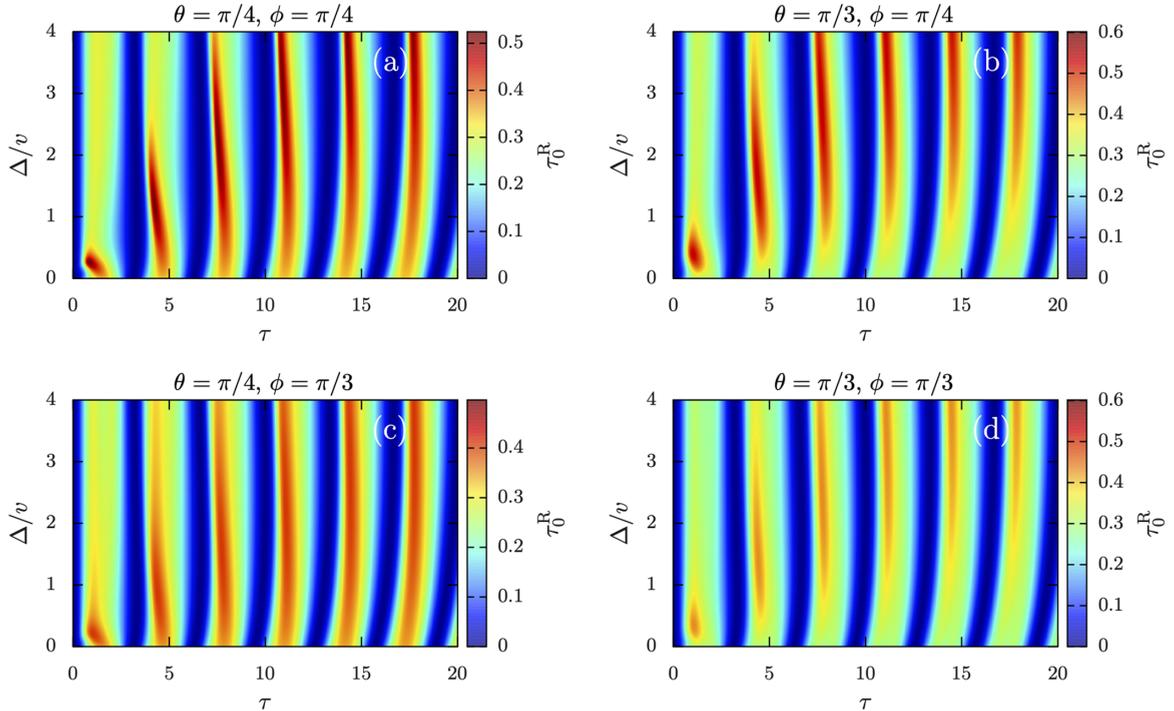
FIG. 6. Density plot of QSL time τ_0^R, as a function of time τ and the ratio Δ/v, respective to the unitary evolution generated by the time-dependent Hamiltonian $H_t = \hat{n}_t \cdot \vec{\sigma}$, with $\hat{n}_t = N^{-1}\{\Delta, 0, vt\}$ and $N := \sqrt{\Delta^2 + (vt)^2}$. Here we choose the initial pure single qubit state $\rho_0 = (1/2)(\mathbb{I} + \hat{r} \cdot \vec{\sigma})$ with $r = 1$ and (a) $\{\theta, \phi\} = \{\pi/4, \pi/4\}$; (b) $\{\theta, \phi\} = \{\pi/3, \pi/4\}$; (c) $\{\theta, \phi\} = \{\pi/4, \pi/3\}$; and (d) $\{\theta, \phi\} = \{\pi/3, \pi/3\}$.

$$= 2[\langle\psi_+|(U_t^\dagger H_t U_t)^2|\psi_+\rangle - \langle\psi_+|U_t^\dagger H_t U_t|\psi_+\rangle^2]$$
$$= 2[1 - (\hat{\mu}_t \cdot \hat{r})^2]. \tag{D21}$$

From Eqs. (D19) and (D21), and also using the Schatten 2-norms $\|\rho_0\|_2 = \|\Pi_{\rho_0}\|_2 = 1$, we thus conclude that $\mathcal{Q}_0^t(\rho_0, \Pi_{\rho_0}) = \mathcal{Q}_0^t(\Pi_{\rho_0}, \rho_0)$ [see Eq. (19)], which implies the following:

$$\langle\!\langle \mathcal{Q}_0^t(\rho_0, \Pi_{\rho_0}) \rangle\!\rangle_\tau = \langle\!\langle \mathcal{Q}_0^t(\Pi_{\rho_0}, \rho_0) \rangle\!\rangle_\tau$$
$$= \frac{\sqrt{2}}{\tau} \int_0^\tau dt \frac{\sqrt{1 - (\hat{\mu}_t \cdot \hat{r})^2}}{|1 - [1 - (\hat{u}_t \cdot \hat{r})^2]\sin^2(|\vec{u}_t|)|}. \tag{D22}$$

Based on Eqs. (D20) and (D22), the QSL time for min-entropy defined in Eqs. (21) and (22) satisfy the following

constraint:

$$\tau_0^R(\rho_\tau\|\rho_0) = \tau_0^R(\rho_0\|\rho_\tau)$$
$$= \frac{|\ln[1 - (1 - (\hat{u}_t \cdot \hat{r})^2]\sin^2(|\vec{u}_t|))|}{\sqrt{2}\left\{\frac{1}{\tau}\int_0^\tau dt \frac{\sqrt{1-(\hat{\mu}_t\cdot\hat{r})^2}}{|1-[1-(\hat{u}_t\cdot\hat{r})^2]\sin^2(|\vec{u}_t|)|}\right\}}, \tag{D23}$$

while the QSL time for the symmetrized min-relative entropy reads [see Eq. (23)]

$$\tau_0^R(\rho_0 : \rho_\tau) = \frac{|\ln\{1 - [1 - (\hat{u}_t \cdot \hat{r})^2]\sin^2(|\vec{u}_t|)\}|}{\sqrt{2}\left\{\frac{1}{\tau}\int_0^\tau dt \frac{\sqrt{1-(\hat{\mu}_t\cdot\hat{r})^2}}{|1-[1-(\hat{u}_t\cdot\hat{r})^2]\sin^2(|\vec{u}_t|)|}\right\}}. \tag{D24}$$

In Fig. 6 we plot the QSL time $\tau_0^R = \max\{\tau_0^R(\rho_\tau\|\rho_0), \tau_0^R(\rho_0\|\rho_\tau), \tau_0^R(\rho_0 : \rho_\tau)\}$, as a function of time τ and Δ/v, for the initial pure single qubit state $\rho_0 = (1/2)(\mathbb{I} + \hat{r} \cdot \vec{\sigma})$ with $r = 1$ and [Fig. 6(a)] $\{\theta, \phi\} = \{\pi/4, \pi/4\}$; [Fig. 6(b)] $\{\theta, \phi\} = \{\pi/3, \pi/4\}$; [Fig. 4(c)] $\{\theta, \phi\} = \{\pi/4, \pi/3\}$; and [Fig. 6(d)] $\{\theta, \phi\} = \{\pi/3, \pi/3\}$.

[1] C. E. Shannon, A mathematical theory of communication, Bell Syst. Tech. J. **27**, 379 (1948).

[2] A. G. Dimitrov, A. A. Lazar, and J. D. Victor, Information theory in neuroscience, J. Comput. Neurosci. **30**, 1 (2011).

[3] A. Kempf, Quantum gravity, information theory and the CMB, Found. Phys. **48**, 1191 (2018).

[4] E. Konishi, Holographic interpretation of Shannon entropy of coherence of quantum pure states, Europhys. Lett. **129**, 11006 (2020).

[5] J. Goold, M. Huber, A. Riera, L. del Rio, and P. Skrzypczyk, The role of quantum information in thermodynamics—A topical review, J. Phys. A: Math. Theor. **49**, 143001 (2016).

[6] R. Zhou, R. Cai, and G. Tong, Applications of entropy in finance: A review, Entropy **15**, 4909 (2013).

[7] L. F. Seoane and R. V. Solé, Information theory, predictability and the emergence of complex life, R. Soc. Open Sci. **5**, 172221 (2018).

[8] N. Shiraishi, K. Funo, and K. Saito, Speed Limit for Classical Stochastic Processes, Phys. Rev. Lett. **121**, 070601 (2018).

[9] C. Tsallis, Possible generalization of Boltzmann-Gibbs statistics, J. Stat. Phys. **52**, 479 (1988).

[10] A. Rényi, On Measures of Entropy and Information, in *Proceedings of the 4th Berkeley Symposium on Mathematics, Statistics, and Probability, Vol. 1: Contributions to the Theory of Statistics* (University of California Press, Berkeley, CA, 1961), pp. 547–561.

[11] H. Umegaki, Conditional expectation in an operator algebra. IV. Entropy and information, Kodai Math. Sem. Rep. **14**, 59 (1962).

[12] T. van Erven and P. Harremos, Rényi Divergence and Kullback-Leibler Divergence, IEEE Trans. Inf. Theory **60**, 3797 (2014).

[13] R. Kawai, J. M. R. Parrondo, and C. Van den Broeck, Dissipation: The Phase-Space Perspective, Phys. Rev. Lett. **98**, 080602 (2007).

[14] V. Vedral, M. B. Plenio, M. A. Rippin, and P. L. Knight, Quantifying entanglement, Phys. Rev. Lett. **78**, 2275 (1997).

[15] K. Bu, U. Singh, S.-M. Fei, A. K. Pati, and J. Wu, Maximum Relative Entropy of Coherence: An Operational Coherence Measure, Phys. Rev. Lett. **119**, 150405 (2017).

[16] L-H Shao, Y.-M. Li, Y. Luo, and Z.-J. Xi, Quantum coherence quantifiers based on Rényi α-relative entropy, Commun. Theor. Phys. **67**, 631 (2017).

[17] N. Shiraishi and K. Saito, Information-Theoretical Bound of the Irreversibility in Thermal Relaxation Processes, Phys. Rev. Lett. **123**, 110603 (2019).

[18] A. Kolchinsky and D. H. Wolpert, Work, entropy production, and thermodynamics of information under protocol constraints, arXiv:2008.10764.

[19] M. Scandi, H. J. D. Miller, J. Anders, and M. Perarnau-Llobet, Quantum work statistics close to equilibrium, Phys. Rev. Research **2**, 023377 (2020).

[20] I. Marvian and R. W. Spekkens, Extending Noether's theorem by quantifying the asymmetry of quantum states, Nat. Commun. **5**, 3821 (2014).

[21] F. Brandão, M. Horodecki, N. Ng, J. Oppenheim, and S. Wehner, The second laws of quantum thermodynamics, Proc. Natl. Acad. Sci. USA **112**, 3275 (2015).

[22] A. Bernamonti, F. Galli, R. C. Myers, and J. Oppenheim, Holographic second laws of black hole thermodynamics, J. High Energ. Phys. 07 (2018) 111.

[23] I. Csiszár, Generalized cutoff rates and Rényi's information measures, IEEE Trans. Inf. Theory **41**, 26 (1995).

[24] K. P. Seshadreesan, L. Lami, and M. M. Wilde, Rényi relative entropies of quantum Gaussian states, J. Math. Phys. **59**, 072204 (2018).

[25] P. J. Coles, V. Katariya, S. Lloyd, I. Marvian, and M. M. Wilde, Entropic Energy-Time Uncertainty Relation, Phys. Rev. Lett. **122**, 100401 (2019).

[26] D. P. Pires, A. Smerzi, and T. Macrì, Relating relative rényi entropies and Wigner-Yanase-Dyson skew information to generalized multiple quantum coherences, Phys. Rev. A **102**, 012429 (2020).

[27] C. Tsallis and E. Brigatti, Nonextensive statistical mechanics: A brief introduction, Contin. Mech. Thermodyn. **16**, 223 (2004).

[28] S. Furuichi, K. Yanagi, and K. Kuriyama, Fundamental properties of Tsallis relative entropy, J. Math. Phys. **45**, 4868 (2004).

[29] A. E. Rastegin, Quantum-coherence quantifiers based on the Tsallis relative α entropies, Phys. Rev. A **93**, 032136 (2016).

[30] A. E. Rastegin, Bounds of the Pinsker and Fannes types on the Tsallis relative entropy, Math. Phys. Anal. Geom. **16**, 213 (2013).

[31] V. Vedral, The role of relative entropy in quantum information theory, Rev. Mod. Phys. **74**, 197 (2002).

[32] J. Baez and B. Pollard, Relative entropy in biological systems, Entropy **18**, 46 (2016).

[33] R. Renner, Security of quantum key distribution, arXiv:quant-ph/0512258.

[34] G. Guarnieri, N. H. Y. Ng, K. Modi, J. Eisert, M. Paternostro, and J. Goold, Quantum work statistics and resource theories: Bridging the gap through Rényi divergences, Phys. Rev. E **99**, 050101(R) (2019).

[35] I. Marvian, R. W. Spekkens, and P. Zanardi, Quantum speed limits, coherence, and asymmetry, Phys. Rev. A **93**, 052331 (2016).

[36] S. Abe and Y. Okamoto, *Nonextensive Statistical Mechanics and Its Applications* (Springer, Berlin, 2001).

[37] M. Müller-Lennert, F. Dupuis, O. Szehr, S. Fehr, and M. Tomamichel, On quantum Rényi entropies: A new generalization and some properties, J. Math. Phys. **54**, 122203 (2013).

[38] K. M. R. Audenaert, Comparisons between quantum state distinguishability measures, Quant. Inf. Comput. **14**, 31 (2014).

[39] D. Petz, Quasi-entropies for finite quantum systems, Rep. Math. Phys. **23**, 57 (1986).

[40] R. L. Frank and E. H. Lieb, Monotonicity of a relative Rényi entropy, J. Math. Phys. **54**, 122201 (2013).

[41] S. Beigi, Sandwiched Rényi divergence satisfies data processing inequality, J. Math. Phys. **54**, 122202 (2013).

[42] N. Datta and F. Leditzky, A limit of the quantum Rényi divergence, J. Phys. A: Math. Theor. **47**, 045304 (2014).

[43] K. M. R. Audenaert and N. Datta, α-z-Rényi relative entropies, J. Math. Phys. **56**, 022202 (2015).

[44] S. Luo and Q. Zhang, Informational distance on quantum-state space, Phys. Rev. A **69**, 032106 (2004).

[45] P. Gibilisco and T. Isola, Wigner-Yanase information on quantum state space: The geometric approach, J. Math. Phys. **44**, 3752 (2003).

[46] A. Jenčová, Geodesic distances on density matrices, J. Math. Phys. **45**, 1787 (2004).

[47] J. M. Amigó, S. G. Balogh, and S. Hernández, A brief review of generalized entropies, Entropy **20**, 813 (2018).

[48] N. Datta, Min- and max-relative entropies and a new entanglement monotone, IEEE Trans. Inf. Theory **55**, 2816 (2009).

[49] M. Mosonyi and F. F. Hiai, On the quantum Rényi relative entropies and related capacity formulas, IEEE Trans. Inf. Theory **57**, 2474 (2011).

[50] T. Sagawa, Second law-like inequalities with quantum relative entropy: An introduction, in *Lectures on Quantum Computing, Thermodynamics and Statistical Physics* (World Scientific, New York, 2012), pp. 125–190.

[51] S. Abe, Monotonic decrease of the quantum nonadditive divergence by projective measurements, Phys. Lett. A **312**, 336 (2003).

[52] S. Abe, Nonadditive generalization of the quantum Kullback-Leibler divergence for measuring the degree of purification, Phys. Rev. A **68**, 032302 (2003).

[53] D. Virosztek, The metric property of the quantum Jensen-Shannon divergence, Adv. Math. **380**, 107595 (2021).

[54] A. Böttcher and D. Wenzel, The Frobenius norm and the commutator, Lin. Algebr. Appl. **429**, 1864 (2008).

[55] K. M. R. Audenaert, Variance bounds, with an application to norm bounds for commutators, Lin. Algebr. Appl. **432**, 1126 (2010).

[56] K.-S. Fong, I.-K. Lok, and C.-M. Cheng, A note on the norm of the commutator and the norm of $XY - YX^T$, Lin. Algebr. Appl. **435**, 1193 (2011).

[57] L. Mandelstam and I. Tamm, The uncertainty relation between energy and time in non-relativistic quantum mechanics, J. Phys. USSR **9**, 249 (1945).

[58] N. Margolus and L. B. Levitin, The maximum speed of dynamical evolution, Physica D **120**, 188 (1998).

[59] L. B. Levitin and T. Toffoli, Fundamental Limit on the Rate of Quantum Dynamics: The Unified Bound is Tight, Phys. Rev. Lett. **103**, 160502 (2009).

[60] M. M. Taddei, B. M. Escher, L. Davidovich, and R. L. de Matos Filho, Quantum Speed Limit for Physical Processes, Phys. Rev. Lett. **110**, 050402 (2013).

[61] A. del Campo, I. L. Egusquiza, M. B. Plenio, and S. F. Huelga, Quantum Speed Limits in Open System Dynamics, Phys. Rev. Lett. **110**, 050403 (2013).

[62] S. Deffner and E. Lutz, Quantum Speed Limit for Non-Markovian Dynamics, Phys. Rev. Lett. **111**, 010402 (2013).

[63] S. Deffner and S. Campbell, Quantum speed limits: From Heisenberg's uncertainty principle to optimal quantum control, J. Phys. A: Math. Theor. **50**, 453001 (2017).

[64] B. Shanahan, A. Chenu, N. Margolus, and A. del Campo, Quantum Speed Limits Across the Quantum-To-Classical Transition, Phys. Rev. Lett. **120**, 070401 (2018).

[65] M. Okuyama and M. Ohzeki, Quantum Speed Limit is Not Quantum, Phys. Rev. Lett. **120**, 070402 (2018).

[66] Y. Shao, B. Liu, M. Zhang, H. Yuan, and J. Liu, Operational definition of a quantum speed limit, Phys. Rev. Research **2**, 023299 (2020).

[67] K. Kobayashi and N. Yamamoto, Quantum speed limit for robust state characterization and engineering, Phys. Rev. A **102**, 042606 (2020).

[68] R. Puebla, S. Deffner, and S. Campbell, Kibble-Zurek scaling in quantum speed limits for shortcuts to adiabaticity, Phys. Rev. Research **2**, 032020(R) (2020).

[69] B. Mohan and A. K. Pati, Reverse quantum speed limit: How slow quantum battery can discharge? arXiv:2006.14523.

[70] A. del Campo, Probing quantum speed limits with ultracold gases, arXiv:2007.15019.

[71] M. R. Lam, N. Peter, T. Groh, W. Alt, C. Robens, D. Meschede, A. Negretti, S. Montangero, T. Calarco, and A. Alberti, Demonstration of quantum brachistochrones between distant states of an atom, Phys. Rev. X **11**, 011035 (2021).

[72] E. O'Connor, G. Guarnieri, and S. Campbell, Action quantum speed limits, Phys. Rev. A **103**, 022210 (2021).

[73] F. Campaioli, C.-S. Yu, F. A. Pollock, and K. Modi, Resource speed limits: Maximal rate of resource variation, arXiv:2004.03078.

[74] D. Girolami, Observable Measure of Quantum Coherence in Finite Dimensional Systems, Phys. Rev. Lett. **113**, 170401 (2014).

[75] E. P. Wigner and M. M. Yanase, Information contents of distributions, Proc. Natl. Acad. Sci. USA **49**, 910 (1963).

[76] F. Campaioli, F. A. Pollock, Felix C. Binder, and K. Modi, Tightening Quantum Speed Limits for Almost all States, Phys. Rev. Lett. **120**, 060409 (2018).

[77] F. Campaioli, F. A. Pollock, and K. Modi, Tight, robust, and feasible quantum speed limits for open dynamics, Quantum **3**, 168 (2019).

[78] J. Yang, S. Pang, and A. N. Jordan, Quantum parameter estimation with the Landau-Zener transition, Phys. Rev. A **96**, 020301(R) (2017).

[79] I. Marvian and R. W. Spekkens, How to quantify coherence: Distinguishing speakable and unspeakable notions, Phys. Rev. A **94**, 052324 (2016).

[80] T. Fogarty, S. Deffner, T. Busch, and S. Campbell, Orthogonality Catastrophe as a Consequence of the Quantum Speed Limit, Phys. Rev. Lett. **124**, 110601 (2020).

[81] S. B. Nicholson, L. P. Garcia-Pintos, A. del Campo, and J. R. Green, Time-information uncertainty relations in thermodynamics, Nat. Phys. **16**, 1211 (2020).

[82] S. B. Nicholson, A. del Campo, and J. R. Green, Nonequilibrium uncertainty principle from information geometry, Phys. Rev. E **98**, 032106 (2018).

[83] H. J. D. Miller, M. Scandi, J. Anders, and M. Perarnau-Llobet, Work Fluctuations in Slow Processes: Quantum Signatures and Optimal Control, Phys. Rev. Lett. **123**, 230603 (2019).

[84] H. J. D. Miller, G. Guarnieri, M. T. Mitchison, and J. Goold, Quantum Fluctuations Hinder Finite-Time Information Erasure Near the Landauer Limit, Phys. Rev. Lett. **125**, 160602 (2020).

[85] H. Zhao and C.-S. Yu, Coherence measure in terms of the Tsallis relative α entropy, Sci. Rep. **8**, 299 (2018).

[86] M. B. Ruskai and F. H. Stillinger, Convexity inequalities for estimating free energy and relative entropy, J. Phys. A: Math. Gen. **23**, 2421 (1990).

[87] K. M. R. Audenaert and J. Eisert, Continuity bounds on the quantum relative entropy, J. Math. Phys. **46**, 102104 (2005).

[88] K. M. R. Audenaert and J. Eisert, Continuity bounds on the quantum relative entropy–II, J. Math. Phys. **52**, 112201 (2011).