Spatial distributions of nonconservatively interacting particles

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Certain types of active systems can be treated as an equilibrium system with excess nonconservative forces driving some of the microscopic degrees of freedom. We derive results for how many particles having both conservative and nonconservative forces will behave. Treating nonconservative forces perturbatively, we show how the probability distribution of the microscopic degrees of freedom is modified from the Boltzmann distribution. We then derive approximate forms of this distribution through analyzing the nature of our perturbations. We compare the perturbative expansion for the microscopic probability distribution to an exactly solvable active system. Finally, we consider how the approximate forms for the microscopic distributions we have derived lead to different macroscopic states when coarse grained for two different kinds of systems, a collection of motile particles, and a system where nonconservative forces are applied in space. In the former, we are able to show that nonconservative forces lead to an effective attractive interaction between motile particles, and in the latter we note that by introducing nonconservative interactions between particles we modify densities through extra terms which couple to surfaces. In this way, we are able to recast certain active problems as the statistical mechanics of nonconservative forces.

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I. INTRODUCTION

Nonequilibrium systems remain a frontier for physics. Indeed, even the term "nonequilibrium" applies to a large number of observed systems, and it is not immediately apparent whether there would exist a single framework which would unify all these disparate phenomena. One particular class of nonequilibrium systems, deemed "active matter," focuses on situations where the microscopic components are being driven through continuous injection of energy [1]. Active matter has generated special interest as continual consumption of energy microscopically is one of the characteristic properties of life [2].

Even as nonequilibrium describes a wide variety of things, active matter also applies to a very diverse variety of phenomena, existing from subcellular length scales to interactions among organisms. Theoretical treatments of these systems often proceed from writing a model of the particular active system in question, usually starting from considering the forces involved [3,4]. One may then either simulate the microscopic equations of motion or coarse grain (either with top down or bottom up approaches) the microscopic degrees of freedom into effective field treatments that capture large scale properties. These methods are usually able to be successfully compared to the real experimental realizations [5]. Whether there exists a more generic conceptual basis for such systems is still a topic of ongoing research [6–8].

One salient difference between an active and an equilibrium system is the existence of nonvanishing microscopic currents [9]. In equilibrium all currents vanish, however, for an active system this does not necessarily hold [10]. Indeed, continual consumption of energy by the microscopic components somewhat implies the existence of currents. Nevertheless, active matter can relax to a nonequilibrium steady state, where the relevant statistical properties of the system are no longer changing in time, though microscopic currents may still be present. However, while for an equilibrium system the steady state properties can be calculated from the Boltzmann distribution, which can relate the microscopic degrees of freedom to the macroscopic properties via the partition function, steady state properties of active matter cannot necessarily be calculated via this method, as they are fundamentally out of equilibrium.

We can interrogate more deeply why exactly this may be. In particular, when one writes a model for an active system it may not be immediately apparent why this microscopic description would describe a nonequilibrium system. Were one to perform a careful analysis of the equations, however, one should find that there are terms in the dynamics which would necessitate the continuous injection of energy in order to be a realistic descriptor of a physical system. The microscopic equations may, for example, break the fluctuation dissipation theorem, corresponding to the fact that there may be active noise driving the components [11]. Another option is that, after one has written a microscopic description, one may see that there exist effective currents in the deterministic force part of the time evolution; a simple example could be that one microscopic degree of freedom x_1 is affecting the time evolution of another microscopic degree of freedom x_2 but that x_2 does not affect the time evolution of x_1 (or does not affect it in a commensurate way). Such a dynamical scheme would

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violate Newton's third law and could only be maintained by external energy input. This simple loop would correspond to there being a *nonconservative force* in the system. The presence of microscopic currents at steady state also implies the existence of effective nonconservative forces in some of the microscopic degrees of freedom. A more complicated example of nonconservative forces in active matter would be the microscopic descriptions of chemically active systems and nonequilibrium systems with switchable interactions [12].

The stationary state of a system where some of the forces are nonconservative can no longer be characterized as an equilibrium one, nor could the microscopic probability distribution be constructed from a Hamiltonian, as the forces in the system no longer arise as the gradient of a Hamiltonian. More broadly, the steady states of the system have to be understood from a consideration of the dynamic processes involved rather than from construction via the Boltzmann distribution.

In this paper, we therefore recast the stationary properties of active systems as the stationary distributions of particles evolving under a nonconservative force. We show how certain active systems can be mapped to time evolution under the interplay of conservative and nonconservative forces. We then describe how the stationary states of systems with nonconservative forces can be written for systems of arbitrary size. Previous approaches to this problem [13–17] are generally considered low-dimensional systems (not many particles). In Sec. II we discuss what effect a nonconservative force perturbing an equilibrium system has in terms of modifications of the ordinary Boltzmann distribution, and obtain formally exact results. As these forms are rather complicated, in Sec. III we derive effective forms of the stationary distribution for arbitrary nonconservative forces. Subsequently in Sec. III, in order to test the validity of the perturbative expansion, we apply these results to a simple active system (particles with an active noise), where the microscopic probability distribution for a force field with conservative and nonconservative parts can be solved exactly using matrix methods. Following on from this in Sec. IV we discuss how microscopic nonconservative forces lead to changes on macroscopic observables such as density, for two different test cases, where the nonconservative forces are "internal" and "external" and demonstrate the qualitative validity of our approach against simulations of particles interacting under a nonconservative pair force. Finally, we discuss in more depth the physics that emerges from statistical ensembles of nonconservatively interacting particles in Sec. V.

Nonconservative forces

Before we discuss our main results, we will briefly review conservative and nonconservative forces, as these concepts will appear repeatedly throughout the rest of the text. The presence of nonconservative forces implies that the system has some form of path dependence. A conservative force is one in which the the total work done around any closed loop is zero:

$$\oint \mathbf{F}_c(\mathbf{r}) \cdot d\mathbf{r} = 0 \tag{1}$$

or, alternatively, the total work done along a path depends only on the end points of the path. Nonconservative forces do not have this property:

$$\oint \mathbf{F}_{nc}(\mathbf{r}) \cdot d\mathbf{r} \neq 0. \tag{2}$$

By the gradient theorem, a conservative vector field can be written as a gradient of a scalar potential:

$$\mathbf{F}_c(\mathbf{r}) = \nabla f(\mathbf{r}). \tag{3}$$

However, nonconservative vector fields cannot be represented in this way. For vector fields in three dimensions, it is well known that any vector field (with proper conditions) can be decomposed into a gradient part and a curl part through the famous Helmholtz decomposition [18]:

$$\mathbf{F}(\mathbf{r}) = -\nabla\psi(\mathbf{r}) + \nabla \times \mathbf{a}(\mathbf{r}), \tag{4}$$

where ψ is some scalar field and $\mathbf{a}$ is a vector field. From now on we will use interchangeably the terminology that gradients are conservative and curls are nonconservative. Certain generalizations to the Helmholtz decomposition exist for vector fields in higher dimensions, through Hodge theory, with mathematical assumptions over the domain and that F is sufficiently well behaved. In equilibrium systems, the interparticle forces arise from potentials, and are thus conservative. However, for nonequilibrium systems this will not be generically true. Additionally, a system which is being driven by nonconservative forces would require there to be some energy source in order to continue its drive. Thermal and other random forces are also sometimes referred to as nonconservative forces; we therefore clarify that when we say nonconservative throughout the text we mean *deterministic* forces, where the force is a specified function of spatial coordinates $\mathbf{r}$.

The existence of vector field decomposition is suggestive when considering some problems in active matter, as the nonconservative part of the forces could be the part that is necessary for driving the system out of equilibrium. These terms could either be imposed or identified by hand or, for more realistic forces, decomposition algorithms could identify which part of the force field is either conservative or nonconservative [19]. Were one to note that the nonconservative part is zero, the system is either equilibrium with an effective Hamiltonian, or is being driven out of equilibrium by another mechanism.

Then, we have a twofold central question to address if this suggestion is to be meaningful: (a) Can the stationary properties of many-body systems with nonconservative forces be written in a way which is tractable for calculation? (b) Can we map real active systems onto a system evolving under a nonconservative force? The greater part of this paper shall be spent addressing these questions.

II. PARTICLES WITH NONCONSERVATIVE FORCES

We write the generic time evolution equations for the microscopic degrees of freedom of a system of N particles evolving under a force field inside a dissipative medium:

$$\frac{d\mathbf{r}}{dt} = \frac{\mathbf{p}}{m}, \tag{5}$$

$$\frac{d\mathbf{p}}{dt} = \mathbf{F}(\mathbf{r}) - \gamma\mathbf{p} + \boldsymbol{\xi}(t), \tag{6}$$

where $\mathbf{r}$ is the vector of all the positions of particles in the system, and $\mathbf{p}$ is the vector of all the momenta in the system:

$$\mathbf{r} = (\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_N), \tag{7}$$

$$\mathbf{p} = (\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_N). \tag{8}$$

We have employed the assumption for this model that the force field $\mathbf{F}$ depends on the position degrees of freedom of the system only $\mathbf{r}$. This assumption could in principle be relaxed in order to describe different classes of active systems where the forces depend on the momenta. We also assume that these particles exist in a media with a dissipative term γ and a white noise term $\boldsymbol{\xi}$ which has the property that $\langle \xi_i(t)\xi_j(t')\rangle = 2k_bT\delta_{ij}\gamma\delta(t-t')$. We assume that the friction coefficient γ is constant. These equations apply well in a few physical situations, such as large Brownian particles in a dense fluid. This is the physical picture we have in mind for this paper. It is also the case that the damping coefficient γ does not depend on the particle speed, which in reality would not be the case [20,21].

We now restrict ourselves to analysis of the stationary distribution of systems with this time evolution. We further focus our attention on the case where the system is overdamped, that is, we ignore inertial terms. We wish to obtain knowledge of the microscopic probability as a function of the positions of all the particles $P(\mathbf{r})$. The corresponding Smoluchowski equation for the evolution of the probability density under these forces is given by [22]

$$\frac{\partial P(\mathbf{r},t)}{\partial t} = \frac{1}{\gamma m}\nabla\cdot[-\mathbf{F}(\mathbf{r})P(\mathbf{r},t) + k_bT\nabla P(\mathbf{r},t)], \tag{9}$$

where we have introduced the ∇ differential operator, which is given by $\nabla = (\frac{\partial}{\partial \mathbf{r}_1}, \frac{\partial}{\partial \mathbf{r}_2}, \ldots, \frac{\partial}{\partial \mathbf{r}_N})$. The validity of the overdamped approximation in leading to this Smoluchowski equation is itself also assumed [23–25], corresponding to situations where the force is considered constant over long periods of time, however, we shall proceed on the assumption that Eq. (9) is accurate. As the probability must be greater than or equal to zero, we represent the probability density as $P(\mathbf{r}) = \exp[-\phi(\mathbf{r})]$. The stationary state distribution is the one in which the time derivative of the probability is zero, in other words, the left hand side of Eq. (9) is equal to zero. From Eq. (9), the stationary probability density in this system must obey the following relationship:

$$\nabla\cdot(\exp[-\phi(\mathbf{r})][-\mathbf{F}(\mathbf{r}) - k_bT\nabla\phi(\mathbf{r})]) = 0. \tag{10}$$

One can see trivially if the forces in the system arise from the gradient of a function (such as the Hamiltonian) $F(\mathbf{r}) = -\nabla H(\mathbf{r})$ that the stationary probability distribution over microstates will be given by the Boltzmann distribution $P(\mathbf{r}) \sim \exp[-H(\mathbf{r})/k_bT]$. For systems in which the forces do not arise as the gradient of a function, the solutions are not so trivial. We remind readers that we use the terms *conservative* to describe forces which are expressible as the gradient of a scalar function, and *nonconservative* to describe forces which cannot be expressed in this way. Following Risken and others [16,22], we imagine that the force can be split into two components:

$$F(\mathbf{r}) = \mathbf{f}^{(c)}(\mathbf{r}) + \mathbf{f}^{(a)}(\mathbf{r}), \tag{11}$$

where the new conditions that have to be specified for the probability distribution to be stationary are given by

$$\mathbf{f}^{(c)}(\mathbf{r}) = -k_bT\nabla\phi(\mathbf{r}), \tag{12}$$

$$\nabla\cdot(e^{-\phi(\mathbf{r})}\mathbf{f}^{(a)}(\mathbf{r})) = 0. \tag{13}$$

However, this is only a redefinition of the problem (10). Were we to be able to find the proper splitting of the force fields, we could in principle solve for the stationary distribution. We will now focus our attention on an example system given by forces

$$\mathbf{F}(\mathbf{r}) = -\nabla H_0(\mathbf{r}) + \mathbf{m}(\mathbf{r}), \tag{14}$$

where H_0 defines some equilibrium Hamiltonian, but we keep $\mathbf{m}(\mathbf{r})$ to be as general as possible, such that it may include conservative and nonconservative elements. We here note that the splitting operation defined in Eqs. (12) and (13) is not the same as merely separating the conservative and nonconservative components of the force, as can be seen from Eq. (13), which will not generally be equal to zero even if $\nabla\cdot\mathbf{m}(\mathbf{r}) = 0$. The challenge is therefore to find the splitting of the force field which satisfies these conditions. This is a difficult problem, but can be recast into the form of a differential equation by making the following addition, introducing the new scalar field χ:

$$f^{(c)}(\mathbf{r}) = -\nabla H_0(\mathbf{r}) - \nabla\chi(\mathbf{r}), \tag{15}$$

$$f^{(a)}(\mathbf{r}) = m(\mathbf{r}) + \nabla\chi(\mathbf{r}). \tag{16}$$

This trivially satisfies the conservative condition of the log probability (12). In order to find the probability distribution, we are then left with satisfying the following differential equation for χ from Eq. (13) [Eq. (17)]:

$$\begin{aligned}\nabla^2\chi(\mathbf{r}) - \beta[\nabla H_0(\mathbf{r}) + \mathbf{m}(\mathbf{r})]\cdot\nabla\chi(\mathbf{r}) - \beta(\nabla\chi(\mathbf{r}))^2 \\ = \beta\nabla H_0(\mathbf{r})\cdot\mathbf{m}(\mathbf{r}) - \nabla\cdot\mathbf{m}(\mathbf{r}),\end{aligned} \tag{17}$$

where we have introduced the notation that $k_bT = \beta^{-1}$. This is an N-dimensional nonlinear partial differential equation in the unknown $\chi(\mathbf{r})$, which is by itself still a challenging proposition. However, if we define the following substitution $\chi(\mathbf{r}) = -\beta^{-1}\log[\mu(\mathbf{r})]$ we are left with the following linear equation in $\mu(\mathbf{r})$:

$$[\nabla^2 - \mathbf{q_1}(\mathbf{r})\cdot\nabla + q_2(\mathbf{r})]\mu(\mathbf{r}) = 0, \tag{18}$$

$$\mathbf{q_1}(\mathbf{r}) = \beta[\nabla H_0(\mathbf{r}) + \mathbf{m}(\mathbf{r})], \tag{19}$$

$$q_2(\mathbf{r}) = \beta^2\nabla H_0(\mathbf{r})\cdot\mathbf{m}(\mathbf{r}) - \beta\nabla\cdot\mathbf{m}(\mathbf{r}), \tag{20}$$

where only solutions with $\mu(\mathbf{r}) > 0$ are physical [$\chi(\mathbf{r})$ is real]. We assume a perturbative expansion exists in the magnitude of the additional forces $\mathbf{m}$, parametrizing the additional forces with a parameter ϵ,

$$\mathbf{m} \to \epsilon\mathbf{m}, \tag{21}$$

and seeking solutions of $\mu(\mathbf{r})$ of the form $\mu(\mathbf{r}) \approx \sum_{n=0}\epsilon^n\mu_n(\mathbf{r})$.

Substitution of this expansion leads to the following differential equation for the *nth* order perturbation:

$$[\nabla^2 - \beta\nabla H_0(\mathbf{r})\cdot\nabla]\mu_n(\mathbf{r}) = [\beta\mathbf{m}(\mathbf{r})\cdot\nabla - q_2(\mathbf{r})]\mu_{n-1}(\mathbf{r}), \tag{22}$$

where, additionally, the perturbative expansion equation (22) can be factorized as

$$\nabla\cdot(\exp[-\beta H_0(\mathbf{r})]\nabla\mu(\mathbf{r})) \tag{23}$$

$$= \exp[-\beta H_0(\mathbf{r})][\beta\mathbf{m}(\mathbf{r})\cdot\nabla - q_2(\mathbf{r})]\mu_{n-1}(\mathbf{r}). \tag{24}$$

The expression on the left is a linear operator acting on $\mu_n(\mathbf{r})$ and the term on the right is some forcing term. This series would be supplemented by the condition that $\mu_0 = \text{const}$. The equation for each individual μ_n is acted upon by the same linear operator for every n, the only differences are the *source* terms on the right hand side of the equations. Therefore, were one to be able to find a Green's function of the linear operator acting on each individual μ_n, given by the following equation

$$[\nabla^2 - \beta\nabla H_0(\mathbf{r})\cdot\nabla]G(\mathbf{r},\mathbf{r}') = \delta(\mathbf{r}-\mathbf{r}') \tag{25}$$

with the appropriate boundary conditions that G vanishes at infinity. Then, one may express the entire series solution in terms of this function as

$$\mu_0(\mathbf{r}) = \text{const}, \tag{26}$$

$$\mu_1(\mathbf{r}) = -\mu_0\int G(\mathbf{r},\mathbf{r}')q_2(\mathbf{r}')d\mathbf{r}', \tag{27}$$

$$\vdots \tag{28}$$

$$\mu_n(\mathbf{r}) = \int G(\mathbf{r},\mathbf{r}')[\beta\mathbf{m}(\mathbf{r}')\cdot\nabla' - q_2(\mathbf{r}')]\mu_{n-1}(\mathbf{r}')d\mathbf{r}'. \tag{29}$$

As the Green's function only depends on the *equilibrium* properties of the system, this representation shows the effect of nonconservative force fields in terms of integrals over effective source terms arising from the imposition of the nonconservative force.

This series would be the steady state solution of the microscopic probability for a system evolving under a generic force field. However, the complexity of this representation is hardly less than the original equation, given the difficulty in evaluating the coordinate space Green's functions. We will revisit this question in a later section and for now also discuss the physical significance of the perturbations. For completeness, we note that the first order perturbation is given by

$$P(\mathbf{r})\approx\exp\left(-\beta H_0(\mathbf{r})+\epsilon\beta\int d\mathbf{r}'G(\mathbf{r},\mathbf{r}')[\beta\nabla' H_0(\mathbf{r}')\cdot\mathbf{m}(\mathbf{r}') - \nabla\cdot\mathbf{m}(\mathbf{r}')]\right), \tag{30}$$

which is already a familiar result in the recent literature [13,14], though we here display it in terms of the Green's function. For extensions to the above for underdamped systems see Ref. [17], as well as cases where one can solve the Green's function by inspection and comparison to real experimental cases [26]. The application of these equations to active systems constitutes the central goal of this paper.

Conservative and nonconservative perturbations

In the previous section, we showed the modification to the Boltzmann distribution from the action of a generic additional vector field of forces $\mathbf{m}$. The modified distribution can be seen to be equal to the Boltzmann distribution with an additional series of perturbations for higher order effects, which are represented as integrals over effective *source terms* with an integral kernel derived from the corresponding equilibrium system (the Green's function). Thus far, we kept the form of the additional forces $\mathbf{m}$ general. We now analyze the nature of this perturbation with respect to two different extremes, one of which is that the additional forces are themselves conservative, and another where the additional forces are divergence free, corresponding to a "purely" nonconservative perturbation.

First, we imagine that the perturbation is solely made up of a gradient part $\mathbf{m}(\mathbf{r}) = -\nabla H_1(\mathbf{r})$. For this special example we can calculate the perturbative terms exactly, for example, the first order perturbation given by

$$\mu_1 = -\mu_0\beta\int G(\mathbf{r},\mathbf{r}')\beta^{-1}q_2(\mathbf{r}')d\mathbf{r}', \tag{31}$$

where for a conservative perturbation the function q_2 from Eq. (20) is given by

$$\beta^{-1}q_2 = \nabla^2 H_1 - \beta\nabla H_0\cdot\nabla H_1. \tag{32}$$

However, the Green's function is just the inverse of this operator acting on H_1, thus the integral in expression (31) becomes simply $-\mu_0\beta H_1$, which is what we would expect from ordinary perturbations to the original Hamiltonian with an extra potential term. In fact, it is possible to prove that the full series expression for μ when the imposed field is conservative is given by

$$\mu = \exp(-\beta H_1). \tag{33}$$

In other words, the Hamiltonians are additive, as we might have expected. When the imposed field is divergence free, $\nabla\cdot\mathbf{m}(\mathbf{r}) = 0$, with no gradient component, simple closed form expressions cannot be found, however, we note the source term has the form $q_2(\mathbf{r}) = \beta^2\nabla H_0(\mathbf{r})\cdot\mathbf{m}(\mathbf{r})$. From here on, we shall refer to this term $\nabla H_0(\mathbf{r})\cdot\mathbf{m}(\mathbf{r})$ simply as "the source," for if this expression is equal to zero, the entire coordinate space distribution will reduce back to the Boltzmann distribution.

From the above, it is clear that conservative perturbations are naturally included within the new perturbed expression in a simple way. However, we have established that the perturbing forces do not necessarily have to be nonconservative to reproduce the probability distribution. In principle, we can do the perturbation around any force, which would modify the Green's function. The particular choice will depend on the ease of the calculation and the physics relating the magnitude of perturbations.

III. STATIONARY PROBABILITY DISTRIBUTIONS IN NONCONSERVATIVE SYSTEMS

In the previous section, we defined the distribution of particles with both conservative and nonconservative forces as integrations over integral kernels. In the following, we

show the full form of the Green's function, and then derive approximate forms of the Green's functions, and discuss what physics they entail. Then, taking an example of a real active system, we then show how the perturbative scheme described in the previous section can be used to derive the first order microscopic probability distribution function and we test its validity by comparing it to an exact numerical calculation.

A. Coordinate space Green's functions and effective distributions

We observed in Sec. II that the full solutions to any system could be represented by integrations of various source terms with Green's function kernels. The kernels are the solutions to the following equation:

$$[\nabla^2 - \beta \nabla H_0(\mathbf{r}) \cdot \nabla] G(\mathbf{r}, \mathbf{r}') = \delta(\mathbf{r} - \mathbf{r}'). \tag{34}$$

Physically, the Green's function is a linear operator which characterizes the modification of a microscopic distribution to the effect of a microscopic nonconservative perturbation. We can think of it as a microscopic response function that tells us how an equilibrium system is modified by the presence of nonconservative forces. A key question is as follows: Can we say anything at all about the states for a general nonconservative system of arbitrary size? Analyzing the form of these Green's function can give us an insight into the physics of microscopic distributions under nonconservative forces. While a full analytic form of these Green's function kernels is difficult to work with, it can be written as a series expansion. The full Green's function can be written as a Liouville-Neumann series in the Boltzmann factor β:

$$G(\mathbf{r}, \mathbf{r}') = \sum_{n=0}^{\infty} \beta^n G_n(\mathbf{r}, \mathbf{r}'). \tag{35}$$

The first few terms in this series can be written as

$$G_0(\mathbf{r}, \mathbf{r}') = \frac{c}{|\mathbf{r} - \mathbf{r}'|^{N-2}}, \tag{36}$$

$$G_1(\mathbf{r}, \mathbf{r}') = \int d\mathbf{r_1} G_0(\mathbf{r}, \mathbf{r_1}) G_0(\mathbf{r_1}, \mathbf{r}') [-\nabla H_0(\mathbf{r_1}) \cdot \bar{\mathbf{q}}(\mathbf{r_1}, \mathbf{r}')], \tag{37}$$

$$G_2(\mathbf{r}, \mathbf{r}') = \int d\mathbf{r_1} d\mathbf{r_2} G_0(\mathbf{r}, \mathbf{r_1}) G_0(\mathbf{r_1}, \mathbf{r_2}) G_0(\mathbf{r_2}, \mathbf{r}') \times [-\nabla H_0(\mathbf{r_1}) \cdot \bar{\mathbf{q}}(\mathbf{r_1}, \mathbf{r_2})][-\nabla H_0(\mathbf{r_2}) \cdot \bar{\mathbf{q}}(\mathbf{r_2}, \mathbf{r}')], \tag{38}$$

where we introduce the vector $\bar{\mathbf{q}}(\mathbf{r_1}, \mathbf{r}') = \frac{\mathbf{r_1} - \mathbf{r}'}{|\mathbf{r_1} - \mathbf{r}'|^2}$. It is perhaps easier to see how each term in this expansion can be written graphically, which we include in Fig. 1. The graphical representation of each term in Fig. 1 makes more clear the physical aspects of the Green's function. Despite the fact we have not introduced any path integrals in our analysis so far, the solution to the deceptively simple Eq. (34) takes the form of integrating over all paths connecting the points $\mathbf{r}$ and $\mathbf{r}'$. This path is weighted by the zeroth order Green's function, which strongly suppresses overly long paths connecting the two points, and by the factor $[-\nabla H_0(\mathbf{r_1}) \cdot \bar{\mathbf{q}}(\mathbf{r_1}, \mathbf{r_2})]$ which weights how "helpful" (work done) the equilibrium forces at each point along the path are, by calculating the magnitude of their projection to the next point along the path. In this figure,

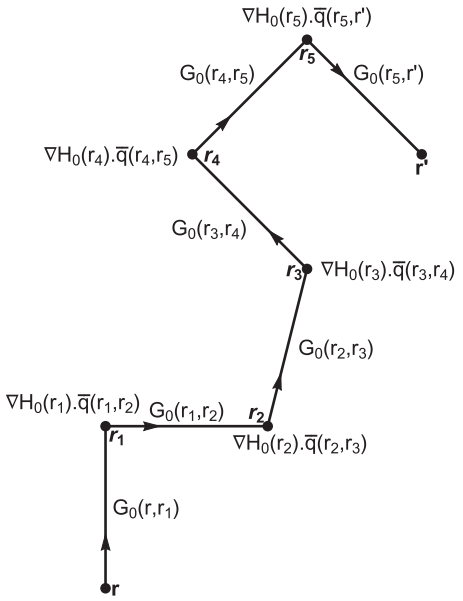

FIG. 1. Graphical representation of the nth term of the Green's function series solution (in this case $n = 6$). One draws a line of n segments connecting the points $\mathbf{r}$ and $\mathbf{r}'$. Associated with each interior vertex is a factor corresponding to the projection of the force at the point along the next line segment. Along each edge is associated a zeroth order Green's function. The final form is integrated along all the interior points, giving an effective sum over paths of n links.

the path dependence of the Green's function becomes clear, in that the effect of a source at point $\mathbf{r}'$ is strong at point $\mathbf{r}$ if there is a path of "helpful" forces connecting the two points. We also mention that as the operator acting on the left hand side of Eq. (34) is not self-adjoint for a general Hamiltonian, the Green's function does not display symmetry in its arguments nor does it necessarily form a complete set such that an eigenfunction expansion is always valid. In other words, $G(\mathbf{r}, \mathbf{r}') \neq G(\mathbf{r}', \mathbf{r})$. Therefore, to first order, the probability of each microstate of the system with nonconservative forces is weighted by the sum over all paths (weighted by the path length) over all space of how much "work" the nonconservative forces are doing on the conservative forces.

Equation (35) is of physical interest as the full solution of the Green's function for any system, however, it is difficult to make use of it in calculation. Therefore, we supplement it with the following two well founded approximations, which allow us to calculate approximate forms of this Green's function that give analytic more tractability for calculation in many-body systems.

The first approximation can be obtained by assuming that the Green's function integrals from Fig. 1 are path independent (see Appendix A for details). This effective Green's function is given by

$$G(\mathbf{r}, \mathbf{r}') = \int d\mathbf{r}'' e^{-[\beta H_0(\mathbf{r}') - \beta H_0(\mathbf{r}'')]} \frac{(\mathbf{r}'' - \mathbf{r}') \cdot (\mathbf{r} - \mathbf{r}'')}{A_N |\mathbf{r}'' - \mathbf{r}'|^N |\mathbf{r} - \mathbf{r}''|^N}, \tag{39}$$

where A_N is some constant related to the surface area of the N sphere. We compare the solutions given by these Green's functions and some real solutions to Eq. (22) in Fig. 2. It can

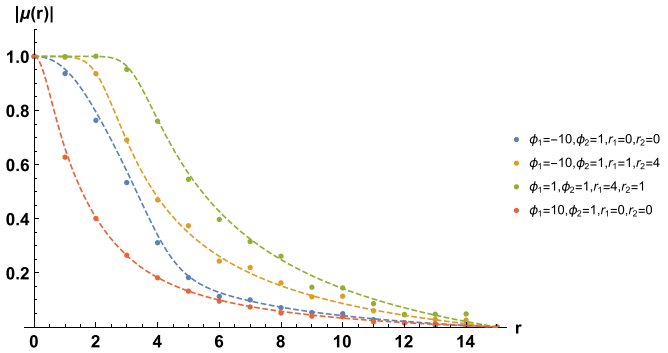

FIG. 2. Comparison of the full numerical solution for a given system compared to the solution calculated from the approximate Green's function in Eq. (A3). Shown are the numerical solutions (solid lines) to the differential equation (22) compared against solutions obtained by integrating the source with the Green's function (points). For all these problems we use a source term that goes as $\rho(r) = \phi_2 \exp[-(r - r_2)^2]$ and a Hamiltonian that goes as $\phi_1 \exp[-(r - r_1)^2]$. We show the comparisons for different values of all these parameters (indicated in the caption). The approximate Green's function correctly captures the features of the solution.

be seen in Fig. 2 that this approximate Green's function does a good job at capturing the features of the solution compared to the full numerical solution (22). The error associated with this Green's function goes as $\sim\beta^2|\nabla H_0(\mathbf{r})|^2$, i.e., this approximation is poor for states which are unlikely anyway. It can also be seen that even for large values of the force in Fig. 2 that the solutions are still reasonably accurate. This approximate form of the Green's function is useful in analyzing the physics due to nonconservative perturbation. Physically, the Green's function characterizes the effect that an impulse at some point $\mathbf{r}'$ has at point $\mathbf{r}$. The asymmetry in the Green's function is then explicative of the structure of the equilibrium landscape, it is clear that if there is a unit impulse at a point $\mathbf{r}'$ the effect at point $\mathbf{r}$ will be minimal if that point $\mathbf{r}'$ was very unlikely in the original system. However, the converse is not true, in that the effect of a unit impulse at a local minimum will strongly affect all the points around it. Due to the contribution of the ∇^2 term, the entire Green's function $G(\mathbf{r}, \mathbf{r}')$ is peaked around $\mathbf{r} = \mathbf{r}'$. As can also be seen from Fig. 2, this is the case even if we have a large repulsive potential at $\mathbf{r} = \mathbf{r}'$.

This form of the Green's function includes all the correlations present in H_0. However, certain other simplifications exist that make this problem much simpler for large systems. One of the main difficulties of problems of this type is that the addition of a single nonconservative force acting on one particle would in principle modify the form of the microscopic probability distribution of all the particles, due to the correlations in the Green's function. Most source terms that we encounter for large systems only exist in a few of the variables of the system, for example, if we were to introduce particles interacting with some pairwise nonconservative source between them, our total source term would go something as

$$\rho_S(\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_N) = \sum_{i \neq j} \rho_{Su}(\mathbf{r}_i, \mathbf{r}_j). \tag{40}$$

For some general function ρ_{Su}. A great simplification can result from considering the fact that the Green's function is a linear operator, it is applied independently to each term on the right hand side of Eq. (40). Therefore, we have the situation where the Green's function is being integrated over a source term which exists in only a few variables, despite the fact that in principle the Green's function depends on the position of every particle in the system. Another very useful assumption that can employed in this case is that the integration over all other variables not present in the source term is given by

$$\begin{aligned}\mu(\mathbf{r}) &= \int d\mathbf{r}' G(\mathbf{r}, \mathbf{r}')\rho_S(\mathbf{r}'_1, \mathbf{r}'_2) \\ &\approx \int dr'_1 dr'_2 G_R(r_1, r_2, r'_1, r'_2)\rho(r'_1, r'_2),\end{aligned} \tag{41}$$

where G_R is the Green's function in reduced dimensions. This assumption is well founded as we approach the thermodynamic limit $N \to \infty$, and is also exact at high temperatures. In Appendix B we show how this assumption can be justified for systems with a pairwise Hamiltonian, relying on the fact that ordinary Hamiltonians are symmetric. In general, there will be some extra three body terms introduced into the statistical weight (exponent), but they go as $O(1/N)$ so we feel safe in discarding them when we are trying to understand the properties of many particle systems. Physically, this assumption amounts to the idea that if we introduce nonconservative pair sources, the dominant contributors to the statistical weight will themselves be pairwise, which means that the relevant correlations in the Hamiltonian when calculating the perturbations are those that exist in the subset of the variables of the perturbing source term, i.e., if we have some operator equation $\mathcal{O}(\mathbf{r})\mu(r) = \rho_S(\mathbf{r}_1, \mathbf{r}_2)$, then $\mu(\mathbf{r})$ is approximately the solution to $\mathcal{O}(\mathbf{r}_1, \mathbf{r}_2)\mu(\mathbf{r}) = \rho_S(\mathbf{r}_1, \mathbf{r}_2)$. It can be seen, for example, if $\mathcal{O}$ is the Laplacian, that this is exactly true.

This leads to a drastic simplification of the problem. For example, if we were to introduce a pair nonconservative source acting between every particle, we would only need to calculate the value of the perturbation for these two particles, and our full solution to (22) would be given by a sum over all pairs over this reduced solution. Thus, the physical meaning of the Green's function becomes clearer in this light. They map the real nonconservative forces in the system to effective "potentials" that enter into a modified distribution given by Eq. (41). One can then use the approximate form of the Green's function given in Eq. (A3) or else solve the associated differential equations directly in order to map a particular form of nonconservative force to the effective "potential" that appears in the exponent of the distribution. This will then allow one to calculate microscopic properties of a system containing nonconservative forces by taking the sum over all these effective potentials arising from the nonconservative force.

The results of the previous sections suggest a possible path about answering questions about the distributions of particles subject to nonconservative forces. The steps can be roughly summarized as follows:

(1) Write the forces for the system of interest.

(2) From these generic force fields, establish the contribution that is conservative, and the contribution that is

nonconservative. This can either be done by inspection, where possible, or through the use of decomposition theorems where it is not so.

(3) From the conservative part, establish the effective Hamiltonian. This is the effective state around which the system is being perturbed by the nonconservative contribution.

(4) Using this Hamiltonian, calculate the modification to the microscopic probability due to perturbation by the nonconservative fields, up to whichever order. For large systems, we can achieve this by splitting the full problem into subproblems over only a few coordinates in the system. We can then combine these results together to form the full perturbation up to whichever order.

B. Validity of first order perturbation: Particles with active noise

As the list at the end of the previous section remains rather abstract, we shall apply it definitively to a simple system describing some active process, and also illustrate how well the perturbative approach is able to capture the features of this system. As we wish to test how accurate the first order perturbation is, we need to compare against a system which is in principle exactly solvable. We take one of the simplest examples of an active system, for example, an optically trapped colloidal bead inside a bath of bacteria, whose time evolution of position x can be modeled through the following equation [11]:

$$\zeta \frac{dx(t)}{dt} = -kx(t) + \xi(t) + A(t), \tag{42}$$

where we assume that this colloid is overdamped and exists in a media with a dissipative term ζ (distinct from γ used in the previous sections, as this is an overdamped equation) and a white noise term ξ which has the property that $\langle \xi(t)\xi(t')\rangle = 2k_bT\zeta\delta(t-t')$. Supplementing this is an active noise $A(t)$ where $\langle A(t)A(0)\rangle = \frac{\alpha}{\tau_a}\exp(-t/\tau_a)$ where τ_a is the timescale of the active noise, establishing the system as breaking the fluctuation-dissipation relation. By representing the active forces in terms of an extra degree of freedom, we get an expression in two degrees of freedom describing the same equation:

$$\zeta \frac{dx(t)}{dt} = -kx(t) + \xi(t) + A(t), \tag{43}$$

$$\frac{dA(t)}{dt} = -\frac{1}{\tau_a}A(t) + \xi_2(t), \tag{44}$$

where we introduce another fluctuating white noise process $\xi_2(t)$, which will have a similar correlation property as $\xi(t)$, $\langle \xi_2(t)\xi_2(t')\rangle = 2k_bT\zeta\delta(t-t')$. This system can be defined in terms of a matrix of forcings:

$$\zeta \frac{d}{dt}\begin{pmatrix} x(t) \\ A(t)\end{pmatrix} = -\begin{pmatrix} k & \beta\alpha/\tau_a^2 \\ 0 & \frac{\zeta}{\tau_a}\end{pmatrix}\begin{pmatrix} x(t) \\ A(t)\end{pmatrix} + \begin{pmatrix} \xi(t) \\ \xi_2(t)\end{pmatrix}. \tag{45}$$

This leads to generic forces corresponding to this process given by

$$\mathbf{F} = \underline{C}\mathbf{r}, \tag{46}$$

where the deterministic forces are nonconservative in the expanded space $\mathbf{r} = (x, A)$, defining the system as an out of equilibrium one in this expanded state space. For a many-body system, this can be written in block matrix form as

$$\underline{C} = -\begin{pmatrix} \underline{K} & \beta\alpha/\tau_a^2\underline{I} \\ 0 & \frac{\zeta}{\tau_a}\underline{I}\end{pmatrix}, \tag{47}$$

where $\underline{I}$ is the identity matrix, and $\underline{K}$ is some matrix of springs connecting our particles. These "forces" act on both the real position degree of freedom and the extra degree of freedom that represents the active force. If the colloidal particles are not interacting with each other in any way, this matrix $\underline{K}$ would be diagonal. If, for example, we allowed the colloidal particles to be tied together by springs, and if we additionally allowed different active process to collectively impact on the colloidal particles, this would serve to make the matrix $\underline{K}$ and the off-diagonal block matrices dense. We shall see that mathematically it makes no difference (as long as the full matrix $\underline{C}$ is positive definite) which particular form we wish to consider, as the system for the linear forces in (46) can be solved for any matrix $\underline{C}$, which we shall proceed to do in order to demonstrate that the perturbative solution reproduces approximately correct solutions. We shall then discuss the physics of system (45).

In order to be able to compare the validity of our perturbative approach, we need to know the real solution. Fortunately, a numerical scheme exists for solving (17) which gives a solution that looks like $\phi \sim \mathbf{r}^T\underline{B}\mathbf{r}$ or, more precisely, that the probability distribution has the form $P(\mathbf{r}) = \exp[-\phi(\mathbf{r})] = \exp(-\beta\frac{1}{2}\mathbf{r}^T\cdot\underline{B}.\mathbf{r})$ where $\underline{B}$ is a matrix. We omit the details for how the numerical solutions are obtained, and direct interested readers to Appendix C instead.

We will compare the exact numerical exact solution for a linear system of size 10 in one dimension, with forces given by

$$\mathbf{F}(\mathbf{r}) = -k\underline{I}\cdot\mathbf{r} + \delta\underline{M_S}\cdot\mathbf{r} + \epsilon\underline{M_{AS}}\cdot\mathbf{r}, \tag{48}$$

where I is the identity matrix and M_S and M_{AS} are, respectively, a symmetric and antisymmetric random matrix of couplings uniformly distributed between 1 and -1 and where every diagonal element is zero, and k, ϵ, and δ are parameters that set the strength of the different matrices and are chosen such that the full matrix is positive definite. For linear systems, identification of conservative and nonconservative forces is very straightforward, symmetric matrices are conservative, and antisymmetric matrices are nonconservative (point 2 of the itemized list in the last section). For the particles with active noise system, it is obvious that the matrix of forcings is not symmetric, thereby establishing it as nonconservative.

The first order perturbation with these forces would be given by

$$P(\mathbf{r}) = \exp\left(-\tfrac{1}{2}\beta\mathbf{r}^T\cdot(k\underline{I} + \delta\underline{M_S})\cdot\mathbf{r} - \tfrac{1}{2}\beta\mu_1\right), \tag{49}$$

where μ_1 is the solution to Eq. (22):

$$[\nabla^2 - \beta\mathbf{r}^T\cdot(k\underline{I} + \delta\underline{M_S})\cdot\nabla]\mu_1(\mathbf{r}) = \beta^2\mathbf{r}^T\cdot(k\underline{I} + \delta\underline{M_S})\cdot(\epsilon\underline{M_{AS}})\cdot\mathbf{r}. \tag{50}$$

Fortunately for this problem, the solution can be written exactly without having to perform the Green's function integral,

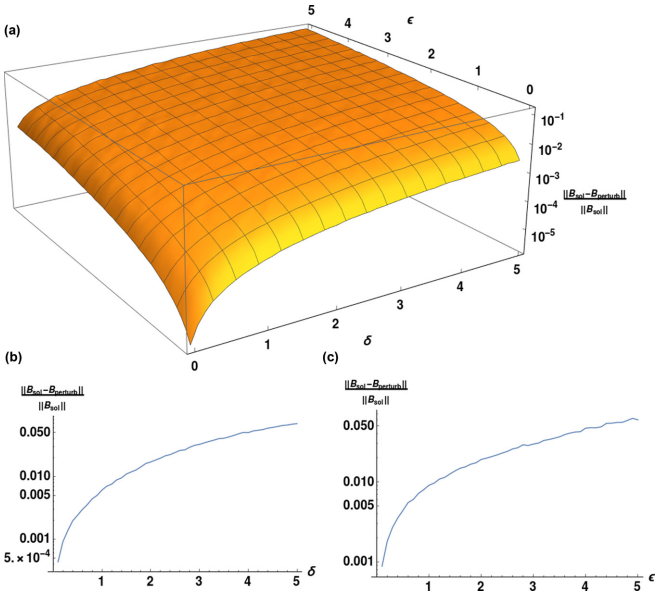

FIG. 3. Average error between perturbative solutions and real solutions for a linear system for 100 different realizations of the perturbation matrices $\underline{M}_S, \underline{M}_{AS}$ for each magnitude of perturbation ϵ, δ (with $k = 10$). (a) Shows the relative error as a function of magnitudes of perturbations ϵ and δ are inreased. (b), (c) Show slices at one point in $\epsilon = 2.5$ and $\delta = 2.5$, respectively. As expected, the perturbation theories become worse for larger perturbations. However, the relative error is acceptable even for large perturbations ($\sim$50% of the magnitude of the equilibrium problem has a 10% relative error in the microscopic probability).

and is given by the solution to the following commutator equation:

$$\mu_1 = \beta \mathbf{r}^T \cdot (\underline{X}) \cdot \mathbf{r}, \tag{51}$$

$$\{(k\underline{I} + \delta \underline{M_S}), \underline{X}\} = [(k\underline{I} + \delta \underline{M_S}), \epsilon \underline{M_{AS}}], \tag{52}$$

where the curly brackets denote an anticommutator and the square brackets are a normal commutator. This equation can be solved using the eigendecomposition of the symmetric force matrix so long as $\underline{X}$ is symmetric and traceless.

The perturbative and exact solutions to this problem can be written in the form $\frac{1}{2}\mathbf{r}^T \underline{B} \cdot \mathbf{r}$ where $\mathbf{r}$ are the coordinates and $\underline{B}$ is the solution. We compare the similarity of the two solutions as a way of probing the effectiveness of the perturbative approach $||\underline{B}_{\text{sol}} - \underline{B}_{\text{perturb}}||/||\underline{B}_{\text{sol}}||$ where $\underline{B}_{\text{sol}}$ is the actual solution and $\underline{B}_{\text{perturb}}$ is the perturbative solution. We calculate the error by taking the matrix norm of the differences divided by the norm of the exact solution. The average error is shown in Fig. 3 for different magnitudes of the perturbation.

As seen in Fig. 3, the error between the real and perturbative solutions for this system becomes larger as the size of the perturbation increases, but it gives good agreement to the real solution even for perturbations which are half the size of the original problem ($\sim$10% error). This is not a rigorous proof that a perturbative expansion always exists in the nonconservative forces, however, we also note that for the case where there are only divergence-free nonconservative forces in the system, the probability is constant across all coordinate space (for any force with this property). Therefore, even in the case where the nonconservative forces were very much stronger than the conservative ones, we could imagine performing an expansion around the homogeneous state in order to obtain physically meaningful results.

Having established how well the first order perturbation does mathematically, we can also use it to describe what happens to our original system of interest, particles with an additional, active, noise. For one, the addition of the term going as β^2 plays havoc with the ordinary potentials of statistical mechanics. For example, there is no simple way of calculating the average energy of the system any more, in fact, it is not clear that energy is meaningful in any sense any more, as the occupation of states is no longer just related to a gradient of a potential, nor does there exist any constant of motion which can be called energy. We shall revisit the question of observables under this distribution in the next section. Performing the integration over the subspace of the "hidden" active parameters can give us the effective distributions in x. In principle, it is possible to get these formulas analytically, but they are very long so will not be reproduced here. The main features arising from the consideration of the nonconservative part of the force for the model system (45) is a broadening of the microscopic distribution, as one would expect. This applies whether the matrix of connections $\underline{K}$ is dense or not. By itself, this effect is not so interesting or difficult to understand, as the nonconservative forces roughly seem to have the same effect as temperature, broadening of the microscopic distribution. In the next section, we shall consider systems where more interesting phenomena can arise, as well as macroscopic effects.

IV. LARGER SCALE EFFECTS: DENSITY IN NONCONSERVATIVE SYSTEMS

Despite having a full microscopic form of the probability distribution in the preceding sections, of more immediate importance in experimental realizations would be the larger scale properties of such systems. In this section we will try to more qualitatively understand how *microscopic* driving will lead to *macroscopic* effects. The correct macroscopic degrees of freedom for nonequilibrium system are not obvious [27], so we will choose to focus on the density of the particles, which always exists and is well defined in and out of equilibrium. Rather than treating the density as a conserved field and reasoning over the form of the density currents from physics in a continuity equation like scheme, we instead wish to focus on how our microstate probability would modify the macroscopic density that one would observe for nonconservative systems. We are particularly interested in the generalities that might arise on a large scale from a consideration of any nonconservative force, as opposed to model-dependent effects. We wish to see if we can qualitatively understand the behavior that we see in real systems this way.

We can do this by transforming our microstate probability into a probability for a macroscopic property, such as the density. Proceeding from the microscopic probability

$$P(\mathbf{r}) \sim \exp[-\beta H_0(\mathbf{r}) - \beta^2 \mu_1(\mathbf{r})]. \tag{53}$$

Analogously to ordinary statistical mechanics, we can take the normalization constant associated with the probability distribution (53) as the integral over all the microstates:

$$\mathcal{Z} = \int d\mathbf{r}\, \exp[-\beta H_0(\mathbf{r}) - \beta^2 \mu_1(\mathbf{r})]. \tag{54}$$

We can then calculate observables in the standard way for a probability distribution:

$$\langle O \rangle = \frac{1}{\mathcal{Z}} \int d\mathbf{r}\, O(\mathbf{r}) \exp[-\beta H_0(\mathbf{r}) - \beta^2 \mu_1(\mathbf{r})]. \tag{55}$$

In the absence of nonconservative driving, $\mathcal{Z}$ would just be the normal partition function. In the presence of it, as we mentioned previously, we can see that simple relationships in statistical mechanics do not hold. For example, if we were to take the derivative of this $\log(\mathcal{Z})$ with respect to β we would no longer have anything that looked like energy, or even the average of the Hamiltonian (which is itself a representation of the conservative part of the force), due to the term which goes as β^2. Thus, as one might expect, a lot of the standard quantities and relationships of statistical mechanics no longer apply in this new situation where nonconservative forces are present.

The observable quantity we are most interested in is the density of a fluid of particles interacting nonconservatively, which corresponds to the average of the density operator under the stationary distribution

$$\rho(\mathbf{x}, \mathbf{r}) = \sum_{i=1}^{N} \delta(\mathbf{x} - \mathbf{r_i}), \tag{56}$$

$$\langle \rho(\mathbf{x}) \rangle = \int d\mathbf{r}\, \rho(\mathbf{x}, \mathbf{r}) P(\mathbf{r}), \tag{57}$$

where $\mathbf{x}$ now refers to ordinary three dimensional space.

Calculation of Eq. (54) is hard, but it is not different in kind to the calculation of partition functions in ordinary statistical mechanics. In fact, we merely have an additional temperature dependent potential that has been introduced into the statistical weight for each state. In order to reason about possible effects of nonconservative driving, we shall use mean field theory. We introduce a mean field into (53) by splitting the contributions due to the interactions. After some derivation (see Appendix D for details) one obtains that the effective bound on (54) can be found using the following equation:

$$-\beta^{-1} \log(\mathcal{Z}) \leqslant F_{\text{eq}}[\rho] + \beta \mu_1[\rho], \tag{58}$$

where we write out in full the the term on the left hand side in full rather than calling it a free energy, even though it has the same form, as terminologically free energy implies some sort of equilibrium situation, which we are not in. However, there is a contribution to the evaluation of the nonequilibrium partition function arising from the equilibrium system which does not contain any nonconservative forces $F_{\text{eq}}[\rho]$ (the mean field theory of the corresponding equilibrium or conservative system). Finding the density field ρ which minimizes the right hand side of Eq. (58) is equivalent to choosing the mean field which best satisfies all the interactions present in the exponent of Eq. (54). In order to reason about the density then, at any particular temperature we map our nonconservative system to an effectively conservative one, but where the interactions depend on the temperature. This is then the first order approximation to the full nonconservative system. Establishing a functional connection between the single particle density $\rho(\mathbf{x})$ and the other terms in the exponent $H_0(\mathbf{r}) = H_0[\rho]$, $\mu(\mathbf{r}) = \mu[\rho]$ would therefore allow us to derive densities of the first order nonconservative system by considering a simpler mean field estimate to the calculation of Eq. (54) [28].

We shall use the above to consider two kinds of nonconservative systems, which we label *internal* and *external*. By internal we mean that, if a particle is described by a set of degrees of freedom, all the nonconservative forces exist within that set, and by external we mean that the nonconservative forces act between particles, so outside just the set of its own degrees of freedom. The next sections should illuminate these two scenarios.

A. Internally imposed nonconservative field: Motile particles

Similarly to the previous section describing particles in an active bath, we can apply the same analysis to motile particles, motile meaning a system where particles are consuming fuel and transforming it to motion.

Applying our method more concretely, we take as an example a system of particles with positions $\mathbf{r} = (x, y)$ and orientation θ. Additionally, they have some motility, where the time evolution of the degrees of freedom is given by [29]

$$\gamma \dot{\mathbf{r}}(t) = -\nabla H_0(\mathbf{r}) + \boldsymbol{\xi}(t) + v_0 \hat{\boldsymbol{\theta}}, \tag{59}$$

$$\dot{\theta}(t) = \xi(t), \tag{60}$$

where $\hat{\boldsymbol{\theta}} = (\cos(\theta), \sin(\theta))$ and ξ is some white noise process. These equations correspond to particles being propelled along their axis with some parameter v_0 characterizing the swim speed. The potentials in the Hamiltonian H_0 are taken to be just hard sphere potentials. Therefore, each particle is described by three numbers (x, y, θ)

At first sight, it would appear that our theory would not be applicable to this situation. However, by thinking carefully about this system it can also be shown to fit within this framework. In fact, we merely have a collection of points in a three-dimensional space, where one of those dimensions is taken to be an orientation. The position along the orientation dimension affects the time evolution in the other dimensions, but not vice versa, establishing the self-propulsion as a nonconservative force in this three-dimensional space. Hence, this is why we refer to this as an internally imposed nonconservative field, as each particle has an extra degree of freedom affecting its evolution in the other degrees of freedom in a way which breaks Newton's third law, but only within the set of its own coordinates (x, y, θ).

From our scheme in Sec. III A, we are left with the question of separating the nonconservative and conservative contributions to the deterministic forces in this space. Fortunately, this is not complicated, and it can be observed that the two contributions for each particle are given by

$$F^{(c)}(\mathbf{r}, \theta) = (-\partial_x H_0(\mathbf{r}), -\partial_y H_0(\mathbf{r}), 0), \tag{61}$$

$$F^{(nc)}(\mathbf{r}, \theta) = (\cos(\theta), \sin(\theta), 0), \tag{62}$$

where it is clear that the divergence of the nonconservative force $F^{(nc)}(\mathbf{r})$ is zero and that the curl of the conservative part is $F^{(c)}(\mathbf{r})$ is zero, thus establishing this as a valid splitting of the forces.

We can thus go through the steps outlined in the previous section, where we now imagine a situation of N such particles all being driven in this way. We use the approximation to the full distribution given in Sec. III A where bulk properties of the system can be understood through the construction of effective pair "potentials." (See Appendix E for more details about this for this system in particular.) For the example where the H_0 is just a hard sphere potential, this will lead to a first order perturbation term of the form (for every particle pair)

$$\mu_1(\mathbf{r}_1-\mathbf{r}_2,\theta_1,\theta_2)=\Phi(\mathbf{r}_1,\mathbf{r}_2)\{(x_1-x_2)[\cos(\theta_1)-\cos(\theta_2)] +(y_1-y_2)[\sin(\theta_1)-\sin(\theta_2)]\}, \quad (63)$$

where $\Phi(\mathbf{r}_1,\mathbf{r}_2)$ is some function arising from the hard sphere potential (depending on our choice of this). Its precise form will depend on the details of the interaction, but it will have a minima at $|\mathbf{r}_1-\mathbf{r}_2|=\sigma$ where σ is the hard sphere interaction. The form of this first order interaction is displayed in Fig. 4(a). The important aspect arising from a consideration of the forces in this way is that there is now an effective attractive interaction arising between particles, solely due to the presence of the nonconservative forces. Moreover, the coupling due to the nonconservative force now explicitly includes a term that depends on the orientations of the particles θ, thus leading to the possibility of alignment effective interactions being present in the stationary distribution [30]. The physics arising from Eq. (63) and seen in Fig. 4 is not too difficult to understand. It corresponds to a situation where states where two particles are being propelled into each other are much more likely to occur that two particles propelling away from each other. In fact, to first order, the effective attraction due to particles propelled into each other is exactly balanced by the effective repulsion from particles propelling away from each other.

What happens to collection of particles interacting under the first order interaction? Were one to calculate the effective perturbation in the mean field theory arising from this first order potential for some density of particles $\rho(r,\theta)$,

$$\mu_1[\rho]\sim\int d\mathbf{r}_1 dr_2\int d\theta_1 d\theta_2\mu_1(\mathbf{r}_1,\mathbf{r}_2,\theta_1,\theta_2) \times\rho(\mathbf{r}_1,\theta_1)\rho(\mathbf{r}_2,\theta_2). \quad (64)$$

It would always be equal to zero due to the linear terms in the trigonometric functions. Thus, the form of pair potential defined here would not in itself lead to any change in density, due to the symmetry in the orientation term. (The equilibrium theory is just a hard sphere gas.) Physically, we cannot arrange particles in ways where there is any extra statistical weight associated with a particular density, as all the attractive interactions in the interaction (63) are exactly counterbalanced by the repulsive interactions. So, to first order, our observation of the density in this system would look like the equilibrium one, just a constant, flat density profile. However, the situation is rather different once we extend the argument to include interactions up to second order. The second order term, as can be seen in Fig. 4, breaks the symmetry between the situations where the particles are propelling towards each other and propelling away from each other, leading to a deeper well for the propelling-into-each-other situation. The second order term introduces both three- and four-point interactions in the statistical weight, but we here only display the two-point interaction in Fig. 4. With such an interaction, the integral in Eq. (64) is no longer zero for any density, and in fact can be minimized by concentrating density as the size of the well is deeper than the size of the peak. However, this effect is obviously weak as the potential is both (a) short ranged and

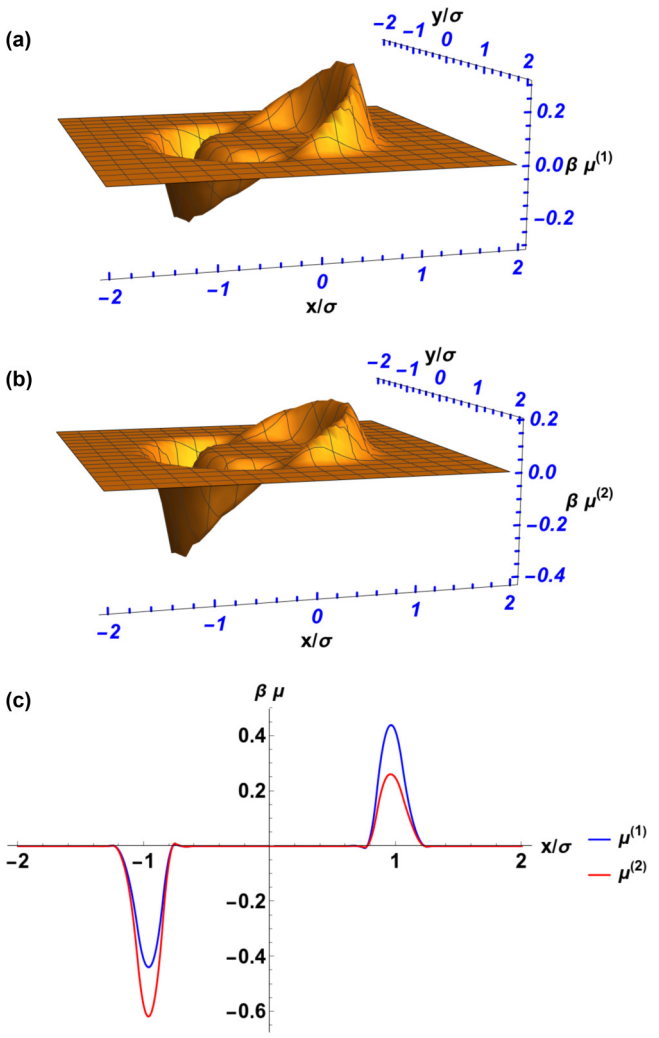

FIG. 4. Effective interactions arising between two motile particles as a function of distance. (a) The interaction in space arising from considering the first order perturbation, (b) the interaction arising from considering what happens to second order. (c) A slice along $y=0$ comparing the interactions for the first and second order perturbations. All interactions arise from a particle placed at (0,0) with an orientation of $\theta_1=\pi$ (swimming in negative x) and particle 2 having an orientation $\pi=0$ (swimming in positive x). The presence of nonconservative interactions biases the system to favor situations where the particles are propelling into one another. This leads to the introduction of an effective attractive interaction between the particles, as can be observed by the presence of wells.

(b) not very deep, the relevant quantity being the difference between the size of the well and the size of the peak. In the second order theory, this difference goes as $\beta^2 v_0^2 \gamma / D_R$, i.e., we can expect that as we increase the swim speed or decrease the rotational diffusion of the motile particle we can transition to a situation where dense phases are favored over dilute ones, thus allowing for phenomena such as motility induced phase separation. In principle, one could use this method to predict effective pair correlation functions for a system of motile particles, however, a full analysis of this system is beyond the scope of this paper and shall be left for future work. We have shown qualitatively that effects seen in active matter systems, such as motility induced phase separation [29], can also be understood through the interplay of conservative and nonconservative forces.

B. Interacting nonconservative systems

In all the previous applications we have made to a specified system, the nonconservative aspect of the problem was in some respect "nonspatial" in that what we had was some internal parameter which affected the evolution of the position degree of freedom but which was not affected explicitly by the real space position. The particles were not interacting nonconservatively in the sense that the presence of absence of particles nearby one another were not exerting nonconservative forces on each other in themselves. We here consider what would happen if instead of being internally driven, the *interactions* between particles were nonconservative in themselves. Therefore, in this section, we shall explore what occurs when this is the case, when we have a "real" nonconservative force acting between any pair of particles, not unlike particles interacting with some potential, which we call "externally driven." This is a better description of systems where a particle might exert a nonconservative force on its neighbor, representing a different class of active systems (for example, subunits that update their motion depending on their neighbors) or systems with modifiable interactions [12].

Statistical mechanics benefits from the intuitive connection we can often draw between the action of the microscopic potentials and the macroscopic structure. For example, we know that attractive forces have a tendency to concentrate density, and repulsive ones have a tendency to reduce density. Can we say something similar about a system of particles interacting with a nonconservative force? As force decomposition for interacting particles is more challenging than for internally driven particles (like in the last section), we introduce by hand such systems interacting with both conservative and nonconservative parts by introducing an extra pair force exerted on particle i by particle j which is given by a curl of some vector $\mathbf{m_i}(\mathbf{r}_{ij}) = \nabla_i \times \boldsymbol{\alpha}(\mathbf{r}_{ij})$, but stress these are not meant to represent real physical systems. In particular, by choosing a pair nonconservative force arising in this way we have chosen axes of the system. The closest physical picture that would correspond to this kind of system would be one in which a global field (such as a magnetic field) induces a constant angular momentum in the particles relative to the imposed field, and then these angular momenta interact with each other according to some effective curl, reminiscent of some experimentally realized systems [31]. In order to model real active matter systems, we would have to introduce particles with their own orientational degrees of freedom, Alternatively, nonconservative three body forces between particles could be introduced. This notwithstanding, these toy models can show us qualitatively the effect that nonconservative microscopic forces have on spatial distributions of the system, as any vector field in three dimensions (3D) can be decomposed into a gradient and curl part, this would at least approach what one may expect in reality. We are particularly interested in whether any generalities arise from the consideration of the modified Boltzmann distribution in (30).

As we mentioned, the observable quantity we are most interested in is the density of a fluid of particles interacting nonconservatively, which corresponds to the statistical average of the density operator under the stationary distribution, which we reproduce here:

$$\rho(\mathbf{x}, \mathbf{r}) = \sum_{i=1}^{N} \delta(\mathbf{x} - \mathbf{r_i}), \tag{65}$$

$$\langle \rho(\mathbf{x}) \rangle = \int d\mathbf{r}\, \rho(\mathbf{x}, \mathbf{r}) P(\mathbf{r}). \tag{66}$$

We can reason about what the effect of pairwise nonconservative driving is by considering the effective source term that appears in the modified microscopic probability $\nabla H \cdot \mathbf{m}$, where H and $\mathbf{m}$ are some functions arising from our microscopic forces (assuming we already performed the Green's function integral), and $\mathbf{m}$ is divergence free. There are several salient facts about this source term, the most important of which is that if it is equal to zero, either from the nonconservative force being zero, or the angle between the nonconservative force and the conservative force is always equal to $\pi/2$, then the stationary properties of the system will be given by a Boltzmann distribution arising from the conservative part. This source can also be described using the density operator in the same way as the equilibrium Hamiltonian, using pairwise additive curl force between particles (ignoring density correlations) and a pairwise potential between particles $U(\mathbf{r}_i - \mathbf{r}_j)$:

$$\mu_1(\mathbf{r}) \approx \sum_{i \neq j,k} \nabla_i U(\mathbf{r}_i - \mathbf{r}_j) \cdot [\nabla_i \times \boldsymbol{\alpha}(\mathbf{r}_i - \mathbf{r}_k)], \tag{67}$$

$$\mu_1(\mathbf{r})[\rho] \approx \iiint d\mathbf{x}\, d\mathbf{x}' d\mathbf{x}'' \rho(\mathbf{x}) \rho(\mathbf{x}') \rho(\mathbf{x}'') [\nabla U(\mathbf{x} - \mathbf{x}') \cdot \nabla \times \boldsymbol{\alpha}(\mathbf{x} - \mathbf{x}'')], \tag{68}$$

where now the differential operators ∇ in the second equation are with respect to ordinary three-dimensional space.

While the contribution to the partition function due to interparticle potentials is fairly unconstrained, the term corresponding to nonconservative pair forces is more interesting. A salient mathematical fact of integrals of this type is that

$$\int_{\text{all space}} d\mathbf{x}\, \nabla U(\mathbf{x}) \cdot \nabla \times \mathbf{A}(\mathbf{x}) = 0. \tag{69}$$

Therefore, the contribution to the first order theory in Eq. (58) arising from *any* nonconservative forces will always be equal to 0 when the density is constant (this was not the case

for the self-propelled particle system as the nonconservative force was not translation invariant in orientation). Taking this logic further, pairwise nonconservative interactions, if they do anything to the density, would be to drive the density profiles of the system from the homogeneous state by coupling to surfaces. For real physical systems with an orientation degree of freedom, we would have to consider higher order terms in order to capture the relevant physics, however, a similar picture would emerge for terms which depend on this source squared and so on. This interesting result arises solely from the fact that the forces are nonconservative and decomposable in terms of curls.

We can understand why this is for particular kinds of density field $\rho(\mathbf{x})$. If the source has a finite length scale and the density is approximately constant over this length scale, then the contributions to $F^{(nc)}$ from these terms are effectively zero (ignoring fluctuations). Therefore, the main contribution to Eq. (69) from nonconservative forces happens due to *surfaces*. This can be proven using the divergence theorem for a constant but finite density field. Due to all the complexities included in the derivations thus far, we wish to demonstrate this for a "real" system. Therefore, we introduce a toy model of particles interacting with pairwise nonconservative forces, via simulation. In order to test this, we simulate 10 000 particles in a box with periodic boundary conditions interacting with both an ordinary Lennard-Jones potential

$$U(r) = 4\epsilon_{\mathrm{LJ}}\left[\left(\frac{\sigma}{r}\right)^{12} - \left(\frac{\sigma}{r}\right)^{6}\right] \tag{70}$$

and where we now introduce an extra divergence-free force between the particles, again not meant to represent a real physical system, but rather as a qualitative representation for how nonconservative forces affect spatial density. We choose this force such that it obeys some simple pair conditions. We would like $\mathbf{F}(\mathbf{x}) = -\mathbf{F}(-\mathbf{x})$, so that the force exerted by particle 1 on particle 2 is the same force as exerted on particle 2 by particle 1 (though in principle this does not need to be the case for a nonconservative system). We also would like this force to be spatially limited, such that it decays quickly as the distance between any two particles increases. Additionally, the force must be divergence free for it to apply to our theory as derived thus far (but not zero). In contrast to the previous section, we do not wish to include self-propulsion, as we are interested in the minimal effect. A simple example would be a force of the following form being exerted on a particle i from a particle with j with orientation $\boldsymbol{\theta}_j$ in three dimensions:

$$\mathbf{F}_{ij}(\mathbf{r}_i - \mathbf{r}_j) = \frac{\epsilon}{\lambda}[\boldsymbol{\theta}_j \times (\mathbf{r}_i - \mathbf{r}_j)] \exp[-(\mathbf{r}_i - \mathbf{r}_j)^2/\lambda^2]$$

which obeys all the conditions above. It corresponds to a particle of exerting an additional nonconservative force on its neighbor due to its orientation. For simulation purposes, we treat this force first by (a) fixing all the particles orientation with respect to a chosen system axis, rather like imposing a strong field aligning all the particles, thus breaking the rotational symmetry of the system. Another modification is due to the fact the nonconservative force in the above Eq. (71) is at a maxima at zero, the effect of the nonconservative force does not extend far beyond the hard sphere potential defined in the Lennard-Jones potential. Thus, we shall also shift the potential

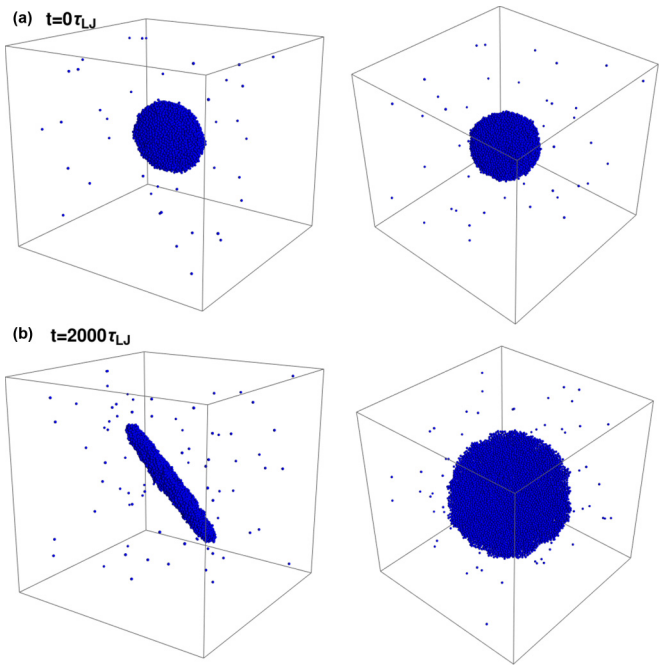

FIG. 5. Snapshots of simulations of particles interacting under a Lennard-Jones potential and a pair nonconservative force given in Eq. (71). In (a) we show different viewpoints of the start point of the simulation with a roughly spherical droplet of the particles, after relaxation under only the Lennard-Jones potential and immediately after imposition of the nonconservative force. In (b) we show the stationary droplet shape after time evolution of the microscopic equations after switching on the nonconservative forces. The particles, which would under the ordinary (equilibrium) Lennard-Jones form a spherical droplet, are instead driven away from the spherical shape through their nonconservative interactions, adopting an elliptical shape with respect to the axes where their internal degrees of freedom are fixed.

so nonconservative effects are observable by including an additional factor going as r_{ij}^2 in our force. For the following, we consider therefore the following nonconservative force:

$$\mathbf{F}_{ij}(x_{ij}, y_{ij}, z_{ij}) = \frac{\epsilon}{\lambda^2}\nabla_i \frac{1}{\lambda^2}\big(\big(y_{ij}^2, z_{ij}^2, x_{ij}^2\big) \times \exp\big[-\big(x_{ij}^2 + y_{ij}^2 + z_{ij}^2\big)/\lambda^2\big]\big). \tag{71}$$

But, we also note that under a range of different choices of this force yields qualitatively similar results.

In order to show qualitative variation from expected equilibrium density profiles, we first simulate the relaxation of the system to an equilibrium state, and then switch on the nonconservative forces. In addition, as we expect nonconservative forces to mainly affect surfaces, we keep the total particle volume fraction of the system small $\frac{N\pi\sigma^3}{6V} = 0.01$. The other parameters we choose are that $k_bT = 1, \lambda = 1.5, \sigma = 1$, $\epsilon_{\mathrm{LJ}} = 2.0,\ \epsilon = 1$.

We show a particular example of the shape adopted by this system in Fig. 5. This simple simulation already shows qualitatively the effects that we were able to reason about through a consideration of the source term. The extra term in the effective "free energy" drives the system away from its equilibrium shape (which would be a spherical droplet), lead-

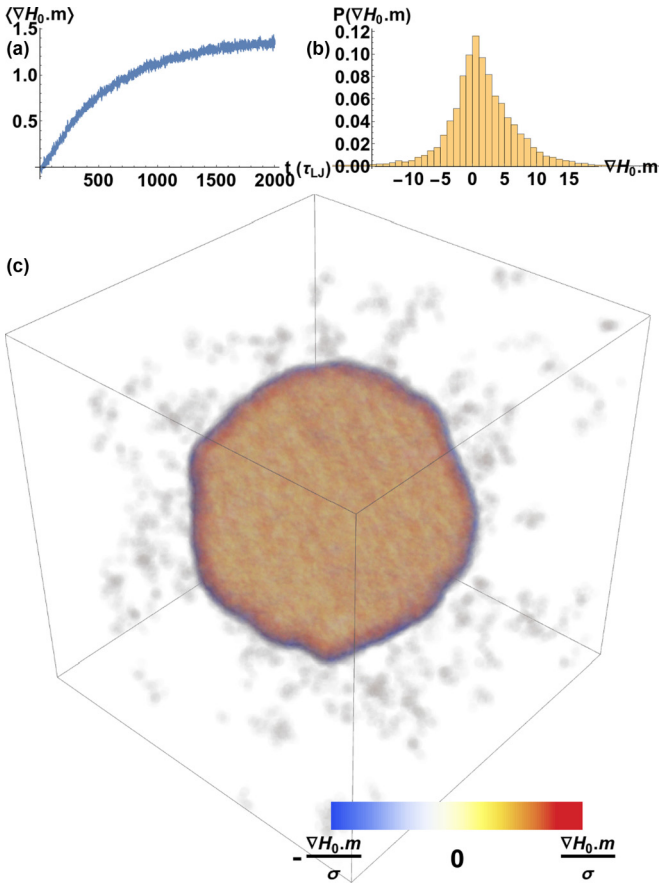

FIG. 6. Characterization of the final states of the system in terms of the effective sources we defined in the main text. (a) The average value of the source term $\nabla H_0 \cdot \mathbf{m}$ during the run of the simulation snapshots seen in Fig. 5. The total value of the source term increases with simulation time, and saturates at the pancakelike shape shown in Fig. 5. (b) The probability distribution of the source term, skewed somewhat towards positive values. (c) How the value of the source is distributed over space, with a color scheme selected from the standard deviation of the distribution. One can observe that the far outer edge of the pancake is blue, corresponding to negative values of the source term, but all the surfaces inside are slightly positive. There exists a subsurface layer that is also positive, below the outer edge.

ing to the formation of a pancakelike shape. It should be noted that while this appears to be similar to elastic deformation of a sphere, the system is heavily overdamped and the states arise from microscopic pair forces. From the preceding discussion we stated that the source term $\nabla H_0 \cdot \mathbf{m}$ would act primarily through surfaces. We can now quantify this effect by looking at the same snapshots of system evolution with the color scale of the particles equal to their total value of this source and also see how the average value of this term changes during our simulation run.

These results are presented in Fig. 6. Several trends are visible from this figure. First, the mean value of $\nabla H_0 \cdot \mathbf{m}$ per particle changes during the simulation as it moves from the spherical shape to the pancake shape. However, the distribution of $\nabla H_0 \cdot \mathbf{m}$ over all the particles is very broad compared to the mean. Moreover, these values are not distributed equally across the final state, but are instead localized in different parts of the pancake. As can be seen, there exists a strong surface layer across one axis of the pancake, and a differently valued one across another surface, according well with the mathematical arguments which we presented earlier.

This simple phenomenology and simulations show us qualitatively the effect nonconservative microscopic interactions can have on larger scale observations. We kept our discussion general up until the point of simulating the particles. Interestingly, a natural result of a framework considering the interplay of microscopic nonconservative interactions was that it will subsequently manifest itself through density inhomogeneities, and most strongly through surfaces. In particular, the imposition of given forms of microscopic driving naturally leads to particular shapes when considering aggregates of such particles, given by the interplay of the conservative and nonconservative forces. In this system, the nonconservative and conservative forces compete against each other over which shape they find preferable, settling on the pancake shown.

V. DISCUSSION

We have in the preceding displayed results relating to both the microscopic and macroscopic treatments of nonconservative forces in a system of particles. We will now proceed to discuss some broader aspects of the physical relevance of these results, in both its microscopic and macroscopic manifestations.

A. Physical intuition of microscopic perturbations to Boltzmann distributions by nonconservative vector fields

The linear system provides a useful avenue to test our intuition regarding the effects of nonconservative force. We imagine the coordinate space of the equilibrium problem as a surface embedded in three dimensions, where the x, y axes correspond to different points in coordinate space and the height in z is the energy. For a harmonic system we plot the coordinate space of the system and imagine what would happen upon imposition of a nonconservative vector field at every point on the surface. It is easy to see visually (Fig. 7) that the average effect of this vector field will be zero if the divergence is zero and the dot product of the nonconservative field with the equilibrium forces is zero. For such a nonconservative vector field, the motion due to that force in coordinate space proceeds along the equilibrium constant energy surface, and the probability along each point on that surface is not modified.

This helps to motivate our understanding of the "source" term $\nabla H_0 \cdot \mathbf{m}$, which we refer to throughout the text. In a system being driven out of equilibrium by nonconservative forces, states where the nonconservative force is balancing the equilibrium one are stabilized with respect to the equilibrium distribution. This is by itself not a particularly complex idea, though its manifestations can lead to rather complicated effects. For an active particle in a well, this simply leads to a broadening of the distribution that looks a lot like increasing temperature, but, for example, we saw that for motile particles, the addition of such a term leads to effective attractive interactions, by favoring states where particles are propelling

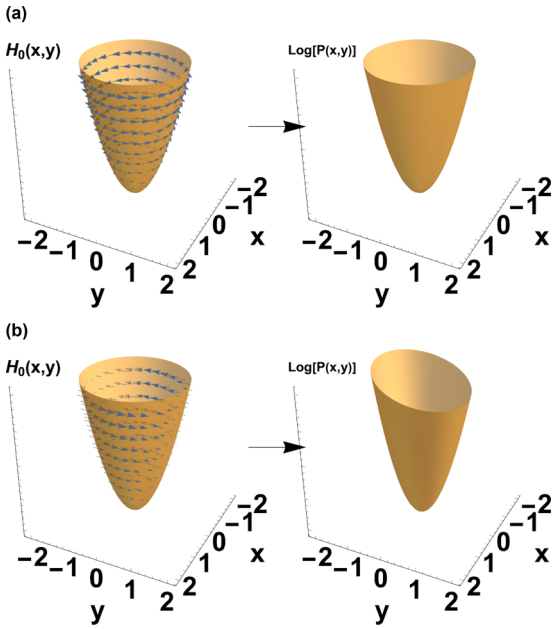

FIG. 7. Impact of imposing a nonconservative field $(-\epsilon y, x)$ on a harmonic equilibrium system (all units arbitrary). On the left hand side we display the potential energy landscape and the nonconservative vector field at each point, which leads to the probability distribution displayed on the right hand side. In (a), for $\epsilon = 1$, the probability distribution is unmodified as the nonconservative field and the equilibrium forces always move orthogonal to each other (the divergence of the nonconservative field is zero) (b) as ϵ is varied, the vector fields are no longer orthogonal, so the distribution is modified. The nonconservative vector field "pushes" against the equilibrium one, leading to modifications of the probability distribution.

into one another, which may be a signature of motility induced phase separation. When the nonconservative force is acting in space it leads to very nonequilibrium density profiles. It is clear from the models considering that nonconservative forces cannot just be considered through renormalizing temperature. Every term in the perturbative expansion depends on this source, so if it is zero the system will relax to an equilibrium distribution over states.

We also note that while this term would appear in the total dissipated power of the system, given by $\langle \sum_i F_i \frac{dx_i}{dt} \rangle$, it is not by itself the total dissipated power of the system [32]. We have in this paper framed our approach towards nonequilibrium steady states in terms of analyzing the dynamics of systems with unspecified nonconservative forces. Another approach, which we did not take in this paper, would be to analyze such systems more generically in terms of quantities such as entropy production, work, and dissipation. Some of the same concepts seem to arise from the microscopic picture, as should be expected, however, future work should relate both the thermodynamic picture and the microscopic picture presented here to each other.

B. Microscopic nonconservative forces and larger scale effects

Proceeding from microscopic consideration of nonconservative forces led to particular features when analyzing average properties such as the density. Our simple phenomenological picture led to the coupling of terms in the effective nonequilibrium "free energy" to surfaces of the density. However, in this simple picture we also ignored the effects of density fluctuations, which nonconservative forces would also affect.

Despite this, even the simplest phenomenological consideration of nonconservative forces results in the system having a preference for selecting certain density profiles, as was seen in the results of Sec. IV. We analyzed the results in terms of two classes of systems, which we referred to as internal or external nonconservative forces. In both cases, the presence of the nonconservative forces in some of the degrees of freedom was seen to be consistent with large scale differences in the density. In the former, by favoring aggregation, and in the latter case, a natural consequence of nonconservative over conservative perturbations is that nonconservative perturbations couple to surfaces (inhomogeneities) rather than to bulks. This is the main difference to a conservative perturbation, which would modify effective free energies in the bulk phase as well.

For the case where the nonconservative driving is internal to the particles, where we reiterate that the meaning of internal here is that when each particle is described by a set of coordinates $(x, y, \ldots)$, that the nonconservative forces exist *solely* within these degrees of freedom, and interactions between particles can be described using a potential. This is an apt descriptor of active systems such as particles in an active bath [internal coordinates (x, A)] or of active Brownian particles [coordinates (x, y, θ)]. In these internally driven systems, which usually mean the particles have some effective motility, we favor states where the motility is maximally working against the conservative forces. For a hard sphere interaction it is not to difficult to see what this means in practice, as the work will be maximized when the hard spheres are being pushed into one another. In principle, this reasoning could be applied to any conservative interactions though, leading to more complex behaviors. This case is slightly modified in the case of external (interactions) nonconservative forces. In the example we have chosen, the conservative (equilibrium) interactions would lead to the formation of a spherical droplet. The introduced pairwise nonconservative interactions, which favor surfaces, lead to the formation of a pancake phase. The interplay of bulk favoring conservative interactions and surface favoring nonconservative interactions could in principle lead to many unique and interesting steady states. We have here shown only one example, but the principle perhaps could be extended further to understand interactions among multiple species. Moreover, the way in which complex equilibrium interactions supplemented with microscopic driving between components could manifest themselves macroscopically could be suggestive for analysis of biological systems. In principle, this defines a route to attempt to understand the formation of complex nonequilibrium patterns from a consideration of microscopic forces. Intriguingly, the fact that nonconservative forces couple strongly to density surfaces suggests that complicated systems, with many species, where there are a lot of effective "surfaces" may behave in ways that are very different to single component fluids which are also being driven out of equilibrium microscopically.

C. Active matter and force decomposition

In this paper, instead of solving for a particular active matter system, we have presented results in terms of the interplay between conservative and nonconservative force fields, and then mapped those results onto real systems. This approach is not generally standard in current treatments of active matter. However, we argue that such analysis is not without merit. Particular models of active matter may display the features we have discussed here. We created nonconservative systems by construction in Sec. IV. For a real system, we would instead proceed by writing the realistic forces, and then we could attempt to decompose the forces mathematically into their conservative and nonconservative parts. While this may sound complicated, the fact that in most realistic systems forces are pairwise between particles would reduce the complexity of the problem massively and leave it amenable to numerical fitting techniques if mathematical solutions are impossible. We would expect such systems to qualitatively obey the distributions we described here. However, an extension of the currently presented theory would have to be derived in order to describe systems coupled to different temperature baths. In this way we treat the problem of finding the steady state distributions of active matter systems as the statistical mechanics of nonconservative vector fields, and consider equilibrium as the statistical mechanics of conservative vector fields.

In many other systems, the nonconservative forces exist in degrees of freedom that are not necessarily spatial, for example, internal degrees of freedom such as orientation may apply effective forces in space. We analyzed two such examples in this paper (active noise driving trapped colloids and motile particles). In both of these cases, the nonconservative nature exists in the presence of an extra variable carried by each particle which affects its evolution in time, such as memory of noise or orientation. In such systems, the decomposition into conservative and nonconservative parts of the force is trivial due to the fact that particles themselves are not coupled to each other nonconservatively. This makes these systems particularly attractive for studying stationary states of many particles with nonconservative forces.

VI. CONCLUSION AND FUTURE DIRECTIONS

In this paper, we have analyzed the steady state distribution that arises from particles interacting with both conservative and nonconservative forces. We have shown that the Boltzmann distribution is modified by the addition of excess convolution integrals over effective source terms that arise when treating the system perturbatively in the nonconservative elements. We have shown results for two different classes of nonconservative system, one of which there is some process leading to additional translation, and another where there are nonconservative interactions. We give some indication for how nonconservative forces in space affect more macroscopic observables, and find that generic nonconservative forces in space couple to density surfaces rather than volume terms. The consequences of this are that microscopic driving implemented between pairs of particles leads to the adoption of different density profiles and shapes than would be expected from the same system with only the conservative interactions. However, the fact that such deformations arise can be understood from the source terms we derived in the microscopic theory, where differences from Boltzmann distributions arise due to operators acting on $\nabla H_0(\mathbf{r}) \cdot \mathbf{m}(\mathbf{r})$, and this perturbation, when course grained, naturally couples to surfaces. We believe the framework we have presented so far is suggestive of future directions.

On the mathematical level, the full microscopic probability distribution is very complicated. The probability distribution due to nonconservative forces can be written as a series of integrals over source terms with Green's functions, where the Green's function is some equilibrium path integral. Nevertheless, certain simplifications exist that make the problem tractable for large numbers of particles.

While in this paper we have mainly focused on the static distributions of matter being driven by nonconservative forces, a whole other aspect that was ignored in this work was the dynamics. The steady states arising from these forms of nonequilibrium systems may still have net currents acting. How these steady states then behave when perturbed by an external field, their response properties and dynamic relaxation are all ongoing topics of research [33]. An additional assumption we employed was that the system was overdamped. Upon relaxation of this assumption, many other interesting phenomena may result.

In the latter sections, we noted that nonconservative systems adopt particular shapes compared to conservative ones. We believe this is an interesting avenue of potential exploration. In particular, one may be able to relate particular mesoscopic patterns or shapes to the interplay of microscopic conservative and nonconservative forces. Additionally, one may be able to reformulate this theory in terms of conservative and nonconservative stress tensors [34].

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APPENDIX A: APPROXIMATE SOLUTION FOR GREEN'S FUNCTION

The Green's function equation is given by

$$(\nabla^2 - \beta \nabla H_0 \cdot \nabla) G(\mathbf{r}, \mathbf{r}') = \delta(\mathbf{r} - \mathbf{r}'). \tag{A1}$$

We supplement this with another different equation for a vector $\mathbf{g}(\mathbf{r}, \mathbf{r}')$:

$$(\nabla - \beta \nabla H_0) \cdot \mathbf{g}(\mathbf{r}, \mathbf{r}') = \delta(\mathbf{r} - \mathbf{r}'). \tag{A2}$$

It is clear that $\mathbf{g}(\mathbf{r}, \mathbf{r}') = \exp[\beta H_0(\mathbf{r}) - \beta H_0(\mathbf{r}')] \nabla G_0(\mathbf{r}, \mathbf{r}')$ solves Eq. (A2) where G_0 is the Green's function in the absence of any Hamiltonian $\nabla \cdot \nabla G_0(\mathbf{r}, \mathbf{r}') = \delta(\mathbf{r} - \mathbf{r}')$.

However, this is not the Green's function as there is no guarantee that $\mathbf{g}$ is expressible as a gradient. An approximate solution involves just taking the gradient part of $\mathbf{g}(\mathbf{r}, \mathbf{r}')$ which leads to the approximate solution in the main text:

$$G(\mathbf{r}, \mathbf{r}') = \int d\mathbf{r}'' e^{-[\beta H_0(\mathbf{r}') - \beta H_0(\mathbf{r}'')]} \frac{(\mathbf{r}'' - \mathbf{r}') \cdot (\mathbf{r} - \mathbf{r}'')}{A_N |\mathbf{r}'' - \mathbf{r}'|^N |\mathbf{r} - \mathbf{r}''|^N}, \tag{A3}$$

which becomes exact if the divergence free part of $\mathbf{g}(\mathbf{r}, \mathbf{r}')$ is equal to zero.

APPENDIX B: CORRELATIONS IN GREEN'S FUNCTION

In the main text we were faced with the problem of calculating the perturbations due to a source term which is given as some pairwise sum over all the particles in the system. In this Appendix we shall demonstrate that this approximation becomes exact for large systems. In order to demonstrate this, we shall use the series form of the Green's function. We wish to calculate

$$[\nabla^2 - \beta \nabla H_0(\mathbf{r}) \cdot \nabla]\mu_n(\mathbf{r}) = \sum_{i \neq j} \rho_{Su}(\mathbf{r}_i, \mathbf{r}_j), \tag{B1}$$

which is the generic form of the perturbative expansion of any order n. All the sums are taken to go over all the particles in the system, which is of size N, corresponding to N particles. We restrict the right hand side to be pairwise in two of the variables of the system, though in principle the following argument could be applied to any subset size. The form of ρ_S is kept general, however, we can additionally employ the assumption that it is translationally invariant $\rho_S(\mathbf{r}_i, \mathbf{r}_j) = \rho_S(\mathbf{r}_i - \mathbf{r}_j)$.

This equation can be formally solved as a series solution in the inverse temperature β, which is how the Green's function in the main text is solved. This leads to an expansion of the form

$$\mu_n(\mathbf{r}) = \sum_m \beta^m \mu_n^{(m)}(\mathbf{r}), \tag{B2}$$

where each term $m > 0$ is given as the solution to

$$\nabla^2 \mu_n^{(m)}(\mathbf{r}) = \nabla H_0(\mathbf{r}) \cdot \nabla \mu_n^{(m-1)}(\mathbf{r}) \tag{B3}$$

with the condition for $m = 0$ being given by

$$\nabla^2 \mu_n^{(0)}(\mathbf{r}) = \sum_{i \neq j} \rho_{Su}(\mathbf{r}_i, \mathbf{r}_j). \tag{B4}$$

For equations of the form (B4), the integral over the Green's function kernels are exactly decomposable into the constituent parts in the sum. This is perhaps easiest seen in the Fourier representation. For a generic system of dimension N but with a source term that depends only on a few variables, the solution (in Fourier space) is given by

$$\hat{\mu}_n^{(0)}(\mathbf{k}) = \int d\mathbf{r} \, \exp(i\mathbf{k} \cdot \mathbf{r}) \frac{1}{|\mathbf{k}|^2} \rho_{Su}(\mathbf{k}), \tag{B5}$$

where we have used that the Green's function for the Laplacian in Fourier space is given by $1/|\mathbf{k}|^2$. For the case that the source term only depends on $\mathbf{r}_1, \mathbf{r}_2$ then the Fourier transform source term $\rho_{Su}(\mathbf{k})$ is given by

$$\rho_{Su}(\mathbf{k}) = \rho_{Su}(\mathbf{k}_1, \mathbf{k}_2)\delta(\mathbf{k}_3)\delta(\mathbf{k}_4)\ldots\delta(\mathbf{k}_N). \tag{B6}$$

We can substitute this into Eq. (B5), where the integrals over all the delta functions can be easily performed, leaving

$$\hat{\mu}_n^{(0)}(\mathbf{k}) = \int d\mathbf{k}_1 \mathbf{k}_2 \exp[i(\mathbf{k}_1, \mathbf{k}_2) \cdot (\mathbf{r}_1, \mathbf{r}_2)] \times \frac{1}{|\mathbf{k}_1|^2 + |\mathbf{k}_2|^2} \rho_{Su}(\mathbf{k}_1, \mathbf{k}_2), \tag{B7}$$

where we can identity the new integral as just the Green's function in reduced dimension. As the Green's function is a linear operator, it can be applied independently to each term in the sum. Following on from this, the solution to the $m = 0$ equation can then be seen to be equal to

$$\mu_n^{(0)}(\mathbf{r}) = \sum_{i \neq j} \Psi(\mathbf{r}_i - \mathbf{r}_j), \tag{B8}$$

where Ψ is just the solution of the reduced dimension problem, where we introduce again the translational symmetry of the problem. What happens when we then try to work out the $m = 1$ term arising from this $m = 0$ term? In the following, we assume that we are working with a pairwise Hamiltonian, such that

$$\nabla H_0(\mathbf{r}) = (\mathbf{F}_1, \mathbf{F}_2, \ldots, \mathbf{F}_N) \tag{B9}$$

$$= \left(\sum_{k \neq 1} \mathbf{f}(\mathbf{r}_1 - \mathbf{r}_k), \sum_{k \neq 2} \mathbf{f}(\mathbf{r}_2 - \mathbf{r}_k), \ldots, \sum_{k \neq N} \mathbf{f}(\mathbf{r}_N - \mathbf{r}_k) \right), \tag{B10}$$

where for any real physical system, the pairwise forces are antisymmetric in the arguments $\mathbf{f}(\mathbf{r}_i) = -\mathbf{f}(-\mathbf{r}_i)$. The function $\mathbf{f}$ is the microscopic forces in the system. Applying the Hamiltonian in Eq. (B9) to the solution for $m = 0$ to obtain the source term for the $m = 1$ perturbation $\nabla H_0(\mathbf{r}) \cdot \nabla \mu_n^{(0)}(\mathbf{r})$ gives us the following equation:

$$\nabla^2 \mu_n^{(1)}(\mathbf{r}) = \nabla H_0(\mathbf{r}) \cdot \nabla \mu_n^{(0)}(\mathbf{r}) \tag{B11}$$

$$= \sum_{i \neq j} [f(\mathbf{r}_i - \mathbf{r}_j) \cdot \nabla_i + \mathbf{f}(\mathbf{r}_i - \mathbf{r}_j) \cdot \nabla_j] \Psi(\mathbf{r}_i - \mathbf{r}_j) + \sum_{i \neq j} \left(\sum_{k \neq i,j} \mathbf{f}(\mathbf{r}_i - \mathbf{r}_k) \cdot \nabla_i + \sum_{k \neq i,j} \mathbf{f}(\mathbf{r}_j - \mathbf{r}_k) \cdot \nabla_j \right) \Psi(\mathbf{r}_i - \mathbf{r}_j). \tag{B12}$$

We see that this source introduces new three point correlation functions, depending on all the positions i, j, and k. However, closer analysis of this equation yields a very useful result. Employing the fact that Ψ are even functions in the arguments, it can be seen that the first sum in the source term expression is summing over symmetric functions, and the second term

is summing over antisymmetric functions (as microscopic force functions are antisymmetric). Therefore, in the limit of $N \to \infty$ the first sum will dominate over the second sum. However, the first term is just a sum of pairwise expressions again, thus, the same analysis we used for the $m = 0$ theory can be used again, and we get that the $m = 1$ term of the solution is pairwise. Moreover, as the $m = 2, 3$, etc., terms also follow the same pattern, it can be shown that we can get similar results for any term of order m. Thus, in the limit $N \to \infty$ we feel justified in assuming that the solution μ is also pairwise. This result means that when we consider a source term in some subset of variables in the system, we can treat it by solving it in that subset rather than solving the differential equation over all N variables in the system. This dramatically reduces the difficulty of problems we encounter as we now only need to solve a two- or three-dimensional partial differential equations (doable) rather than N-dimensional partial differential equations (not doable).

We can interrogate this assumption as a function of N by reinserting it into the full equation. For example, in the case that an operator equation is given by

$$\mathcal{L}\mu(\mathbf{x}, \mathbf{X}) = [\mathcal{L}(\mathbf{x}) + \mathcal{L}(\mathbf{X})]\mu(\mathbf{x}, \mathbf{X}) = \rho(\mathbf{x}), \tag{B13}$$

we have introduced the generic operator $\mathcal{L}$, where $\mathbf{x}$ and $\mathbf{X}$ refer to different degrees of freedom. In this case, the solutions we posited can be seen to be (for a linear operator $\mathcal{L}$)

$$\mu(\mathbf{x}) = \mathcal{L}^{-1}(\mathbf{x})\rho(x) \tag{B14}$$

if the action of the operator $\mathcal{L}(X)$ on this function is zero, this will be a solution to this equation.

Returning again to the fact that our total equation is given by

$$\mathcal{L}\mu(\mathbf{r}) = \rho(\mathbf{r}), \tag{B15}$$

where $\mathbf{r} = (\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_N)$ and now the operator is equal to $\mathcal{L} = \nabla^2 - \beta\nabla H_0 \cdot \nabla$. Again, assume that the density is usually in terms of sums over terms involving only a few of the total degrees of freedom of the system:

$$\rho(\mathbf{r}) = \sum_{i,j} \rho_{Su}(\mathbf{r}_i, \mathbf{r}_j). \tag{B16}$$

The operator $\mathcal{L}$ is a linear operator, and so is its inverse (the Green's function), therefore, the full solution $\mu(r)$ is given by

$$\mu(\mathbf{r}) = \mathcal{L}^{-1} \sum_{i,j} \rho_{Su}(\mathbf{r}_i, \mathbf{r}_j), \tag{B17}$$

$$\mu(\mathbf{r}) = \sum_{i,j} \mathcal{L}^{-1} \rho_{Su}(\mathbf{r}_i, \mathbf{r}_j), \tag{B18}$$

where the second equation arises due to the linearity of the operator. $\mathcal{L}^{-1}\rho_{Su}(\mathbf{r}_i, \mathbf{r}_j)$ is merely the solution to the full equation where the only source is $\rho_{Su}(\mathbf{r}_i, \mathbf{r}_j)$

In order to progress with this we represent the full solution to an equation with a subset source as

$$(\mathcal{L}_{i,j} + \mathcal{L}_{\text{corr}} + \mathcal{L}_R)\mu_{i,j}(\mathbf{r}) = \rho_{Su}(\mathbf{r}_i, \mathbf{r}_j), \tag{B19}$$

which arises from splitting the contributions of our full operator into the terms that depend on $(\mathbf{r}_i, \mathbf{r}_j)$, $\mathcal{L}_{i,j}$, all the other terms with no dependence on $\mathbf{r}_i$ and $\mathbf{r}_j$, $\mathcal{L}_{\mathcal{R}}$ and all the cross terms are in $\mathcal{L}_{\text{corr}}$. For our Green's function system each of these operators can be written out in full as

$$\mathcal{L}_{i,j} = \nabla_i^2 + \nabla_j^2 + \mathbf{f}(\mathbf{r}_i - \mathbf{r}_j) \cdot \nabla_i + \mathbf{f}(\mathbf{r}_j - \mathbf{r}_i) \cdot \nabla_j, \tag{B20}$$

$$\mathcal{L}_R^{(i,j)} = \sum_{k \neq (i,j)} \nabla_k^2 + \sum_{k \neq (i,j)} \sum_{l \neq (i,j,k)} \mathbf{f}(\mathbf{r}_k - \mathbf{r}_l) \cdot \nabla_k, \tag{B21}$$

$$\mathcal{L}_{\text{corr}}^{(i,j)} = \sum_{k \neq (i,j)} \mathbf{f}(\mathbf{r}_i - \mathbf{r}_k) \cdot \nabla_k + \sum_{k \neq (i,j)} \mathbf{f}(\mathbf{r}_j - \mathbf{r}_k) \cdot \nabla_k. \tag{B22}$$

It is clear that in the absence of correlation $\mathcal{L}_{\text{corr}} = 0$, the solution will be given by

$$\mu_{i,j}^{(0)}(r) = \mathcal{L}_{i,j}^{-1} \rho_{Su}(\mathbf{r}_i, \mathbf{r}_j) \tag{B23}$$

as the action of the operator $\mathcal{L}_R^{(i,j)}$ on this function is equal to zero. We can thus interrogate the full solution by setting every term in the series as being equal to sum over this result:

$$\mu_{\text{app}}(r) = \sum_{i \neq j} \mu_{i,j}^{(0)}(\mathbf{r}). \tag{B24}$$

The action of the operator $\mathcal{L}$ on this approximate solution is given by

$$\mathcal{L} \sum_{i \neq j} \mu_{i,j}^{(0)}(\mathbf{r}) = \sum_{i \neq j} \rho_{Su}(\mathbf{r}_i, \mathbf{r}_j) + \sum_{i \neq j} \mathcal{L}_{\text{corr}}^{(i,j)} \mu_{i,j}^{(0)}(\mathbf{r}). \tag{B25}$$

The action of the correlation operator on $\mu_{i,j}^{(0)}$ is zero if it is being acted on by the derivative which does not contain either i or j, therefore, we are left with the following:

$$\sum_{i \neq j} \mathcal{L}_{\text{corr}}^{(i,j)} \mu_{i,j}^{(0)}(\mathbf{r}) = \sum_{i \neq j} \left(\sum_k \mathbf{f}(\mathbf{r}_i - \mathbf{r}_k) \cdot \nabla_i + \sum_k \mathbf{f}(\mathbf{r}_j - \mathbf{r}_k) \cdot \nabla_j \right) \mu_{i,j}. \tag{B26}$$

The relative error between the action of the operator on the real solution and the approximate one goes as

$$\frac{|\mathcal{L}\mu_{\text{app}} - \mathcal{L}\mu|}{|\mathcal{L}\mu|} = \frac{\sum_{i \neq j} \left(\sum_k \mathbf{f}(\mathbf{r}_i - \mathbf{r}_k) \cdot \nabla_i + \sum_k \mathbf{f}(\mathbf{r}_j - \mathbf{r}_k) \cdot \nabla_j \right) \mathcal{L}_{i,j}^{-1} \rho_{Su}(\mathbf{r}_i, \mathbf{r}_j)}{\sum_{i \neq j} \rho_{Su}(\mathbf{r}_i, \mathbf{r}_j)} \tag{B27}$$

which decays rapidly as a function of N again due to the antisymmetry of the forces. The solution scales with N in the same way as the operator acting on the solution does.

APPENDIX C: SOLUTION OF SMOLUCHOWSKI EQUATION FOR LINEAR SYSTEMS

Starting from the stationarity condition:

$$\nabla \cdot [\exp(-\phi(\mathbf{r})][-\nabla H_0 + \mathbf{m} + k_b T \nabla \phi)] = 0 \tag{C1}$$

and introducing a new vector $\mathbf{L} = -\nabla H_0 + \mathbf{m} + k_b T \nabla \phi$, leading to $\mathbf{L} - \mathbf{F} = k_b T \nabla \phi$ where $\mathbf{F}$ is the vector of all the forces in the system we obtain the condition that

$$k_b T \nabla \cdot \mathbf{L} - \mathbf{L}.(\mathbf{L} - \mathbf{F}) = 0 \tag{C2}$$

with the additional condition that $\mathbf{L} - \mathbf{F}$ is conservative. Now, we say that $\mathbf{F} = \underline{A}\mathbf{x}$ where $\underline{A}$ is a matrix which does not depend on position and searching for solutions $\mathbf{L} = \underline{B}\mathbf{x}$ we have the following conditions that must be satisfied in order for Eq. (C1) to hold:

$$\mathrm{Tr}(\underline{B}) = 0, \tag{C3}$$

$$\mathbf{x}^T(\underline{B}^T\underline{B} - \underline{B}^T\underline{A})\mathbf{x} = 0, \tag{C4}$$

$$(\underline{A} - \underline{B})^T = \underline{A} - \underline{B}. \tag{C5}$$

Condition (C4) is true if the combined matrix is antisymmetric. Upon decomposition of the two matrices $\underline{A}, \underline{B}$ into their symmetric and antisymmetric components

$$\underline{A} = \underline{S} + \underline{Q}, \tag{C6}$$

$$\underline{B} = \underline{\Sigma} + \underline{\Theta}, \tag{C7}$$

we can find the elements of matrix B as a function of the given matrix A, where condition (C5) immediately leads to $\underline{\Theta} = \underline{Q}$, and we have upon expansion the following equation for $\underline{\Sigma}$:

$$2\underline{\Sigma}^2 - \underline{\Sigma} \cdot (\underline{S} + \underline{Q}) - (\underline{S} + \underline{Q})^T \cdot \underline{\Sigma} + [\underline{Q}, \underline{S}] = 0, \tag{C8}$$

where the square brackets denote a commutator. This equation is known as the continuous time algebraic Riccati equation and has well developed methods of solution, for example, by eigendecomposition of a larger system. We omit full details of how this equation is solved and let the reader consult [35]. Any matrix Σ that solves this equation in addition to being symmetric and traceless will be a solution to the Smoluchowski equation. We can then compare these solutions to the perturbative expansion and see how well it describes the true solutions to the equations.

APPENDIX D: MEAN FIELD THEORY FOR FIRST ORDER NONCONSERVATIVE SYSTEMS

We show here that the Gibbs-Bogoliubov inequality as applied to nonconservative systems still produces an effective mean field description. The probability distribution for the first order nonconservative system is given by

$$P(\mathbf{r}) = \frac{1}{\mathcal{Z}} \exp[-\beta H_0(\mathbf{r}) - \beta^2 \mu_1(\mathbf{r})]. \tag{D1}$$

The statistical weight (the factor inside the exponential) of each state goes as

$$W(\mathbf{r}) = H_0(\mathbf{r}) + \beta \mu_1(\mathbf{r}). \tag{D2}$$

We introduce an external field acting on all the particles:

$$V(\mathbf{r}) = \sum_i V_{\mathrm{ext}}(\mathbf{r_i}). \tag{D3}$$

We then consider the total statistical weight to be made up of two parts:

$$W(\mathbf{r}) = W_1(\mathbf{r}) + W_2(\mathbf{r}), \tag{D4}$$

where $W_1(\mathbf{r}) = V(\mathbf{r})$ and $W_2(\mathbf{r}) = H_0(\mathbf{r}) + \beta\mu(\mathbf{r}) - V(\mathbf{r})$. The partition function is given by

$$\mathcal{Z} = \int d\mathbf{r} \, \exp(-\beta[W_1(\mathbf{r}) + W_2(\mathbf{r})]). \tag{D5}$$

Dividing through by $\mathcal{Z}_1 = \int d\mathbf{r} \, \exp[-\beta W_1(\mathbf{r})]$ leads to

$$\frac{\mathcal{Z}}{\mathcal{Z}_1} = \frac{1}{\mathcal{Z}_1} \int d\mathbf{r} \, \exp(-\beta[W_1(\mathbf{r}) + W_2(\mathbf{r})]). \tag{D6}$$

The term on the right hand side is just the average of $\exp(-\beta W_2)$ in the system with statistical weight W_1:

$$\frac{\mathcal{Z}}{\mathcal{Z}_1} = \langle \exp[-\beta W_2(\mathbf{r})] \rangle_1. \tag{D7}$$

Using the property of exponentials $\langle \exp(x) \rangle \geqslant \exp(\langle x \rangle)$ we obtain

$$-\beta^{-1} \log(\mathcal{Z}) \leqslant -\beta^{-1} \log(\mathcal{Z}_0) + \langle W_2(\mathbf{r}) \rangle_1. \tag{D8}$$

The terms depending explicitly on $V(\mathbf{r})$ in the above equation cancel with each other, and due to the additive nature of the contributions in W_2 (i.e., when the nonconservative forces are zero we just have the theory for the equilibrium system) we are left with

$$-\beta^{-1} \log(\mathcal{Z}) \leqslant F_0[\rho] + \beta \mu_1[\rho], \tag{D9}$$

where we make explicit that this is a functional of the density $[\rho]$. F_0 contains the contributions due to the equilibrium system, and μ_1 is the perturbation. Finding the density field ρ which minimizes the right hand side of Eq. (D9) is equivalent to finding the mean field which best captures all the interactions in $W_2((\mathbf{r}))$, so long as we are able to establish a functional connection between the single particle density and the interactions in the system.

APPENDIX E: SOLUTION OF PERTURBATION EQUATION FOR SELF-PROPELLED PARTICLES

We start from the factorized form of the perturbation equation

$$\nabla \cdot [e^{-\beta H_0(r)} \nabla \mu_1(r)] = e^{-\beta H_0(r)} \rho_S(r), \tag{E1}$$

where the interaction between any two particles is given by a hard sphere repulsion

$$\phi_{HS}(r) = \begin{cases} 4\epsilon_{\mathrm{LJ}}\left(\left(\frac{\sigma}{r}\right)^{12} - \left(\frac{\sigma}{r}\right)^{6}\right) + \epsilon & \text{if } r < 2^{1/6}\sigma, \\ 0 & \text{otherwise} \end{cases} \tag{E2}$$

with the total equilibrium Hamiltonian being given by

$$H_0 = \sum_{i \neq j} \phi_{HS}(r_{ij}), \tag{E3}$$

where r_{ij} is the radial distance between particles i and j. Each particle has three numbers identifying its state: two positions

and an orientation (x, y, θ). Therefore, the conservative force between any two particles i and j in these coordinates will be given by

$$\nabla_i H_0(r_{ij}) = (x_i - x_j, y_i - y_j, 0)\phi'(r_{ij}); \tag{E4}$$

there is no force acting on the orientation degree of freedom. Between two particles, we can extend this vector to encompass all six coordinates:

$$\nabla H_0(\mathbf{r}_1, \mathbf{r}_2) = (x_i - x_j, y_i - y_j, 0, x_j - x_i, y_j - y_i, 0)\phi'(r_{ij}). \tag{E5}$$

The nonconservative forces in this space are given by

$$\mathbf{F}^{(nc)}(\mathbf{r}_1, \mathbf{r}_2) = v_0(\cos(\theta_1), \sin(\theta_1), 0, \cos(\theta_2), \sin(\theta_2), 0). \tag{E6}$$

The dot product of these two is then the source term entering the right of Eq. (E1):

$$\begin{aligned}\rho_S(x_1, x_2, y_1, y_2, \theta_1, \theta_2) &= (x_i - x_j, y_i - y_j, 0, x_j - x_i, y_j - y_i, 0)\phi'(r_{ij}) \\ &\cdot (\cos(\theta_1), \sin(\theta_1), 0, \cos(\theta_2), \sin(\theta_2), 0),\end{aligned} \tag{E7}$$

where as the expansion is being performed in v_0 (per the main text), we suppress its appearance in the above.

To make progress, we assume that the exponential of the hard sphere term is approximately a Heaviside function

$$e^{-\beta\phi_{HS}(r)} = 0 \text{ if } r < \sigma, \tag{E8}$$

i.e., we treat the hard sphere energy as an effective boundary condition in the solution to Eq. (E1). Expanding out Eq. (E1) leads to

$$\begin{aligned}&\left(\nabla^2_{x_1,y_1,x_2,y_2} + \frac{\gamma}{D_R}\nabla^2_{\theta_1,\theta_2}\right)\mu(x_1, y_1, x_2, y_2, \theta_1, \theta_2) \\ &\quad = \rho_S(x_1, x_2, y_1, y_2, \theta_1, \theta_2)\end{aligned} \tag{E9}$$

with the boundary conditions that $\mu(|r_i - r_j| = \sigma) = 0$, $\mu(\infty) = 0$. We have additionally included the approximation that as there is translational symmetry in x, y, we can set one of these positions to zero and solve for the other. The orientation dependent coordinates can be solved by using a Fourier series expansion for Eq. (E9) in θ_1 and θ_2. Once the solution has been obtained for one perturbation, one can then continually reapply it to get the solution to any order for the pairwise interactions.

[1] M. C. Marchetti, J. F. Joanny, S. Ramaswamy, T. B. Liverpool, J. Prost, Madan Rao, and R. Aditi Simha, Hydrodynamics of soft active matter, Rev. Mod. Phys. **85**, 1143 (2013).

[2] D. Needleman and Z. Dogic, Active matter at the interface between materials science and cell biology, Nat. Rev. Mater. **2**, 17048 (2017).

[3] É. Fodor and M. C. Marchetti, The statistical physics of active matter: From self-catalytic colloids to living cells, Phys. A (Amsterdam) **504**, 106 (2018).

[4] M. E. Cates, Active field theories, arXiv:1904.01330 [cond-mat.stat-mech].

[5] A. Bricard, J.-B. Caussin, D. Das, C. Savoie, V. Chikkadi, K. Shitara, O. Chepizhko, F. Peruani, D. Saintillan, and D. Bartolo, Emergent vortices in populations of colloidal rollers, Nat. Commun. **6**, 7470 (2015).

[6] S. C. Takatori and J. F. Brady, Towards a thermodynamics of active matter, Phys. Rev. E **91**, 032117 (2015).

[7] U. M. B. Marconi and C. Maggi, Towards a statistical mechanical theory of active fluids, Soft Matter **11**, 8768 (2015).

[8] T. Speck, Stochastic thermodynamics for active matter, Europhys. Lett. **114**, 30006 (2016).

[9] X.-J. Zhang, H. Qian, and M. Qian, Stochastic theory of nonequilibrium steady states and its applications. part i, Phys. Rep. **510**, 1 (2012).

[10] C. Battle, C. P. Broedersz, N. Fakhri, V.F. Geyer, J. Howard, C. F. Schmidt, and F. C. MacKintosh, Broken detailed balance at mesoscopic scales in active biological systems, Science **352**, 604 (2016).

[11] L. Dabelow, S. Bo, and R. Eichhorn, Irreversibility in Active Matter Systems: Fluctuation Theorem And Mutual Information, Phys. Rev. X **9**, 021009 (2019).

[12] D. Osmanović and Y. Rabin, Chemically active nanodroplets in a multi-component fluid, Soft Matter **15**, 5965 (2019).

[13] H. W. Moyses, R. O. Bauer, A. Y. Grosberg, and D. G. Grier, Perturbative theory for brownian vortices, Phys. Rev. E **91**, 062144 (2015).

[14] J. D. Noh and J. Lee, On the steady-state probability distribution of nonequilibrium stochastic systems, J. Korean Phys. Soc. **66**, 544 (2015).

[15] E. M. Bertin, Theoretical approaches to the steady-state statistical physics of interacting dissipative units, J. Phys. A: Math. Theor. **50**, 083001 (2017).

[16] R. S. Wedemann, A. R. Plastino, and C. Tsallis, Curl forces and the nonlinear fokker-planck equation, Phys. Rev. E **94**, 062105 (2016).

[17] M. Mangeat, Y. Amarouchene, Y. Louyer, T. Guérin, and D. S. Dean, Role of nonconservative scattering forces and damping on brownian particles in optical traps, Phys. Rev. E **99**, 052107 (2019).

[18] D. J. Griffiths, The Helmholtz theorem, in *Introduction to Electrodynamics* (Cambridge University Press, Cambridge, 2017), pp. 582–584.

[19] E. Deriaz and V. Perrier, Orthogonal Helmholtz decomposition in arbitrary dimension using divergence-free and curl-free wavelets, Appl. Comput. Harmon. Anal. **26**, 249 (2009).

[20] L. Ferrari, Particles dispersed in a dilute gas. II. from the langevin equation to a more general kinetic approach, Chem. Phys. **428**, 144 (2014).

[21] M. Hohmann, F. Kindermann, T. Lausch, D. Mayer, F. Schmidt, E. Lutz, and A. Widera, Individual Tracer Atoms in an Ultracold Dilute Gas, Phys. Rev. Lett. **118**, 263401 (2017).

[22] H. Risken, *Fokker-Planck equation for several variables; methods of solution*, in *The Fokker-Planck Equation*, Volume 18 of Springer Series in Synergetics, edited by Hermann Haken (Springer, Berlin, 1989), pp. 133–162.

[23] L. Ferrari, Heavy (or large) ions in a fluid in an electric field: The diffusion equation exactly following from the Fokker-Planck equation, J. Chem. Phys. **129**, 044903 (2008).

[24] L. Ferrari, Test particles in a gas: Markovian and non-markovian langevin dynamics, Chem. Phys. **523**, 42 (2019).

[25] G. Wilemski, On the derivation of smoluchowski equations with corrections in the classical theory of brownian motion, J. Stat. Phys. **14**, 153 (1976).

[26] Y. Amarouchene, M. Mangeat, B. V. Montes, L. Ondic, T. Guérin, D. S. Dean, and Y. Louyer, Nonequilibrium Dynamics Induced by Scattering Forces for Optically Trapped Nanoparticles in Strongly Inertial Regimes, Phys. Rev. Lett. **122**, 183901 (2019).

[27] A. P. Solon, Y. Fily, A. Baskaran, M. E. Cates, Y. Kafri, M. Kardar, and J. Tailleur, Pressure is not a state function for generic activeÂ fluids, Nat. Phys. **11**, 673 (2015).

[28] J.-P. Hansen and I. R. McDonald, Static properties of liquids: Thermodynamics and structure, *Theory of Simple Liquids* (Elsevier, Amsterdam, 2013), pp. 61–104.

[29] M. E. Cates and J. Tailleur, Motility-induced phase separation, Annu. Rev. Condens. Matter Phys. **6**, 219 (2015).

[30] L. Caprini, U. Marini B. Marconi, and A. Puglisi, Spontaneous Velocity Alignment in Motility-Induced Phase Separation, Phys. Rev. Lett. **124**, 078001 (2020).

[31] V. Soni, E. S. Bililign, S. Magkiriadou, S. Sacanna, D. Bartolo, M. J. Shelley, and W. T. M. Irvine, The odd free surface flows of a colloidal chiral fluid, Nat. Phys. **15**, 1188 (2019).

[32] T. Tomé, Entropy production in nonequilibrium systems described by a Fokker-Planck equation, Braz. J. Phys. **36**, 1285 (2006).

[33] C. Maes and K. Netoĉnỳ, Canonical structure of dynamical fluctuations in mesoscopic nonequilibrium steady states, Europhys. Lett. **82**, 30003 (2008).

[34] M. Krüger, A. Solon, V. Démery, C. M. Rohwer, and D. S. Dean, Stresses in non-equilibrium fluids: Exact formulation and coarse-grained theory, J. Chem. Phys. **148**, 084503 (2018).

[35] P. Lancaster and L. Rodman, *Algebraic Riccati Equations* (Clarendon, Oxford, 1995).