

Enhanced robustness of single-layer networks with redundant dependenciesG. Timár,^{1,*} Gy. Kovács,² and J. F. F. Mendes¹¹*Departamento de Física da Universidade de Aveiro & I3N, Campus Universitário de Santiago, 3810-193 Aveiro, Portugal*²*Analytical Minds Limited, Árpád Street 5, 4933 Beregsurány, Hungary*

(Received 1 December 2020; accepted 28 January 2021; published 26 February 2021)

Dependency links in single-layer networks offer a convenient way of modeling nonlocal percolation effects in networked systems where certain pairs of nodes are only able to function together. We study the percolation properties of the weak variant of this model: Nodes with dependency neighbors may continue to function if at least one of their dependency neighbors is active. We show that this relaxation of the dependency rule allows for more robust structures and a rich variety of critical phenomena, as percolation is not determined strictly by finite dependency clusters. We study Erdős-Rényi and random scale-free networks with an underlying Erdős-Rényi network of dependency links. We identify a special “cusp” point above which the system is always stable, irrespective of the density of dependency links. We find continuous and discontinuous hybrid percolation transitions, separated by a tricritical point for Erdős-Rényi networks. For scale-free networks with a finite degree cutoff we observe the appearance of a critical point and corresponding double transitions in a certain range of the degree distribution exponent. We show that at a special point in the parameter space, where the critical point emerges, the giant viable cluster has the unusual critical singularity $S - S_c \propto (p - p_c)^{1/4}$. We study the robustness of networks where connectivity degrees and dependency degrees are correlated and find that scale-free networks are able to retain their high resilience for strong enough positive correlation, i.e., when hubs are protected by greater redundancy.

DOI: [10.1103/PhysRevE.103.022321](https://doi.org/10.1103/PhysRevE.103.022321)**I. INTRODUCTION**

The desire for an increasingly accurate description of networked systems has resulted in various useful generalizations of the classical percolation theory of random graphs. One branch of these generalizations stems from the notion that the functioning of a particular node may depend on the functioning of certain other nodes in the system to which the node in question may not be directly connected. This idea has led to the definition of mutually connected components in interdependent (or multiplex) networks [1–3]. Such a network is composed of various network layers and a node in one layer may have various interdependency neighbors in other layers. According to the most common definition, a mutually connected component is one that is connected on all layers, i.e., the interdependency neighbors of nodes in a connected cluster in one layer must also form connected clusters on all other layers. The giant mutually connected component shows an increased vulnerability to random damage, compared to the giant components of the individual layers and, for random uncorrelated layers, collapses in a discontinuous hybrid transition [3], such as the one also seen in, e.g., k -cores [4]. Interdependent and multiplex networks have enjoyed considerable popularity in recent years due to their ability to model important and easily observed real-world systems such as online social networks, transportation networks, neuronal networks, and many more [5].

The notion of dependencies between nodes in a network was exploited in a somewhat simpler generalization

of ordinary percolation by Parshani *et al.* in Ref. [6]. Here, a single-layer network of connectivity links is considered, where there may also be “dependency links” between certain pairs of nodes. The percolation rule is given as a deactivation process. Initially, a given fraction of nodes in the network is activated, and nodes can only remain active if (i) they belong to the giant connected component of active nodes and (ii) all of their dependency neighbors are also active. This deactivation process either leads to a stable situation, where a fraction of nodes in the network remain active, or all nodes in the network are deactivated. Using the fraction of initially active nodes as the control parameter, it was shown in Ref. [6] that the stable fraction of active nodes may undergo a continuous or a discontinuous transition, depending on the density and configuration of dependency links in the network. The continuous transition regime is essentially characterized by the fraction of nodes without dependencies and the discontinuous one by the distribution of the sizes of dependency groups (finite clusters of nodes connected by dependency links). Scale-free networks, that are highly robust in the continuous transitions regime, were found to be particularly fragile in the discontinuous transitions regime. The effect of various different dependency group size distributions is explored in Refs. [7–9]. Several generalizations of this model have been investigated in recent years, such as networks with directed [10] and time-varying dependency links [11] and multilayer networks with dependency links both between and within the individual layers [12].

The dependency rules assumed in most of the initial work on interdependent networks and single-layer networks with internal dependency links are too restrictive to describe certain

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systems that do not exhibit the predicted fragility. A relaxation of the standard multiplex percolation rule was explored in Refs. [13,14] where a node is defined to belong to a component if it has at least one neighbor on each layer in the same component—without the requirement that the component be connected on each layer. It was found that a two-layer network, of uncorrelated random networks, in this case still exhibits a continuous percolation transition, as opposed to the discontinuous hybrid transition of the standard giant mutually connected component. A different approach was studied in Ref. [15] where mutually connected components in a multiplex network were required to be connected on at least two layers, as opposed to all of them. For numbers of layers greater than or equal to two, the addition of new layers in this case increases the robustness of the system.

In this paper we consider single-layer networks with internal dependency links and propose a relaxation of the percolation rule introduced in Ref. [6]. We consider a node to belong to a component if it has at least one connectivity link to the given component and—if it has dependency links—at least one of its dependency neighbors is also in the same component. This weaker dependency rule may be suitable for modeling systems where individual nodes require some kind of input from other nodes to function, but this input may be supplied by various different nodes, not just one. This model has the interesting feature that the size of the giant component is nonmonotonic as a function of the density of dependency links. Very few dependencies, as well as a high number of redundant dependencies both correspond to robust structures, with a “valley” of more fragile states in between. We investigate the percolation properties of such systems, with Erdős-Rényi and scale-free connectivity networks, where dependency links are placed randomly, i.e., the dependency network is Erdős-Rényi with a given mean dependency degree. We find continuous and discontinuous percolation transitions separated by a tricritical point for Erdős-Rényi connectivity networks. For scale-free networks with a finite degree cutoff we show that in a certain range of the degree distribution exponent a critical point appears, which is accompanied by a nonsmooth switch between continuous and discontinuous transitions, as well as double percolation transitions. We show that at the point where a critical point appears the giant component has the critical singularity $S - S_c \propto (p - p_c)^{1/4}$. We also consider the situation where connectivity and dependency networks are correlated and find that robustness can be greatly improved by positive correlations between connectivity and dependency degrees.

The paper is organized as follows. In Sec. II we introduce our model and discuss some implications and important differences compared to the definitions of Ref. [6]. In Sec. III we set up self-consistency equations to solve our model for uncorrelated random connectivity and dependency networks, and present numerical solutions compared with simulation results. In Sec. IV we explore the various possible forms of critical behavior. We derive conditions for critical thresholds and obtain the order parameter exponent for the various cases analytically. In Sec. V we present results for scale-free connectivity networks with a finite degree cutoff, explaining the origin of double percolation transitions and the unique order parameter exponent $\beta = 1/4$. In Sec. VI we study the effect

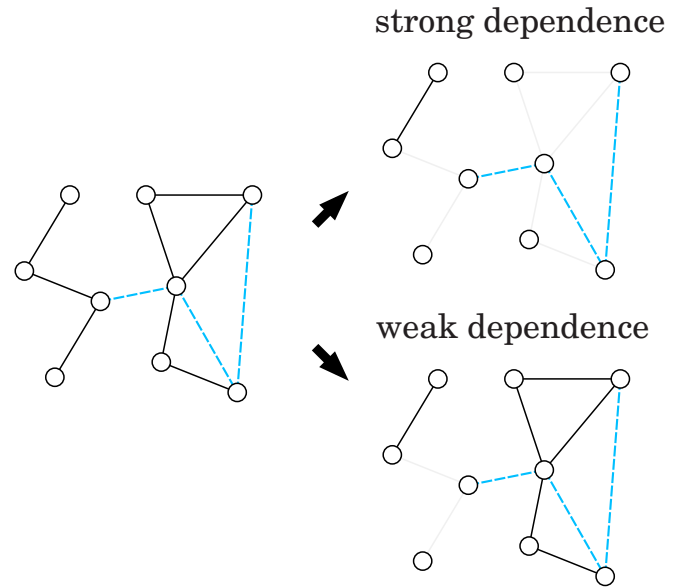


FIG. 1. Example of a simple network with dependencies consisting of two connected components (in the standard percolation sense). Solid black lines represent connectivity links, and dashed blue lines represent dependency links. Strongly and weakly dependent components (of the same network) are shown on the right.

of correlations between connectivity and dependency degrees. We give our conclusions in Sec. VII.

II. CONNECTED COMPONENTS IN NETWORKS WITH REDUNDANT DEPENDENCY LINKS

We consider an arbitrary undirected network of connectivity links between nodes and an arbitrary undirected network of dependency links between the same nodes. Connectivity links establish reachability relationships between nodes and serve as the backbone for connected components. Dependency links signify the conditions that certain nodes can only function if they are able to reach certain other nodes. Connected components in our percolation model of “weak dependencies” may be defined as the stable state of the following iterative process:

- (1) Identify all connected components based on connectivity links.
- (2) Remove all connectivity links of all nodes i that have at least one dependency neighbor, and none of these dependency neighbors are in the connected component of node i .
- (3) Repeat steps 1 and 2 until no further changes are made.

We will refer to the resulting components as *weakly dependent components* (WDCs) to distinguish from “ordinary” connected components based purely on connectivity links. Similarly we will refer to the components defined in Ref. [6] as *strongly dependent components* (SDCs). Note that in the strong dependence model step 2 of the above iterative process is replaced with “remove all connectivity links of all nodes i that have at least one dependency neighbor, and at least one of these dependency neighbors is not in the connected component of node i .” A schematic representation of strongly and weakly dependent components is shown in Fig. 1 for a

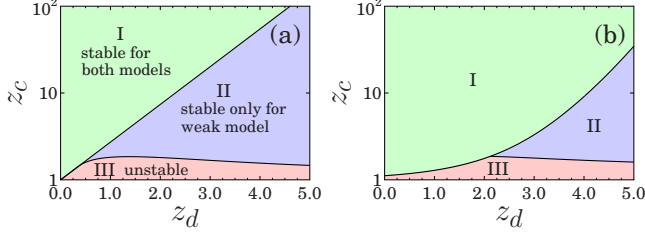


FIG. 2. Phase diagrams of (a) Erdős-Rényi and (b) scale-free connectivity networks with mean connectivity degree z_c , both with an Erdős-Rényi dependency network of mean degree z_d . The scale-free connectivity network is an uncorrelated random network with a degree distribution of the form $P_c(k) \sim (k+B)^{-\gamma}$, with lower and upper degree cutoffs $k_{\min} = 1$ and $k_{\max} = 1000$, respectively. The parameter B was chosen to achieve a given mean degree z_c . The degree distribution exponent was $\gamma = 3$. The phase separation curves were obtained by numerical analysis of the self-consistency equations presented in Sec. III.

simple network of two connected components (in the standard percolation sense).

From the definitions it follows that the SDC that node i belongs to is always a subgraph of its WDC. Also, the sizes of WDCs may increase or decrease as a result of the addition of a dependency link, while the sizes of SDCs cannot increase. For large, sparse, random uncorrelated connectivity and dependency networks the giant connected component (GCC) of the dependency network constitutes a barrier to the existence of SDCs: None of the nodes that belong to the GCC of the dependency network can belong to an SDC. For this reason it is the finite clusters (dependency groups) of the dependency network that determine the percolation properties under the strong dependency rule [6–9]. Such a restriction does not apply in the case of the weak dependency rule. This might make the weak dependency model (or some combination of the weak and strong models) a better candidate to describe the behavior of certain real-world systems with dependencies. Similarly to k -cores and mutually connected components in multiplex networks, the probability that a random node belongs to a finite dependent component that contains dependencies is negligible in large, sparse, uncorrelated random networks, according to both the weak and strong definition. Finite components, with non-negligible probability, exist in both models only if none of the nodes in the given component have dependencies.

In this paper we focus on the properties of the giant weakly dependent component (GWDC) for Erdős-Rényi and random scale-free connectivity networks, with an Erdős-Rényi dependency network. To demonstrate the effect of the weak model (compared to the strong model), in Fig. 2 we present phase diagrams of these two types of connectivity networks, indicating the regions where a giant dependent component—strong or weak—exists. The threshold value of z_c in the strong model is a monotonically increasing function of z_d , while it initially increases, then decreases in the weak model due to an increasing number of redundant dependency links. This qualitative behavior applies to both types of connectivity networks. The weak model allows for stable structures in a much wider range of parameters. (Note the logarithmic scale on the y axis.)

III. SELF-CONSISTENCY EQUATIONS

For infinite sparse random uncorrelated connectivity and dependency networks, exploiting their local tree-likeness, the problem of finding the relative size of the GWDC may be solved by setting up appropriate self-consistency equations. Let us consider a connectivity network with degree distribution $P_c(k)$ and a dependency network with degree distribution $P_d(k)$. We consider each connectivity link to be active with probability p . We introduce two probabilities that will allow us to write exact self-consistency equations for this problem. First, let x be the probability that following a random connectivity link in a random direction we can reach the GWDC. Second, let y be the probability that we encounter a node in the GWDC by following a random dependency link emanating from a node in the GWDC. We can write the following equation for x ,

$$x = p \left[1 - \sum_{k=1}^{\infty} \frac{kP_c(k)}{z_c} (1-x)^{k-1} \right] \left[1 - \sum_{k=1}^{\infty} P_d(k)(1-y)^k \right], \quad (1)$$

where z_c denotes the mean degree of the connectivity network. The first factor in square brackets in Eq. (1) gives the probability that following a random connectivity link, the node encountered has at least one outgoing connectivity link to the GWDC. The second factor gives the probability that following a random connectivity link, the node encountered has either no dependency neighbors, or has at least one dependency neighbor in the GWDC. These two factors must be multiplied by p , the probability that the link on which we arrived is active. The equation for y is simpler,

$$y = \left[1 - \sum_{k=0}^{\infty} P_c(k)(1-x)^k \right]. \quad (2)$$

Note that the right-hand side of Eq. (2) does not depend on y , i.e., x is the only independent variable in this problem. We can solve Eqs. (1) and (2) numerically by iteration. Once the solutions x and y are found, we can express the relative size of the GWDC,

$$S = \left[1 - \sum_{k=0}^{\infty} P_c(k)(1-x)^k \right] \left[1 - \sum_{k=1}^{\infty} P_d(k)(1-y)^k \right].$$

The self-consistency equations (1) and (2) can be written in more compact form using probability generating functions. Let us introduce the generating function for the connectivity degree distribution, the dependency degree distribution, and the connectivity “excess degree” distribution, respectively,

$$G_c(x) = \sum_{k=0}^{\infty} P_c(k)x^k, \quad (3)$$

$$G_d(x) = \sum_{k=0}^{\infty} P_d(k)x^k, \quad (4)$$

$$H_c(x) = \sum_{k=0}^{\infty} \frac{(k+1)P_c(k+1)}{z_c} x^k. \quad (5)$$

Using these generating functions, Eqs. (1) and (2) can be written as

$$x = p[1 - H_c(1 - x)][1 - G_d(1 - y) + P_d(0)], \quad (6)$$

$$y = [1 - G_c(1 - x)]. \quad (7)$$

Substituting Eq. (7) into Eq. (6) we arrive at one single self-consistency equation for x ,

$$\begin{aligned} x &= p[1 - H_c(1 - x)][1 + P_d(0) - G_d(G_c(1 - x))] \\ &\equiv p\Psi(x). \end{aligned} \quad (8)$$

The relative size of the GWDC may be expressed as

$$S = [1 - G_c(1 - x)][1 + P_d(0) - G_d(G_c(1 - x))]. \quad (9)$$

Figure 3(a) shows numerical solutions for S obtained using Eqs. (8) and (9), compared with simulations. Both the connectivity and dependency networks are Erdős-Rényi, with mean connectivity degree z_c and mean dependency degree z_d . For high enough z_c [$z_c = 2$ in Fig. 3(a)] we see that the GWDC exists in the entire range of z_d values, although its size exhibits a minimum for an intermediate value of z_d . With decreasing z_c , below the point $z_c \approx 1.848$, the $S(z_d)$ curve breaks up into two separate regions that correspond to a low and high density of dependencies. The region in the middle is not able to support a GWDC. It can also be seen that, with increasing z_d , the GWDC disappears and reappears in a discontinuous transition [$z_c = 1.6$ in Fig. 3(a)]. At the point $z_c \approx 1.422$ the first transition changes to continuous. All four numerical curves show very good agreement with simulations except close to the critical regions, where large fluctuations are expected. To further demonstrate the different types of transitions, Fig. 3(b) shows a phase diagram of the same network class, with S overlaid as a color map.

For the sake of completeness we present here also the exact self-consistency equations necessary to solve the strong variant of the model, in the same settings. Let the probabilities x and y have the same meaning as above (except for replacing the word ‘‘GWDC’’ with ‘‘GSDC’’ in the definition). The two equations for the strong model are

$$x = p \left[1 - \sum_{k=1}^{\infty} \frac{kP_c(k)}{z_c} (1-x)^{k-1} \right] \left[\sum_{k=0}^{\infty} P_d(k)y^k \right],$$

$$y = \left[1 - \sum_{k=0}^{\infty} P_c(k)(1-x)^k \right] \left[\sum_{k=1}^{\infty} \frac{kP_d(k)}{z_d} y^{k-1} \right],$$

and the relative size of the GSDC is expressed as

$$S = \left[1 - \sum_{k=0}^{\infty} P_c(k)(1-x)^k \right] \left[\sum_{k=0}^{\infty} P_d(k)y^k \right].$$

IV. CRITICAL BEHAVIOR

We explore the various types of critical behavior that occur in the weak dependency model, associated with the appearance of the GWDC. To study the behavior of Eq. (8) it will be useful to introduce the function $f(x) = \Psi(x)/x$. (We should remember that apart from x , f depends on the distributions P_c

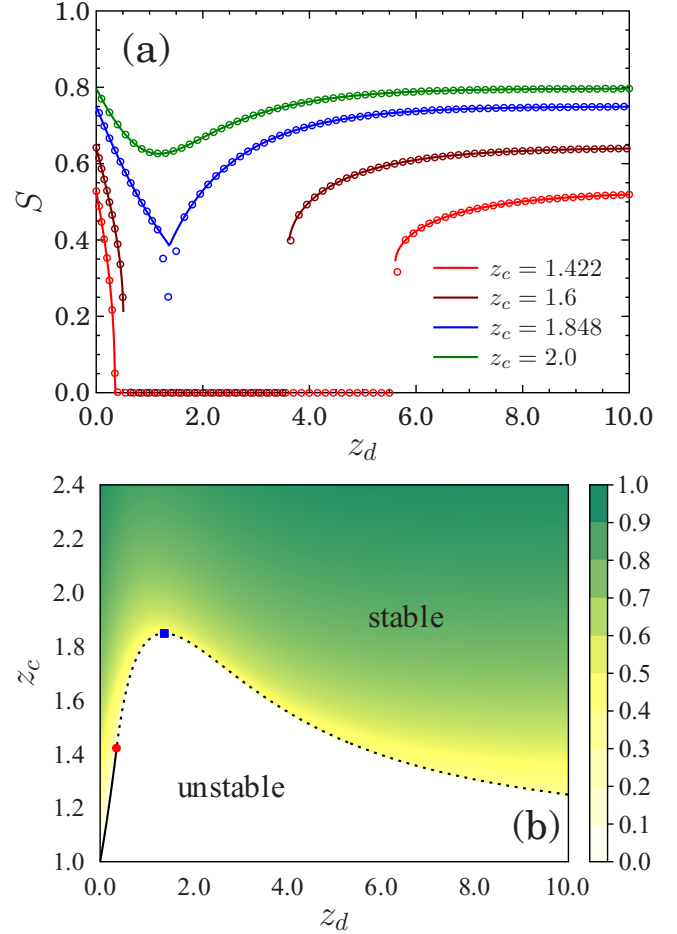


FIG. 3. (a) Relative size S of the GWDC as a function of mean dependency degree z_d , where both the connectivity and dependency networks are Erdős-Rényi networks. Curves are shown for four different values of the mean connectivity degree z_c . Open circles represent simulation results, and solid lines correspond to the numerical solution of Eqs. (8) and (9). (The number of nodes was $N = 10^6$ in all cases and results were averaged over 100 realizations.) (b) Phase diagram of the same network class, with S overlaid as a color map. Solid and dashed black lines correspond to continuous and discontinuous transitions, respectively. The tricritical point and cusp point (see Secs. IV A and IV B) are marked by a solid red circle and solid blue square, respectively.

and P_d .) Equation (8) now reads

$$pf(x) = 1. \quad (10)$$

Note that Eq. (10) is equivalent to Eq. (8) only for $x > 0$. $x = 0$ is always a solution of Eq. (8). Apart from this trivial solution, all other solutions can be found using Eq. (10). The first nonzero solution x^* occurs when the maximum of curve $pf(x)$ first becomes 1. If the maximum of f in the range $[0, 1]$ is denoted by f_{\max} , then the value of p at which this happens is given by $p_c = 1/f_{\max}$. We now discuss the various possible situations that correspond to different types of phase transitions with different critical singularities.

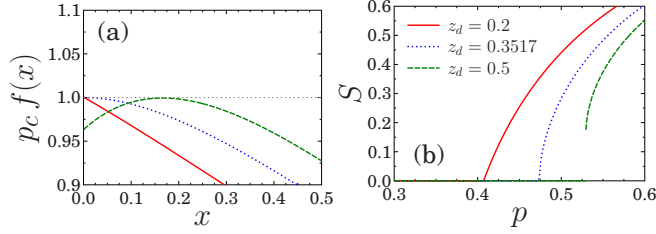


FIG. 4. (a) Function $f(x)$ (scaled to be at the critical threshold) for different values of the mean dependency degree z_d . (b) Relative size S of the GWDC as a function of connectivity link activation probability p , for different values of z_d . The three curves on each panel correspond to a continuous transition ($z_d = 0.2$), the tricritical point ($z_d = 0.3517$), and a discontinuous transition ($z_d = 0.5$). The mean connectivity degree is $z_c = 3$ in all cases. Both the connectivity and the dependency network is Erdős-Rényi.

A. Continuous transitions, discontinuous transitions, and tricritical point

Let us first consider the situation where $f(x)$ is monotonically decreasing, i.e., $f'(x) < 0$ for all $x > 0$ [solid red line in Fig. 4(a)]. In this case $f_{\max} = \lim_{x \rightarrow 0} f(x)$, which corresponds to a continuous transition. [Remember that $f(0)$ is not defined.]

This limit can be evaluated using L'Hospital's rule,

$$\lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} \frac{\Psi(x)}{x} = \lim_{x \rightarrow 0} \frac{\Psi'(x)}{x'} = \Psi'(0).$$

Using Eqs. (3)–(5) and (8), and the basic properties of probability generating functions, we obtain

$$\Psi'(0) = P_d(0) \frac{\langle k(k-1) \rangle_c}{\langle k \rangle_c},$$

resulting in the threshold for a continuous phase transition,

$$p_c = \frac{\langle k \rangle_c}{P_d(0) \langle k(k-1) \rangle_c}. \quad (11)$$

We see that only $P_d(0)$ plays a role, i.e., the shape of the dependency degree distribution is irrelevant and only the fraction of nodes with no dependencies matters. For the case where both the connectivity and dependency networks are Erdős-Rényi (with mean degrees z_c and z_d , respectively) and $p_c = 1$, the condition (11) can be written simply as

$$z_c = e^{z_d}.$$

The corresponding curve is plotted in Fig. 3(b) as a solid black line. Using the probability p as a control parameter, the behavior of S near the transition is given by $S \propto (p - p_c)^1$ [see the solid red line in Fig. 4(b)]. This can be shown by expanding Eq. (8) about the point $(x = 0, p = p_c)$ and using Eq. (10) (see the Appendix for details).

As noted above, a continuous transition can only happen if $f_{\max} = \lim_{x \rightarrow 0} f(x)$. If $\lim_{x \rightarrow 0} f'(x) > 0$, then this is not the case and $f_{\max} = f(x^*)$ for some $x^* > 0$, meaning that the nontrivial solution emerges with a jump [see the dashed green line in Figs. 4(a) and 4(b)]. The condition for this to happen is $f'(x^*) = 0$, and the corresponding threshold for the

discontinuous transition is

$$p_c = \frac{1}{f(x^*)}.$$

The behavior of S near such a transition is given by $S - S_c \propto (p - p_c)^{1/2}$. This type of phase transition is also frequently referred to as a hybrid transition, involving a discontinuity and a critical singularity. The critical behavior can be derived by expanding Eq. (8) about the point $(x = x^*, p = p_c)$ and using the condition $f'(x^*) = 0$ (see the Appendix for details).

Assuming for now that f has only one maximum in the interval $]0, 1]$, we saw that $\lim_{x \rightarrow 0} f'(x) < 0$ results in a continuous transition, while $\lim_{x \rightarrow 0} f'(x) > 0$ produces a discontinuous hybrid transition. The two types of transitions meet at a tricritical point where $\lim_{x \rightarrow 0} f'(x) = 0$ [see the dotted blue line in Figs. 4(a) and 4(b)]. Using L'Hospital's rule we find

$$\begin{aligned} \lim_{x \rightarrow 0} f'(x) &= \lim_{x \rightarrow 0} \left(\frac{\Psi(x)}{x} \right)' = \lim_{x \rightarrow 0} \frac{x\Psi'(x) - \Psi(x)}{x^2} \\ &= \lim_{x \rightarrow 0} \frac{[x\Psi'(x) - \Psi(x)]'}{(x^2)'} = \frac{\Psi''(0)}{2}. \end{aligned}$$

Using the properties of generating functions, $\Psi''(0)$ can be expressed as

$$\Psi''(0) = 2\langle k(k-1) \rangle_c \langle k \rangle_d - P_d(0) \frac{\langle k(k-1)(k-2) \rangle_c}{\langle k \rangle_c},$$

resulting in the condition for a tricritical point,

$$\frac{P_d(0)}{2\langle k \rangle_d} = \frac{\langle k(k-1) \rangle_c \langle k \rangle_c}{\langle k(k-1)(k-2) \rangle_c}. \quad (12)$$

For Erdős-Rényi networks, and setting $p_c = 1$, Eqs. (12) and (11) are equivalent to

$$\begin{aligned} z_c &= e^{1/(2z_d)}, \\ z_d &= 1/(2e^{z_d}), \end{aligned}$$

which have the solution $z_c^{\text{tc}} \approx 1.4215$ and $z_d^{\text{tc}} \approx 0.3517$. This tricritical point is shown as a solid red circle on Fig. 3(b). Using the probability p as a control parameter, the behavior of S near such a transition is given by $S \propto (p - p_c)^{1/2}$. Thus the tricritical point has the same type of critical singularity as hybrid transitions, only here the jump size is zero. The critical behavior can be derived by expanding Eq. (8) about the point $(x = 0, p = p_c)$ and using the condition $\lim_{x \rightarrow 0} f'(x) = 0$ (see the Appendix for details). Equation (12) also means that infinite scale-free connectivity networks with a degree distribution exponent $\gamma < 4$ (and no finite degree cutoff) cannot have a tricritical point for any finite value of the mean dependency degree [as long as $P_d(0) > 0$].

B. “Cusp” point

Assuming an Erdős-Rényi dependency network, let us now set $p = 1$ and take z_d , the mean dependency degree, to be our control parameter. As can be seen in Fig. 3, for Erdős-Rényi connectivity networks, for small enough values of z_c there are two percolation transitions, when varying z_d . The GWDC first disappears, and then reappears for large enough z_d . When $z_c < z_c^{\text{tc}}$, the first transition of the two is continuous and for

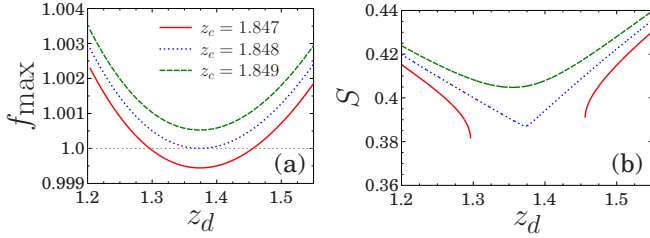


FIG. 5. (a) The maximum (with respect to x) of the function $f(x, z_d)$, as a function of z_d , for three different values of z_c . (b) The relative size S of the GWDC as a function of z_d for the same three values of z_c as in (a). The three curves in each panel correspond to a case with two discontinuous transitions with $S = 0$ in between ($z_c = 1.847$), the cusp point ($z_c = 1.848$), and a situation with no transitions ($z_c = 1.849$). Both the connectivity and the dependency network are Erdős-Rényi.

$z_c > z_c^{\text{tc}}$ it is discontinuous (z_c^{tc} denoting the value of z_c at the tricritical point). An interesting consequence of the weak dependency model is that for large enough z_c , percolation transitions disappear altogether, i.e., a GWDC always exists, for any z_d . For the particular value of z_c where this happens, the $S(z_d)$ curve has a unique “cusp” shape. To find the values of z_c and z_d where this cusp point occurs, let us consider also the z_d dependence of the function $f: f = f(x, z_d)$. It is easy to see that the cusp point is a saddle point of the function $f(x, z_d)$: At this point f is maximal with respect to x and minimal with respect to z_d [see Fig. 5(a)]. The cusp point, therefore, has the following conditions,

$$\begin{aligned} \left. \frac{\partial f}{\partial x} \right|_{x^{\text{cp}}, z_d^{\text{cp}}} &= \left. \frac{\partial f}{\partial z_d} \right|_{x^{\text{cp}}, z_d^{\text{cp}}} = 0, \\ f(x^{\text{cp}}, z_d^{\text{cp}}) &= 1. \end{aligned} \quad (13)$$

These are three equations for the three unknowns x^{cp} , z_d^{cp} , and z_c^{cp} . For Erdős-Rényi networks the cusp point occurs at $z_c^{\text{cp}} \approx 1.848$ and $z_d^{\text{cp}} \approx 1.371$. Using z_d as a control parameter the behavior of S near the cusp point is given by $S - S^{\text{cp}} \propto |z_d - z_d^{\text{cp}}|^1$ (for $z_c = z_c^{\text{cp}}$), i.e., the critical exponent changes from $1/2$ to 1 at this point. Note also that $S - S^{\text{cp}}$ is proportional to the distance from z_d^{cp} on both sides of the cusp point [see Fig. 5(b)]. This critical behavior can be derived by expanding Eq. (8) about the point $(x = x^{\text{cp}}, z_d = z_d^{\text{cp}})$ and using the conditions (13) (see the Appendix for details).

C. Critical point and double transitions

Up to now we have assumed that the function f has one maximum in the interval $]0, 1]$. This is true for Erdős-Rényi connectivity networks, but not necessarily so for broader degree distributions. We consider scale-free connectivity networks with degree distributions of the form

$$P_c(k) = A(k + B)^{-\gamma}, \quad (14)$$

with finite lower and upper degree cutoffs k_{\min} and k_{\max} . The parameter A is a normalization constant and B is adjusted in order to achieve a given mean connectivity degree z_c . Figure 6(a) shows the phase diagram of a network with a connectivity degree distribution of the form (14) with $z_c = 3$,

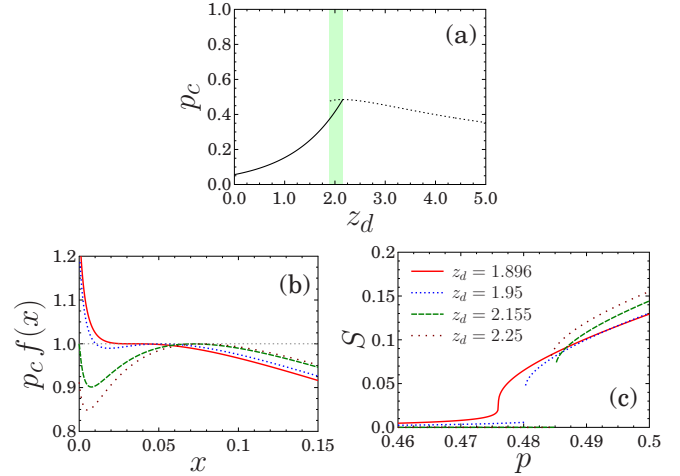


FIG. 6. (a) Phase diagram for a scale-free connectivity network ($z_c = 3$, $\gamma = 3$, $k_{\min} = 1$, $k_{\max} = 1000$) and the Erdős-Rényi dependency network. The solid line represents continuous transitions, and the dashed line corresponds to discontinuous transitions. The shaded green region indicates double percolation transitions. (b) Function $f(x)$ (scaled to be at the critical threshold) for different values of the mean dependency degree z_d . (c) Relative size S of the GWDC as a function of connectivity link activation probability p , for different values of z_d . The four curves on (b) and (c) correspond to the critical point ($z_d = 1.896$), a double transition ($z_d = 1.95$), the point where continuous transitions disappear ($z_d = 2.155$), and a discontinuous transition ($z_d = 2.25$).

$\gamma = 3$, $k_{\min} = 1$, and $k_{\max} = 1000$. The diagram was obtained by a numerical solution of Eq. (8). (The dependency degree distribution is Erdős-Rényi with mean dependency degree z_d , as before.) The transition is continuous for low values and discontinuous for high values of z_d , similar to the case of Erdős-Rényi connectivity networks. However, here there is no smooth switch between the two types of transitions and the discontinuous line ends in a critical point. Also, in this case there is a region [shaded green in Fig. 6(a)] where double percolation transitions occur.

For low values of z_d the function f has only one maximum (at $x \rightarrow 0$), corresponding to a continuous transition as discussed already in Sec. IV A. With increasing z_d the function f develops a second local maximum at some point $x^* > 0$ [see the solid red line in Fig. 6(b)], corresponding to a singularity on the $S(p)$ curve [the solid red line in Fig. 6(c)]. The conditions for this point (a critical point) are

$$f'(x^*) = f''(x^*) = 0. \quad (15)$$

Expanding Eq. (8) about the point $(x = x^*, p = p_c)$ [with $p_c = 1/f(x^*)$] and using the conditions (15) we find the behavior $S - S_c \propto (p - p_c)^{1/3}$ close to the critical point (see the Appendix for details). This type of singularity has been shown to appear in heterogeneous threshold models (see, e.g., Refs. [16,17]).

Increasing z_d beyond the critical point we find that f has two maxima, one at $x \rightarrow 0$ and one at some $x^* > 0$. The first maximum corresponds to a continuous transition, and the second corresponds to a subsequent discontinuous transition [see the dotted blue line in Figs. 6(b) and 6(c)]. The second,

discontinuous transition has the critical singularity $S - S_c \propto (p - p_c)^{1/2}$ already described in Sec. IV A. The double transition exists in the region where f has two local maxima and $\lim_{x \rightarrow 0} f(x) > f(x^*)$, where x^* is the position of the second maximum. This region is shown shaded green in Fig. 6(a). The thresholds of the two transitions become equal when $\lim_{x \rightarrow 0} f(x) = f(x^*)$ [the dashed green line in Figs. 6(b) and 6(c)], and above this point only discontinuous transitions can happen [the dotted maroon line in Figs. 6(b) and 6(c)], as the local maximum at $x \rightarrow 0$ now corresponds to a nonphysical solution.

Similar double transitions were also found in a mixed contagion model of a simple and complex contagion [17]. The situation is similar in our case: The continuous transition signifies an ordinary percolation phase, determined by the fraction of nodes without dependencies. The second, discontinuous transition corresponds to a “complex” phase, where at a certain point enough nodes have enough redundant dependencies to participate in a larger giant weakly dependent component.

D. Emergence of the critical point

For scale-free connectivity networks—i.e., ones with a degree distribution of the form (14)—we find that a critical point, and corresponding double transitions, only exist in a certain range $[\gamma^{(low)}, \gamma^{(high)}]$ of the degree distribution exponent. For $z_c = 3$, $k_{min} = 1$, and $k_{max} = 1000$ we find numerically that $\gamma^{(low)} \approx 2.146$ and $\gamma^{(high)} \approx 6.33$ (to a precision of 0.001). Starting at high values of γ (coming from narrower connectivity degree distributions) the first point at which a critical point appears is $(\gamma = \gamma^{(high)}, z_d = z_d^{(high)}, p_c = p_c^{(high)})$ for some $z_d^{(high)}$ and $p_c^{(high)}$. It can be shown that at this point S has the critical behavior $S \propto (p - p_c^{(high)})^{1/3}$ (see the Appendix for the derivation). At the other extreme, however, at the point $(\gamma = \gamma^{(low)}, z_d = z_d^{(low)}, p_c = p_c^{(low)})$ we find

$$S - S_c^{(low)} \propto (p - p_c^{(low)})^{1/4}.$$

This unusual critical behavior is explained in Sec. V.

V. RESULTS FOR SCALE-FREE NETWORKS

As we saw in Sec. IV A, Erdős-Rényi connectivity networks exhibit a tricritical point where continuous transitions switch smoothly to discontinuous ones. For scale-free connectivity networks this switch may be nonsmooth for certain values of γ , due to the existence of a critical point. A critical point and corresponding double percolation transitions occur because the function $f(x)$ has two maxima (with respect to x) in a certain range of z_d and γ values. Figure 7(a) shows phase diagrams for scale-free connectivity networks with different values of γ . For relatively small and relatively large γ values the switch between continuous and discontinuous transitions is smooth, as for Erdős-Rényi connectivity networks. A critical point exists only for intermediate γ values.

It is important to note that the networks considered here have a finite degree cutoff, therefore all moments of the degree distribution are finite. We are, therefore, not discussing the effect of asymptotically power-law degree distributions.

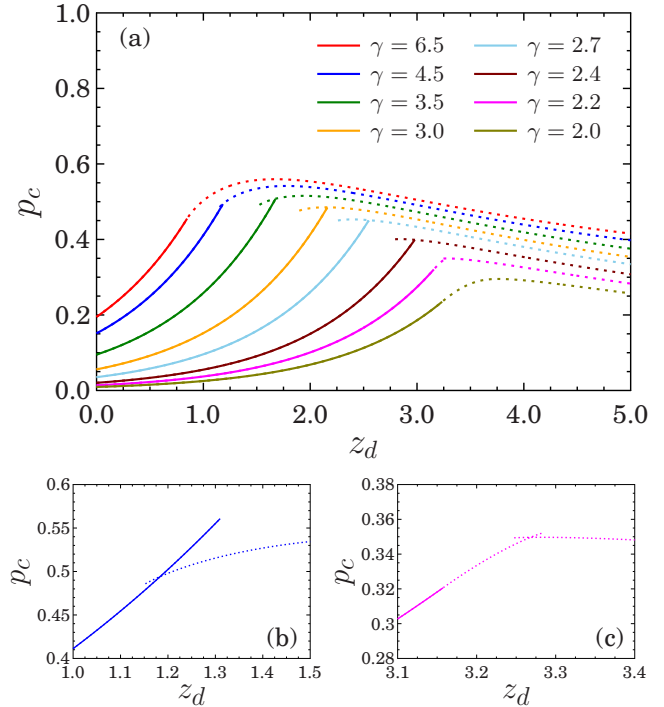


FIG. 7. (a) Phase diagrams for scale-free connectivity networks ($z_c = 3$, $k_{min} = 1$, $k_{max} = 1000$) for different values of the degree distribution exponent γ . Continuous transitions are represented by solid lines and discontinuous ones by dashed lines. A critical point and double transitions appear for intermediate values of γ , while there is a smooth switch (at a tricritical point) between continuous and discontinuous transitions for low and high γ values. (b), (c) Parts of two phase diagrams zoomed in from (a): (b) $\gamma = 4.5$ and (c) $\gamma = 2.2$. In (b) and (c) nonphysical solutions of Eqs. (8) and (9) are also shown.

We are using bounded power-law degree distributions to study the effect of broad distributions, that may also occur in real-world networks, where, of course, a finite cutoff always exists.

To understand how the critical point emerges it is useful to first look at zoomed-in versions of the phase diagrams for $\gamma = 4.5$ [Fig. 7(b)] and $\gamma = 2.2$ [Fig. 7(c)]. (In these figures the transition lines corresponding to the initial maximum are continued as long as this initial maximum exists, however they do not represent physical solutions after the intersection with the second transition line.) Figure 7(c) shows that for $\gamma = 2.2$ a tricritical point actually exists and the critical point occurs in the discontinuous range, corresponding to discontinuous-discontinuous double transitions. Considering that the jump size in discontinuous transitions goes to zero at the critical point, we know that close to the point where the critical point emerges the two local maxima—of $f(x)$ —must be close together. In other words, the second local maximum, and the corresponding critical point, is a result of the initial maximum splitting into two local maxima. Figures 7(b) and 7(c) suggest that the critical point emerges differently for high and low γ .

According to numerical analysis of the function $f(x)$, the highest γ value for which $f(x)$ has two maxima for some z_d , i.e., a critical point exists, is $\gamma^{(high)} \approx 6.33$. The lowest such γ value is $\gamma^{(low)} \approx 2.146$. (These values apply to degree

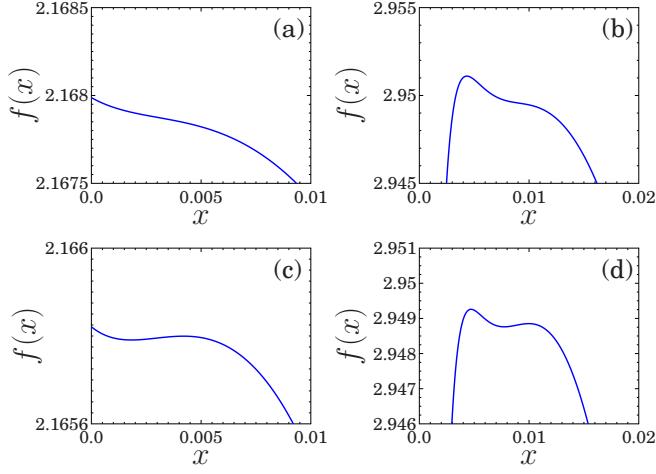


FIG. 8. Appearance of the second maximum of the function $f(x)$ for scale-free networks ($z_c = 3$, $k_{\min} = 1$, $k_{\max} = 1000$). (a) and (c) show the appearance of the second maximum at $x = 0$ [$\gamma = 6.1$, $z_d = 0.888$, and $z_d = 0.889$ for (a) and (c), respectively]. (b) and (d) show the appearance of the second maximum at some $x > 0$ [$\gamma = 2.15$, $z_d = 3.357$, and $z_d = 3.359$ for (b) and (d), respectively].

cutoffs $k_{\min} = 1$, $k_{\max} = 1000$.) We find that the emergence of the second maximum at $\gamma^{(\text{high})}$ corresponds to the tricritical point “breaking up,” i.e., happens at $x^* = 0$. At $\gamma^{(\text{low})}$ the second maximum is a result of the initial maximum (at some $x^* > 0$) splitting into two. These two different ways in which the second maximum can emerge are explained graphically in Fig. 8.

The fact that at $\gamma^{(\text{low})}$ the second maximum emerges at some $x^* > 0$ means that the following conditions must all hold,

$$f'(x^*) = f''(x^*) = f'''(x^*) = 0, \quad (16)$$

unlike at $\gamma^{(\text{high})}$, where the emergence of the second maximum only has the conditions

$$f'(0) = f''(0) = 0.$$

Having a zero condition also for the third derivative of f results in an unusual critical behavior at $\gamma^{(\text{low})}$. Expanding Eq. (8) about the point ($x = x^*$, $p = p_c$) and using the conditions (16) leads to

$$S - S_c \propto (p - p_c)^{1/4} \quad (17)$$

(see the Appendix for details). This type of transition, in our current setup, has little practical significance, however, as the emergence of the second maximum at $\gamma^{(\text{low})}$ happens at an x^* , which, although clearly positive, is still quite close to zero. Also, the maximum of f is very sharp, and $f'''(x^*) \approx 0$ holds only in a very close vicinity of x^* , making the type of singularity difficult to observe. It is nevertheless an interesting phenomenon and its analysis sheds light on the necessary ingredients for this unique critical behavior to occur. Here, we considered a scale-free connectivity degree distribution with a finite upper cutoff, therefore the double transitions and the unique critical exponent are not a consequence of power-law asymptotics, and would probably also occur for different types of broad degree distributions in this weak dependency model.

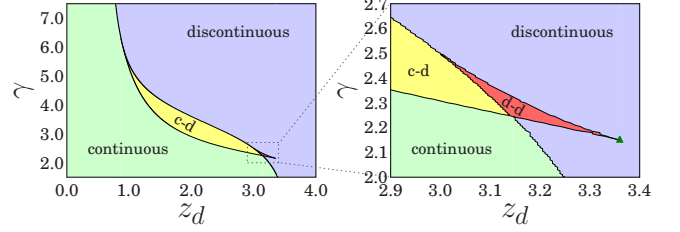


FIG. 9. Map of the different types of phase transitions occurring in a model of random scale-free connectivity network coupled with an Erdős-Rényi dependency network. The scale-free connectivity network had a degree distribution of the form of Eq. (14), the parameter B always adjusted to achieve a mean connectivity degree of $z_c = 3$. The green (blue) region corresponds to a single continuous (discontinuous) transition. The yellow (red) region corresponds to a double transition of the type “continuous–discontinuous” (“discontinuous–discontinuous”). The point marked with a green triangle corresponds to the conditions (16) and the critical behavior (17). The right panel is a zoomed-in version of the framed rectangular area of the left panel.

Investigating the effect of “truly” scale-free networks, with no degree cutoff in the infinite network size limit, is left for future work.

Figure 9 presents a map of the complex landscape of various types of transitions that are possible in the scale-free networks considered here. The boundary curves were determined numerically.

VI. EFFECT OF CORRELATION BETWEEN CONNECTIVITY AND DEPENDENCY DEGREE

Degree-degree correlations are a common feature among many real-world networks, both natural and artificial. In single-layer networks, without dependencies, positive (assortative) correlations between nearest-neighbor degrees increase the robustness of networks [18], i.e., decrease the percolation threshold in random percolation (in terms of the fraction of active links). Similar results were found for multiplex networks [19–21]: Positive correlations between degrees of nodes on different layers made networks more robust against random damage. Here, we investigate the effect of positive correlations between the connectivity and dependency degrees of nodes on the weakly dependent percolation threshold. There are various ways in which such a correlated network model could be built. As before we want to start with an uncorrelated random connectivity network with arbitrary degree distribution $P_c(k)$. We also fix the mean dependency degree z_d , which may be different from z_c . To allow for simple, exact, self-consistency equations (in the infinite network size limit) we assume that the distribution of dependency degrees for a node of given connectivity degree is given by

$$P(k_d = k' | k_c = k) = (1 - f) \frac{z_d^{k'} e^{-z_d}}{k'!} + f \frac{\left(k \frac{z_d}{z_c}\right)^{k'} e^{-\left(k \frac{z_d}{z_c}\right)}}{k'!}, \quad (18)$$

where f is a tuning parameter: It allows us to interpolate between an Erdős-Rényi dependency network which has no correlation with the connectivity network ($f = 0$), and a

network in which a node's dependency degree is Poisson distributed, with the node's rescaled connectivity degree as the mean (maximal correlation, $f = 1$). This kind of network construction resembles the "hidden variable" model of Chung and Lu [22,23], where a unique degree distribution is prescribed for each node, parametrized by the expected degree, a hidden variable of the given node. It is easy to check that the mean degree of the dependency network, using Eq. (18), is indeed equal to z_d , for any f . (Note that $f = 1$ in this model still does not mean complete positive correlation in, e.g., the Pearson correlation coefficient sense, as the connectivity and dependency degrees will not match exactly. An exact match would not be attainable if we want to maintain the possibility of $z_c \neq z_d$.)

To write exact self-consistency equations for the weakly dependent percolation model, we start by writing the degree distribution of the dependency network,

$$P_d(k') = \sum_{k=0}^{\infty} P_c(k) \left[(1-f) \frac{z_d^{k'} e^{-z_d}}{k'!} + f \frac{\left(k \frac{z_d}{z_c}\right)^{k'} e^{-\left(k \frac{z_d}{z_c}\right)}}{k'!} \right]. \quad (19)$$

Since the connectivity network is uncorrelated, the dependency network is also uncorrelated, with the degree distribution given by Eq. (19), and correlations only exist between the connectivity and dependency degrees of nodes. Let $P(k, k')$ denote the probability that the connectivity degree of a randomly chosen node is k and its dependency degree is k' . This can be expressed in two equivalent ways,

$$\begin{aligned} P(k, k') &= P_c(k)P(k_d = k' | k_c = k) \\ &= P_d(k')P(k_c = k | k_d = k'). \end{aligned} \quad (20)$$

Using Eqs. (18) and (20) we can write the following,

$$\begin{aligned} P(k_c = k | k_d = k') &= \frac{P_c(k)}{P_d(k')} \left[(1-f) \frac{z_d^{k'} e^{-z_d}}{k'!} + f \frac{\left(k \frac{z_d}{z_c}\right)^{k'} e^{-\left(k \frac{z_d}{z_c}\right)}}{k'!} \right]. \end{aligned}$$

To proceed it will be useful to express $P_c(k, k')$, the probability that a node arrived at by a random connectivity link, has connectivity degree k and dependency degree k' :

$$\begin{aligned} P_c(k, k') &= \frac{k P_c(k)}{\langle k \rangle_c} P(k_d = k' | k_c = k) \\ &= \frac{k P_c(k)}{\langle k \rangle_c} \left[(1-f) \frac{z_d^{k'} e^{-z_d}}{k'!} + f \frac{\left(k \frac{z_d}{z_c}\right)^{k'} e^{-\left(k \frac{z_d}{z_c}\right)}}{k'!} \right]. \end{aligned}$$

Similarly, let $P_d(k, k')$ be the probability that a node arrived at by a random dependency link, has connectivity degree k and dependency degree k' :

$$\begin{aligned} P_d(k, k') &= \frac{k' P_d(k')}{\langle k \rangle_d} P(k_c = k | k_d = k') \\ &= \frac{k' P_d(k')}{\langle k \rangle_d} \frac{P_c(k)}{P_d(k')} \left[(1-f) \frac{z_d^{k'} e^{-z_d}}{k'!} + f \frac{\left(k \frac{z_d}{z_c}\right)^{k'} e^{-\left(k \frac{z_d}{z_c}\right)}}{k'!} \right] \\ &= \frac{k' P_c(k)}{\langle k \rangle_d} \left[(1-f) \frac{z_d^{k'} e^{-z_d}}{k'!} + f \frac{\left(k \frac{z_d}{z_c}\right)^{k'} e^{-\left(k \frac{z_d}{z_c}\right)}}{k'!} \right]. \end{aligned}$$

Finally, let us express the probability $P_{d \rightarrow c}(k)$ that a node arrived at by a random dependency link, has connectivity degree k ,

$$\begin{aligned} P_{d \rightarrow c}(k) &= \sum_{k'=1}^{\infty} P_d(k, k') \\ &= \sum_{k'=1}^{\infty} \frac{k' P_c(k)}{\langle k \rangle_d} \left[(1-f) \frac{z_d^{k'} e^{-z_d}}{k'!} + f \frac{\left(k \frac{z_d}{z_c}\right)^{k'} e^{-\left(k \frac{z_d}{z_c}\right)}}{k'!} \right]. \end{aligned}$$

With these quantities we can now set up exact self-consistency equations for this correlated model. The probabilities x and y have the same meaning as before:

$$\begin{aligned} x &= p \sum_{k=1}^{\infty} \sum_{k'=0}^{\infty} \{P_c(k, k') [1 - (1-x)^{k-1}] \\ &\quad \times [\delta_{k',0} + 1 - (1-y)^{k'}]\}, \end{aligned} \quad (21)$$

$$y = \sum_{k=0}^{\infty} P_{d \rightarrow c}(k) [1 - (1-x)^k]. \quad (22)$$

The probability that a random node belongs to the GWDC is now given as

$$S = \sum_{k=0}^{\infty} \sum_{k'=0}^{\infty} P(k, k') [1 - (1-x)^k] [\delta_{k',0} + 1 - (1-y)^{k'}].$$

Plugging Eq. (22) into Eq. (21) results in a single self-consistency equation, and allows for the same kind of analysis as for the uncorrelated model in Secs. III and IV.

We considered the above model for different connectivity networks and different degrees of positive correlation between connectivity and dependency degrees. To assess the robustness of these networks, from Eqs. (21) and (22) we numerically identified p_c , the lowest link activation probability that provides a nonzero value of S . This threshold may correspond to a continuous or a discontinuous transition, but here we are only interested in the value p_c where the transition occurs. Figure 10 shows p_c as a function of the mean dependency degree z_d , for different values of the correlation parameter f . As expected, in Erdős-Rényi connectivity networks [Fig. 10(a)], correlations only have a moderate effect, since the connectivity and dependency degrees in this case cannot be too different, irrespective of the degree of correlation. It is clear, however, that increasing positive correlation increases robustness (i.e., decreases p_c) for a relatively dense dependency network. (Note that the opposite must be true for the strong dependency model of Ref. [6].) This can be easily understood qualitatively: Higher degree nodes, which are more important for percolation, have more dependency links, i.e., have a higher probability of surviving. Interestingly, the opposite is true for low-density dependency networks (z_d smaller than ≈ 0.5): Here, higher positive correlation means that high degree nodes have a higher probability of actually having at least one dependency link, as most dependency degrees are now 0 or 1. The same dual phenomenon is seen amplified for random scale-free networks: Positive correlation dramatically increases robustness for a higher density of dependency links, while it decreases robustness for low

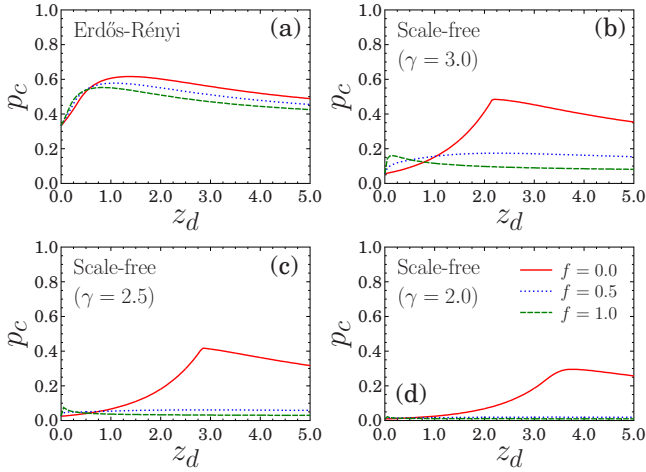


FIG. 10. Phase diagrams for the model of correlated connectivity and dependency degrees defined by the distribution (18). The percolation threshold values p_c were determined by numerical analysis of Eqs. (21) and (22). The mean connectivity degree was $z_c = 3$ for all networks. For scale-free connectivity networks the degree distribution (14) was used, with degree cutoff values $k_{\min} = 1$ and $k_{\max} = 1000$. [For the dependency degree distributions the cutoff values were changed to $k_{\min} = 0$ and $k_{\max} = 1200$, because in the model described by Eq. (18) a dependency degree equal to zero is possible and also the biggest dependency degree may be slightly greater than the biggest connectivity degree.]

dependency density [Figs. 10(b)–10(d)]. The effect becomes greater for smaller values of the degree distribution exponent γ . The low dependency density regime also becomes narrower with decreasing γ . For maximal positive correlation ($f = 1$) and $\gamma = 2$, the value of p_c is practically the same as the percolation threshold for ordinary percolation (without any dependencies), irrespective of the density of dependency links. This suggests that networks, in this model setting, can be efficiently protected from random damage by strong enough positive correlation between connectivity and dependency degrees. The scale-free networks considered here had a finite degree cutoff, so p_c always remains nonzero, i.e., there is no true “hyper-resilience,” which is a hallmark of a true, asymptotically power-law degree distribution. It is an interesting problem for future work to check if the percolation threshold can indeed go to zero for the maximally correlated case, for any value of z_d , when considering a power-law connectivity degree distribution without a degree cutoff.

VII. DISCUSSION AND CONCLUSIONS

We have investigated the percolation properties of random networks where nodes can have dependency neighbors, but only require that at least one of them be in the same component. Assuming also a random network of dependency links, we have demonstrated that networks in this model are considerably more robust than in the strong dependency model of Ref. [6], due to the redundancy of dependency links for dense enough dependency networks. Our weak dependency model predicts a nonmonotonic behavior of the size of the

GWDC as a function of the mean dependency degree: Both low and high density of dependency links allows for robust structures, with more fragile structures in between. This also gives a natural scale for “maximum fragility” in terms of the mean dependency degree.

Studying Erdős-Rényi and scale-free connectivity networks we have revealed a wide variety of percolation critical phenomena. Varying the mean dependency degree, continuous and discontinuous hybrid transitions were found for Erdős-Rényi connectivity networks, separated by a tricritical point. We have derived expressions for the different threshold conditions in terms of the moments of the connectivity and dependency degree distributions. We found standard mean-field critical behavior for continuous transitions, and the same type of critical behavior for the discontinuous transitions as already seen in k -cores and multiplex networks, with an order parameter exponent $1/2$. We have identified a special “cusp” point, above which the system is always stable, irrespective of the density of dependency links. At this point the order parameter exponent of discontinuous transitions was found to change from $1/2$ to 1 .

We have found continuous and discontinuous hybrid transitions also for scale-free connectivity networks, in the low and high dependency regimes, respectively. For a certain range of the degree distribution exponent γ the switch between the two types of transitions was found to be nonsmooth, corresponding to the existence of a critical point marking the end of the line of discontinuous transitions. In this range double percolation transitions were observed. The order parameter exponent at the critical point was found to be $1/3$, as seen also in a heterogeneous k -core, and bootstrap percolation. For the smallest γ value where a critical point first appears, we found the exponent to change from $1/3$ to $1/4$ at the (emergent) critical point. The power-law degree distribution used here had a finite cutoff, and hence did not represent a “truly” scale-free network. The critical point, double transitions, and the unique $1/4$ exponent would likely also appear in other forms of broad degree distributions, not necessarily power law. These effects can be attributed mainly to the structure of the weak dependency model. Studying the effect of asymptotically power-law degree distributions is an interesting problem for future work.

We have investigated the effect of correlation between connectivity and dependency degrees, and found that positive correlation enhances robustness, except for networks with a low density of dependency links. The robustness enhancing effect is amplified for scale-free connectivity networks. For a low enough value of γ , strong enough positive correlation between connectivity and dependency degree appears to completely negate the effect of dependencies: The percolation threshold is practically the same as for the network without any dependencies. The study of asymptotically power-law connectivity degree distributions is left for future work.

In addition to degree-degree correlations, network models with a certain fraction of connectivity and dependency links overlapping, may provide a more realistic representation of many real-world systems. Our weak dependency model may also be combined with the strong variant: Certain nodes may follow the former rule, and others may follow the latter. Such

generalizations may be interesting avenues to consider for future research.

ACKNOWLEDGMENTS

This work was developed within the scope of the project i3N, UIDB/50025/2020 and UIDP/50025/2020, financed by national funds through the FCT/MEC–Portuguese Foundation for Science and Technology. G.T. was supported by FCT Grant No. CEECIND/03838/2017.

APPENDIX: DERIVATION OF ORDER PARAMETER EXPONENTS

Here, we derive the order parameter exponent for the different types of phase transitions that are discussed in the main text. Our starting point is the self-consistency equation

$$x = p[1 - H_c(1-x)][1 + P_d(0) - G_d(G_c(1-x))] \equiv p\Psi(x). \quad (\text{A1})$$

In the main text we defined the function $f(x) = \Psi(x)/x$ and discussed the various threshold conditions in terms of the derivatives of f . To derive the order parameter exponents it is more convenient to consider the function

$$g(x, p) = p\Psi(x) - x.$$

The conditions in terms of the derivatives of f can be easily transformed into conditions in terms of the derivatives of g . Equation (A1) can be written as

$$g(x, p) = 0.$$

Note that $\Psi(0) = 0$, which implies that the “unmixed” partial derivatives of g with respect to p are all zero at $x = 0$. We will use this fact extensively.

1. Continuous transitions

A continuous transition happens at $x = 0$, $p = p_c$, so we expand $g(x, p)$ about the point $(0, p_c)$:

$$\begin{aligned} g(0 + \delta x, p_c + \delta p) &= g(0, p_c) + \left. \frac{\partial g}{\partial x} \right|_{0, p_c} \delta x + \left. \frac{\partial g}{\partial p} \right|_{0, p_c} \delta p + \frac{1}{2} \left. \frac{\partial^2 g}{\partial x^2} \right|_{0, p_c} (\delta x)^2 \\ &+ \frac{1}{2} \left. \frac{\partial^2 g}{\partial p^2} \right|_{0, p_c} (\delta p)^2 + \left. \frac{\partial^2 g}{\partial x \partial p} \right|_{0, p_c} (\delta x)(\delta p) + \dots \quad (\text{A2}) \end{aligned}$$

We know that $g(0, p_c) = 0$. Requiring also that $g(0 + \delta x, p_c + \delta p) = 0$ (i.e., we are expanding along the solution curve) we have that the expansion terms of equal powers on the right-hand side of Eq. (A2) must cancel out. The unmixed partial derivatives of g with respect to p are all 0 at $x = 0$. The condition for a continuous transition, as expressed in terms of the function f , is

$$p_c \left[\lim_{x \rightarrow 0} f(x) \right] = 1.$$

This condition is equivalent to

$$g(0, p_c) = \left. \frac{\partial g}{\partial x} \right|_{0, p_c} = 0.$$

With the above conditions, the leading terms of Eq. (A2) are

$$\frac{1}{2} \left. \frac{\partial^2 g}{\partial x^2} \right|_{0, p_c} (\delta x)^2 + \left. \frac{\partial^2 g}{\partial x \partial p} \right|_{0, p_c} (\delta x)(\delta p) = 0,$$

which results in

$$\delta x \propto \delta p.$$

2. Discontinuous transitions

A discontinuous transition happens at some $x = x^*$, $p = p_c$, so we expand $g(x, p)$ about the point (x^*, p_c) :

$$\begin{aligned} g(x^* + \delta x, p_c + \delta p) &= g(x^*, p_c) + \left. \frac{\partial g}{\partial x} \right|_{x^*, p_c} \delta x + \left. \frac{\partial g}{\partial p} \right|_{x^*, p_c} \delta p + \frac{1}{2} \left. \frac{\partial^2 g}{\partial x^2} \right|_{x^*, p_c} (\delta x)^2 \\ &+ \frac{1}{2} \left. \frac{\partial^2 g}{\partial p^2} \right|_{x^*, p_c} (\delta p)^2 + \left. \frac{\partial^2 g}{\partial x \partial p} \right|_{x^*, p_c} (\delta x)(\delta p) + \dots \quad (\text{A3}) \end{aligned}$$

In this case, since $x^* > 0$, the unmixed derivatives of g with respect to p are nonzero. The conditions for a discontinuous transition, as expressed in terms of the function f , are

$$\begin{aligned} p_c f(x^*) &= 1, \\ f'(x^*) &= 0. \end{aligned}$$

These conditions are equivalent to

$$g(x^*, p_c) = \left. \frac{\partial g}{\partial x} \right|_{x^*, p_c} = 0.$$

With the above conditions, the leading terms in Eq. (A3) are

$$\left. \frac{\partial g}{\partial p} \right|_{x^*, p_c} \delta p + \frac{1}{2} \left. \frac{\partial^2 g}{\partial x^2} \right|_{x^*, p_c} (\delta x)^2 = 0,$$

which results in

$$\delta x \propto (\delta p)^{1/2}.$$

3. Tricritical point

A tricritical point occurs at $x = 0$, $p = p_c$, so we expand $g(x, p)$ about the point $(0, p_c)$:

$$\begin{aligned} g(0 + \delta x, p_c + \delta p) &= g(0, p_c) + \left. \frac{\partial g}{\partial x} \right|_{0, p_c} \delta x + \left. \frac{\partial g}{\partial p} \right|_{0, p_c} \delta p + \frac{1}{2} \left. \frac{\partial^2 g}{\partial x^2} \right|_{0, p_c} (\delta x)^2 \\ &+ \frac{1}{2} \left. \frac{\partial^2 g}{\partial p^2} \right|_{0, p_c} (\delta p)^2 + \left. \frac{\partial^2 g}{\partial x \partial p} \right|_{0, p_c} (\delta x)(\delta p) \\ &+ \frac{1}{6} \left. \frac{\partial^3 g}{\partial x^3} \right|_{0, p_c} (\delta x)^3 + \frac{1}{6} \left. \frac{\partial^3 g}{\partial p^3} \right|_{0, p_c} (\delta p)^3 \\ &+ \frac{1}{2} \left. \frac{\partial^3 g}{\partial x^2 \partial p} \right|_{0, p_c} (\delta x)^2(\delta p) + \frac{1}{2} \left. \frac{\partial^3 g}{\partial x \partial p^2} \right|_{0, p_c} (\delta x)(\delta p)^2 + \dots \quad (\text{A4}) \end{aligned}$$

As before, for continuous transitions, the unmixed derivatives of g with respect to p are all 0. The conditions for a tricritical

point, as expressed in terms of the function f , are

$$p_c \left[\lim_{x \rightarrow 0} f(x) \right] = 1,$$

$$\lim_{x \rightarrow 0} f'(x) = 0.$$

These conditions are equivalent to

$$g(0, p_c) = \left. \frac{\partial g}{\partial x} \right|_{0, p_c} = \left. \frac{\partial^2 g}{\partial x^2} \right|_{0, p_c} = 0.$$

With the above conditions, the leading terms in Eq. (A4) are

$$\left. \frac{\partial^2 g}{\partial x \partial p} \right|_{0, p_c} (\delta x)(\delta p) + \frac{1}{6} \left. \frac{\partial^3 g}{\partial x^3} \right|_{0, p_c} (\delta x)^3 = 0,$$

which results in

$$\delta x \propto (\delta p)^{1/2}.$$

4. Cusp point

To describe this point, as discussed in the main text, we assume $p = 1$ and we consider f to be a function of x and z_d (the mean dependency degree). The function g is now also assumed to be a function of x and z_d : $g(x, z_d) = \Psi(x, z_d) - x = xf(x, z_d) - x$. The cusp point occurs at some $x = x^{\text{cp}}$, $z_d = z_d^{\text{cp}}$, so we expand $g(x, z_d)$ about the point $(x^{\text{cp}}, z_d^{\text{cp}})$:

$$g(x^{\text{cp}} + \delta x, z_d^{\text{cp}} + \delta z_d)$$

$$= g(x^{\text{cp}}, z_d^{\text{cp}}) + \left. \frac{\partial g}{\partial x} \right|_{x^{\text{cp}}, z_d^{\text{cp}}} \delta x + \left. \frac{\partial g}{\partial z_d} \right|_{x^{\text{cp}}, z_d^{\text{cp}}} \delta z_d$$

$$+ \frac{1}{2} \left. \frac{\partial^2 g}{\partial x^2} \right|_{x^{\text{cp}}, z_d^{\text{cp}}} (\delta x)^2 + \frac{1}{2} \left. \frac{\partial^2 g}{\partial z_d^2} \right|_{x^{\text{cp}}, z_d^{\text{cp}}} (\delta z_d)^2$$

$$+ \left. \frac{\partial^2 g}{\partial x \partial z_d} \right|_{x^{\text{cp}}, z_d^{\text{cp}}} (\delta x)(\delta z_d) + \dots \quad (\text{A5})$$

The conditions for the cusp point, as expressed in terms of the function f , are

$$f(x^{\text{cp}}, z_d^{\text{cp}}) = 1,$$

$$\left. \frac{\partial f}{\partial x} \right|_{x^{\text{cp}}, z_d^{\text{cp}}} = 0,$$

$$\left. \frac{\partial f}{\partial z_d} \right|_{x^{\text{cp}}, z_d^{\text{cp}}} = 0.$$

These conditions are equivalent to

$$g(x^{\text{cp}}, z_d^{\text{cp}}) = \left. \frac{\partial g}{\partial x} \right|_{x^{\text{cp}}, z_d^{\text{cp}}} = \left. \frac{\partial g}{\partial z_d} \right|_{x^{\text{cp}}, z_d^{\text{cp}}} = 0.$$

With the above conditions, the leading terms in Eq. (A5) are

$$\frac{1}{2} \left. \frac{\partial^2 g}{\partial x^2} \right|_{x^{\text{cp}}, z_d^{\text{cp}}} (\delta x)^2 + \frac{1}{2} \left. \frac{\partial^2 g}{\partial z_d^2} \right|_{x^{\text{cp}}, z_d^{\text{cp}}} (\delta z_d)^2$$

$$+ \left. \frac{\partial^2 g}{\partial x \partial z_d} \right|_{x^{\text{cp}}, z_d^{\text{cp}}} (\delta x)(\delta z_d) = 0,$$

which results in

$$\delta x \propto \delta z_d.$$

5. Critical point

A critical point occurs at some $x = x^*$, $p = p_c$, so we expand $g(x, p)$ about the point (x^*, p_c) :

$$g(x^* + \delta x, p_c + \delta p)$$

$$= g(x^*, p_c) + \left. \frac{\partial g}{\partial x} \right|_{x^*, p_c} \delta x + \left. \frac{\partial g}{\partial p} \right|_{x^*, p_c} \delta p$$

$$+ \frac{1}{2} \left. \frac{\partial^2 g}{\partial x^2} \right|_{x^*, p_c} (\delta x)^2 + \frac{1}{2} \left. \frac{\partial^2 g}{\partial p^2} \right|_{x^*, p_c} (\delta p)^2$$

$$+ \left. \frac{\partial^2 g}{\partial x \partial p} \right|_{x^*, p_c} (\delta x)(\delta p) + \frac{1}{6} \left. \frac{\partial^3 g}{\partial x^3} \right|_{x^*, p_c} (\delta x)^3$$

$$+ \frac{1}{6} \left. \frac{\partial^3 g}{\partial p^3} \right|_{x^*, p_c} (\delta p)^3 + \frac{1}{2} \left. \frac{\partial^3 g}{\partial x^2 \partial p} \right|_{x^*, p_c} (\delta x)^2 (\delta p)$$

$$+ \frac{1}{2} \left. \frac{\partial^3 g}{\partial x \partial p^2} \right|_{x^*, p_c} (\delta x)(\delta p)^2 + \dots \quad (\text{A6})$$

Here, as in the case of discontinuous transitions, the unmixed derivatives of g with respect to p are nonzero. The conditions for a critical point, as expressed in terms of the function f , are

$$p_c f(x^*) = 1,$$

$$f'(x^*) = 0,$$

$$f''(x^*) = 0.$$

These conditions are equivalent to

$$g(x^*, p_c) = \left. \frac{\partial g}{\partial x} \right|_{x^*, p_c} = \left. \frac{\partial^2 g}{\partial x^2} \right|_{x^*, p_c} = 0.$$

With the above conditions, the leading terms in Eq. (A6) are

$$\left. \frac{\partial g}{\partial p} \right|_{x^*, p_c} \delta p + \frac{1}{6} \left. \frac{\partial^3 g}{\partial x^3} \right|_{x^*, p_c} (\delta x)^3 = 0,$$

which results in

$$\delta x \propto (\delta p)^{1/3}.$$

6. Birth point of critical point (at γ^{high})

The critical point at γ^{high} emerges from a tricritical point, which always occurs at $x = 0$, $p = p_c$, so we expand $g(x, p)$ about the point $(0, p_c)$:

$$g(0 + \delta x, p_c + \delta p)$$

$$= g(0, p_c) + \left. \frac{\partial g}{\partial x} \right|_{0, p_c} \delta x + \left. \frac{\partial g}{\partial p} \right|_{0, p_c} \delta p + \frac{1}{2} \left. \frac{\partial^2 g}{\partial x^2} \right|_{0, p_c} (\delta x)^2$$

$$+ \frac{1}{2} \left. \frac{\partial^2 g}{\partial p^2} \right|_{0, p_c} (\delta p)^2 + \left. \frac{\partial^2 g}{\partial x \partial p} \right|_{0, p_c} (\delta x)(\delta p)$$

$$+ \frac{1}{6} \left. \frac{\partial^3 g}{\partial x^3} \right|_{0, p_c} (\delta x)^3 + \frac{1}{6} \left. \frac{\partial^3 g}{\partial p^3} \right|_{0, p_c} (\delta p)^3$$

$$+ \frac{1}{2} \left. \frac{\partial^3 g}{\partial x^2 \partial p} \right|_{0, p_c} (\delta x)^2 (\delta p) + \frac{1}{2} \left. \frac{\partial^3 g}{\partial x \partial p^2} \right|_{0, p_c} (\delta x)(\delta p)^2 + \dots \quad (\text{A7})$$

The unmixed derivatives of g with respect to p are 0. The conditions for the birth point of the tricritical point at $x = 0$, as expressed in terms of the function f , are

$$\begin{aligned} p_c \left[\lim_{x \rightarrow 0} f(x) \right] &= 1, \\ \lim_{x \rightarrow 0} f'(x) &= 0, \\ \lim_{x \rightarrow 0} f''(x) &= 0. \end{aligned}$$

These conditions are equivalent to

$$g(0, p_c) = \frac{\partial g}{\partial x} \Big|_{0, p_c} = \frac{\partial^2 g}{\partial x^2} \Big|_{0, p_c} = \frac{\partial^3 g}{\partial x^3} \Big|_{0, p_c} = 0.$$

With the above conditions, the leading terms in Eq. (A7) are

$$\frac{\partial^2 g}{\partial x \partial p} \Big|_{0, p_c} (\delta x)(\delta p) + \frac{1}{24} \frac{\partial^4 g}{\partial x^4} \Big|_{0, p_c} (\delta x)^4 = 0,$$

which results in

$$\delta x \propto (\delta p)^{1/3}.$$

7. Birth point of critical point (at $\gamma^{(\text{low})}$)

The critical point, at $\gamma^{(\text{low})}$, emerges at some $x = x^*$, $p = p_c$, so we expand $g(x, p)$ about the point (x^*, p_c) :

$$\begin{aligned} &g(x^* + \delta x, p_c + \delta p) \\ &= g(x^*, p_c) + \frac{\partial g}{\partial x} \Big|_{x^*, p_c} \delta x + \frac{\partial g}{\partial p} \Big|_{x^*, p_c} \delta p \\ &\quad + \frac{1}{2} \frac{\partial^2 g}{\partial x^2} \Big|_{x^*, p_c} (\delta x)^2 + \frac{1}{2} \frac{\partial^2 g}{\partial p^2} \Big|_{x^*, p_c} (\delta p)^2 \end{aligned}$$

$$\begin{aligned} &+ \frac{\partial^2 g}{\partial x \partial p} \Big|_{x^*, p_c} (\delta x)(\delta p) + \frac{1}{6} \frac{\partial^3 g}{\partial x^3} \Big|_{x^*, p_c} (\delta x)^3 \\ &+ \frac{1}{6} \frac{\partial^3 g}{\partial p^3} \Big|_{x^*, p_c} (\delta p)^3 + \frac{1}{2} \frac{\partial^3 g}{\partial x^2 \partial p} \Big|_{x^*, p_c} (\delta x)^2 (\delta p) \\ &+ \frac{1}{2} \frac{\partial^3 g}{\partial x \partial p^2} \Big|_{x^*, p_c} (\delta x)(\delta p)^2 + \dots \end{aligned} \quad (\text{A8})$$

The unmixed derivatives of g with respect to p are nonzero. The conditions for the birth point of the critical point, as expressed in terms of the function f , are

$$\begin{aligned} p_c f(x^*) &= 1, \\ f'(x^*) &= 0, \\ f''(x^*) &= 0, \\ f'''(x^*) &= 0. \end{aligned}$$

These conditions are equivalent to

$$g(x^*, p_c) = \frac{\partial g}{\partial x} \Big|_{x^*, p_c} = \frac{\partial^2 g}{\partial x^2} \Big|_{x^*, p_c} = \frac{\partial^3 g}{\partial x^3} \Big|_{x^*, p_c} = 0.$$

With the above conditions, the leading terms in Eq. (A8) are

$$\frac{\partial g}{\partial p} \Big|_{x^*, p_c} \delta p + \frac{1}{24} \frac{\partial^4 g}{\partial x^4} \Big|_{x^*, p_c} (\delta x)^4 = 0,$$

which results in

$$\delta x \propto (\delta p)^{1/4}.$$

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