Synchronization in asymmetrically coupled networks with homogeneous oscillators

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Synchronization among coupled oscillators is a common feature of symmetrically coupled networks with homogeneous, i.e., identical, oscillators. Recently, it was reported [T. Nishikawa and A. Motter, Phys. Rev. Lett. **117**, 114101 (2016) and Y. Zhang, T. Nishikawa, and A. E. Motter, Phys. Rev. E **95**, 062215 (2017)], however, that in networks with asymmetrically coupled oscillators, synchronization can only be found to be stable when the oscillators are heterogenous or nonidentical. In this manuscript, it is proven, mathematically, that the conclusions in those works are incorrect, and that stable synchronization states can, and do, exist in asymmetrically coupled homogeneous oscillators. Theoretical results are confirmed with numerical simulations.

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I. INTRODUCTION

Complex networks have gained considerable interest, especially over the past two decades, through the development of interdisciplinary methods to model and analyze collective behavior [1–10]. The complexity arises from the fact that individual units or cells cannot exhibit on their own the collective behavior of the entire network. Examples include: laser arrays [11,12], digital communication [13,14], microbiology [15–18], neuroanatomy [19], Josephson junctions [20–23], central pattern generators in biological systems [24–26], coupled laser systems [27,28], chaotic oscillators [29,30], collective behavior of bubbles in fluidization [31], the flocking of birds [32], and even psychology [33–35]. Multiple overviews of the subject have been written [36–42].

In most cases, three factors are normally considered when studying the collective behavior of a complex system: the internal dynamics of each individual unit or cell; the topology of cell connections, i.e., which cells communicate with each other; and the type of coupling. More recently, a fourth factor has gained further attention—*Symmetry*. It is well known that symmetry alone can restrict the type of solutions of systems of ordinary- and partial-differential equations, which often serve as models of complex networks. So it is reasonable to expect that certain features of the collective behavior of a complex network can be inferred from the presence of symmetry alone. One of those features of great interest is *synchronization*. Typically, synchronization states are found in symmetrically coupled networks, with homogeneous units, i.e., identical units. However, a recent set of papers [43,44], claimed an opposite situation to be true: synchronization (being the symmetric state) can only be found to be stable in asymmetrically coupled networks with heterogeneous oscillators. Specifically, the claims (on page 3) include: "the stability of the synchronous state can only be supported by nonidentical oscillators", also, "no homogeneous oscillators can be stably synchronized", and, "significant differences between the oscillators are required to achieve stable synchronization" [43].

In this manuscript, it is demonstrated that these claims are incorrect. They are incorrect, because, from a mathematics standpoint, showing the existence of a certain type of behavior—in this case, stable synchronization in networks of asymmetrically coupled heterogeneous oscillators—does not necessarily exclude the existence of the same type of behavior but with the opposite type of oscillators. In fact, it is proven, analytically and computationally, that stable synchronization states can, and do, occur in asymmetrically coupled networks of homogeneous oscillators. To do that, ideas and methods from perturbation analysis and equivariant bifurcation theory are employed. From the stand point of view of equivariant bifurcation theory, bifurcation problems have been, typically, studied under the assumption that the state variables of a model exhibit *spontaneous symmetry-breaking*. This means that solutions of such bifurcation problems lose symmetry as some parameters are varied, even though the equations that such solutions satisfy retain the full symmetry of the system. A less common perspective of equivariant bifurcation theory is when the bifurcation equations possess less symmetry when some parameters are nonzero. This less common scenario is known as *forced symmetry breaking*. In this manuscript, it is also shown that the existence and stability of the synchronization state, in asymmetrically coupled networks of homogeneous oscillators, falls, and can be studied, under the category of equivariant bifurcation problems with parameter symmetry [45].

The manuscript is organized as follows. In Sec. II, a review of the main findings in Refs. [43,44] is provided. Mainly, computer simulations of a network of $N = 7$ oscillators are reproduced to show the emergence of unstable synchronization in a network with homogeneous oscillators, while stable synchronization are shown to appear with heterogenous oscillators. But, then, additional computer simulations reveal that the same network can also support stable synchronization with homogeneous oscillators, thus contradicting the findings in

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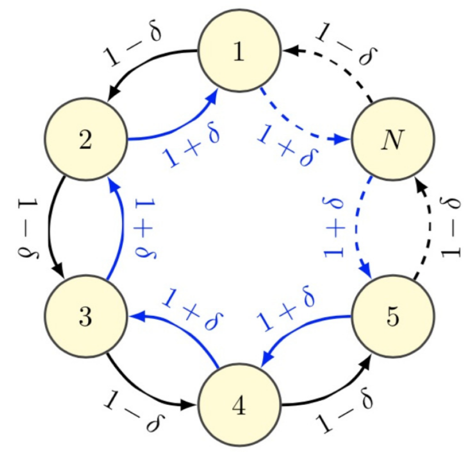

FIG. 1. Asymmetrically coupled network of N nonlinear oscillators. When $\delta = 0$, the network is, however, symmetrically, coupled and its geometry possesses $\mathbf{D}_N$ symmetry, where $\mathbf{D}_N$ is the orthogonal group of symmetries of an N-gon.

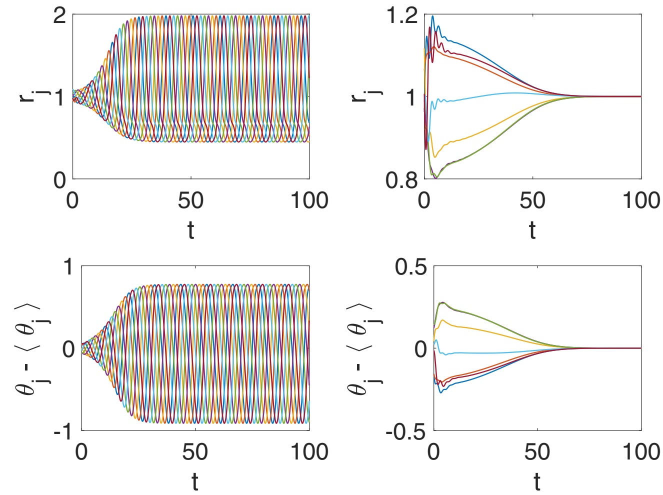

FIG. 2. Amplitude and phase evolution of an asymmetrically coupled network with $N = 7$ oscillators. Parameters are: $\omega = 1.0$, $\varepsilon = 2.0$, $\delta = 0.3$, and $\gamma = 0.1$. (Left) When the oscillators are homogeneous, with $b_j = 1.868$, $j = 1, \dots, 7$, the synchronization state $r_j = 1$ is unstable. (Right) For a heterogeneous network, with $b_1 = 1.187$, $b_2 = 7.229$, $b_3 = 1.467$, $b_4 = 0.787$, $b_5 = 4.062$, $b_6 = 3.041$, $b_7 = 1.204$, the synchronization state is, however, stable.

Refs. [43,44]. In Sec. III, a perturbation analysis is carried out to unravel the source of the stability (or instability for that matter) of the synchronization state. The main results lead to a Hopf bifurcation, which determines the existence and stability of the synchronization state for both types of networks, with homogeneous and heterogenous oscillators. In Sec. IV, ideas and methods from equivariant bifurcation theory are employed to study the force symmetry-breaking bifurcations, with parameter symmetry, which serve to explain the observations of synchronization states in asymmetrically coupled networks.

II. BACKGROUND AND EVIDENCE THAT HOMOGENEOUS OSCILLATORS CAN BE STABLY SYNCHRONIZED

The network considered by Nishikawa *et al.* [43,44] consists of N heterogeneous oscillators, coupled asymmetrically in a ring configuration, as is shown in Fig. 1.

The time evolution of the network is measured by the amplitude, $r_j(t)$, and phase, $\theta_j(t)$, variables, which are governed by a model of the form

$$\begin{aligned} \dot{r}_j &= b_j r_j (1 - r_j) + \varepsilon r_j \sum_{k=1}^{N} A_{jk} \sin(\theta_k - \theta_j), \\ \dot{\theta}_j &= \omega + r_j - 1 - \gamma r_j \sum_{k=1}^{N} \sin(\theta_k - \theta_j), \end{aligned} \tag{1}$$

where, b_j are the growth rate for each individual j^{th} oscillator, ω is a common oscillating frequency, ε and γ are coupling strengths for radial and azimuthal coupling, respectively, and A is an $N \times N$ matrix of coupling connectivity. In this work it is assumed that $b_j > 0$. Observe that the coupling topology between the phase dynamics is all-to-all, while that of the radial elements is determined by the entries in the matrix A. For the case of Fig. 1, with bidirectional coupling, the matrix A can be expressed as

$$A = \begin{bmatrix} 0 & 1+\delta & 0 & 0 & \dots & 1-\delta \\ 1-\delta & 0 & 1+\delta & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \dots & \vdots \\ \vdots & \dots & & \ddots & \ddots & 0 \\ 1-\delta & \dots & \dots & & 0 & 1+\delta \\ 1+\delta & 0 & \dots & & 1-\delta & 0 \end{bmatrix}.$$

The network Eq. (1) has a trivial equilibrium $r_j = 0$ and a symmetric equilibrium $r_j = 1$, $j = 1, \dots, N$, which corresponds to synchronous oscillations with amplitude one, and phase $\theta_j = \theta_0 + \omega t$. The primary interest in this manuscript is in the synchronous state.

Computer simulations with a network of $N = 7$ oscillators, were reported in Ref. [43]. The simulations revealed that when the oscillators are homogeneous, i.e., identical values of the b_j parameters, the synchronous state, $r_j = 1$, $j = 1, \dots, 7$, is unstable. But when heterogeneity is introduced into the oscillators, i.e., nonidentical values of the parameter b_j, the synchronous state becomes stable. For completeness purposes, computer simulations, see Fig. 2, have been conducted to reproduce the transition from unstable to stable synchronization, as it was just described in Ref. [43].

Subsequently, the authors of Ref. [43] claim that "the stability of the synchronous state can only be supported by nonidentical oscillators". However, additional computer simulations (carried out with a slight change in one of the parameters), show otherwise. Indeed, Fig. 3 shows that a stable synchronous state can also be supported by the same network of $N = 7$ homogeneous oscillators, which had previously led to an unstable synchronous state in Fig. 2(left).

One is then left to wonder about the source of the discrepancy. Here is a hint. All parameter values used in Fig. 3 are the same as those used in Fig. 2(left), except for a slightly lower

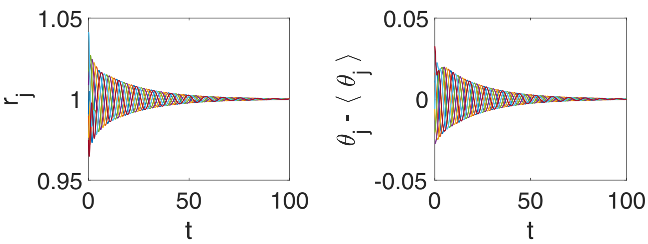

FIG. 3. Opposite behavior, i.e., stable synchronization, to the one observed in Fig. 2(left). Same asymmetrically coupled network with $N = 7$ homogeneous oscillators and same parameters, as the one used to produce Fig. 2(left), except for a lower value of the phase coupling strength, $\gamma = 0.07$. This shows that an asymmetrically coupled network of homogeneous oscillators can support stable synchronization.

value of the phase coupling strength, e.g., $\gamma = 0.07$, instead of $\gamma = 0.1$. This provides a clue for the source of discrepancies, which are investigated in more detail in the next section.

III. SYNCHRONIZATION WITH HOMOGENEOUS AND HETEROGENEOUS OSCILLATORS

In this section, a perturbation analysis of the synchronization state in a network of N oscillators is conducted. Both type of oscillators, heterogeneous and homogeneous, are considered. In both cases, the analysis shows that the stability properties of the synchronous state are related to a simple Hopf bifurcation. The analysis yields a threshold value of the phase-coupling strength, γ_c, which is associated with the Hopf bifurcation point. When $\gamma < \gamma_c$, the synchronous state is stable. At $\gamma = \gamma_c$, a Hopf bifurcation occurs, which leads to periodic orbits in the amplitude dynamics. These periodic orbits correspond to quasiperiodic oscillations in Cartesian coordinates. Past the Hopf bifurcation point, the synchronous state becomes unstable. When the growth rate parameters, b_j, become all equal, the critical Hopf bifurcation parameter, γ_c, for a network with heterogenous oscillators reduces to that of homogenous oscillators.

A. Perturbation Analysis

As a case study, a network with $N = 3$ heterogenous oscillators is considered first. In this case, the matrix, A, of coupling connectivity is a circular matrix given by

$$A = \begin{bmatrix} 0 & 1+\delta & 1-\delta \\ 1-\delta & 0 & 1+\delta \\ 1+\delta & 1-\delta & 0 \end{bmatrix}.$$

Let $\phi_{12} = \theta_2 - \theta_1$, $\phi_{13} = \theta_3 - \theta_1$, and $\phi_{23} = \theta_3 - \theta_2$ represent the phase differences among all three oscillators. In these coordinates, the original model Eq. (1) (after substituting A) can be written as

$$\begin{aligned} \dot{r}_1 &= b_1 r_1(1 - r_1) + \varepsilon r_1[(1+\delta)s_{12} + (1-\delta)s_{13}], \\ \dot{r}_2 &= b_2 r_2(1 - r_2) + \varepsilon r_2[(1+\delta)s_{23} - (1-\delta)s_{12}], \\ \dot{r}_3 &= b_3 r_3(1 - r_3) - \varepsilon r_3[(1+\delta)s_{13} + (1-\delta)s_{23}], \\ \dot{\phi}_{12} &= r_2 - r_1 - \gamma r_2(s_{23} - s_{12}) + \gamma r_1(s_{12} + s_{13}), \\ \dot{\phi}_{13} &= r_3 - r_1 + \gamma r_3(s_{13} + s_{23}) + \gamma r_1(s_{12} + s_{13}), \\ \dot{\phi}_{23} &= r_3 - r_2 + \gamma r_2(s_{13} + s_{23}) + \gamma r_2(s_{23} - s_{12}), \end{aligned} \tag{2}$$

where, for brevity, the notation $s_{jk} = \sin\phi_{jk}$ is used. When $r_j = 0$, the right-hand side of Eq. (2) vanishes identically. Consequently, $r_j = 0$, $\phi_{jk} = \phi^*_{jk}$ is an equilibrium, where the phase differences, ϕ^*_{jk}, are all arbitrary constants, which can be set by initial conditions. Regardless of the actual values, the system does not oscillate since the amplitudes are all equal to zero. For simplicity, we can choose $r_j = 0$, $\phi_{jk} = 0$ as the trivial equilibrium.

The linearization of Eq. (2) about the trivial equilibrium, $(r_j = 0, \phi_{jk} = 0)$, leads to a 6×6 Jacobian matrix, where three of its eigenvalues are zero, while the other three eigenvalues are equal to b_j. It follows that the trivial solution is stable when $b_j < 0$ and unstable when $b_j > 0$. Since the interest is in the synchronization state, the growth rate parameters are assumed to be $b_j > 0$. The synchronous state is $r_j = 1$, $j = 1, 2, 3$, $\phi_{12} = \phi_{13} = \phi_{23} = 0$.

To study the stability properties of the synchronized solution, $(r_j, \phi_{jk}) = (1, 0)$, small perturbations of amplitudes and phase differences are considered through $r_j = 1 + \eta_{r_j}$ and $\phi_{jk} = 0 + \eta_{\phi_{jk}}$, where $\eta_{r_j} \ll 1$ and $\eta_{\phi_{jk}} \ll 1$. Substituting into Eq. (2), the dynamics for the amplitude components is given by

$$\begin{aligned} \dot{\eta}_{r_1} &= -b_1(1 + \eta_{r_1})\eta_{r_1} + \varepsilon[(1+\delta)s_{\eta_{12}} + (1-\delta)s_{\eta_{13}}]\eta_{r_1} \\ &\quad \times \varepsilon[(1+\delta)s_{\eta_{12}} + (1-\delta)s_{\eta_{13}}], \\ \dot{\eta}_{r_2} &= -b_2(1 + \eta_{r_2})\eta_{r_2} + \varepsilon[(1+\delta)s_{\eta_{23}} - (1-\delta)s_{\eta_{12}}]\eta_{r_2} \\ &\quad \times \varepsilon[(1+\delta)s_{\eta_{23}} - (1-\delta)s_{\eta_{12}}], \\ \dot{\eta}_{r_3} &= -b_3(1 + \eta_{r_3})\eta_{r_3} + \varepsilon[(1+\delta)s_{\eta_{13}} + (1-\delta)s_{\eta_{23}}]\eta_{r_3} \\ &\quad \times \varepsilon[(1+\delta)s_{\eta_{13}} + (1-\delta)s_{\eta_{23}}], \end{aligned}$$

where $s_{\eta_{jk}} = \sin\eta_{\phi_{jk}}$. Observe that the last term in each of the equations above is bounded because the sine functions are bounded. Thus, the coefficients of the amplitudes, η_{r_j}, can be compared directly, and conclude that if $b_j \gg \varepsilon[\pm(1+\delta)s_{\eta_{jk}} \pm (1-\delta)s_{\eta_{lm}}]$, where $jk, lm \in \{12, 13, 23\}$, then the system is significantly damped. The term in brackets is bounded by 2δ, so that the damping condition becomes $b_j \gg 2\varepsilon\delta$. When this condition is satisfied, it can be assumed that $\dot{\eta}_{r_j} = 0$, then we can solve for η_{r_j}, up to first order, to get

$$\begin{aligned} \eta_{r_1} &= \frac{\varepsilon}{b_1}[(1+\delta)\eta_{\phi_{12}} + (1-\delta)\eta_{\phi_{13}}], \\ \eta_{r_2} &= \frac{\varepsilon}{b_2}[(1+\delta)\eta_{\phi_{23}} - (1-\delta)\eta_{\phi_{12}}], \\ \eta_{r_3} &= \frac{\varepsilon}{b_3}[(1+\delta)\eta_{\phi_{13}} + (1-\delta)\eta_{\phi_{23}}], \end{aligned}$$

where the fact that $\sin\phi_{jk} \approx \phi_{jk}$ was used. Substituting η_{r_j} back into Eq. (2), yields a set of equations in which the perturbations, $\eta_{\phi_{jk}}$, of the phase difference, decouple from those of the radial components. Up to first-order, the amplitude equations are

$$\begin{aligned} \dot{\eta}_{r_1} &= -b_1\eta_{r_1} + \varepsilon[(1+\delta)\eta_{\phi_{12}} + (1-\delta)\eta_{\phi_{13}}], \\ \dot{\eta}_{r_2} &= -b_2\eta_{r_2} + \varepsilon[(1+\delta)\eta_{\phi_{23}} - (1-\delta)\eta_{\phi_{12}}], \\ \dot{\eta}_{r_3} &= -b_3\eta_{r_3} + \varepsilon[-(1+\delta)\eta_{\phi_{13}} - (1-\delta)\eta_{\phi_{23}}], \end{aligned} \tag{3}$$

while the phase dynamics can be written in matrix form

$$\begin{bmatrix} \dot{\eta}_{\phi_{12}} \\ \dot{\eta}_{\phi_{13}} \\ \dot{\eta}_{\phi_{23}} \end{bmatrix} = J_{\eta_\phi} \begin{bmatrix} \eta_{\phi_{12}} \\ \eta_{\phi_{13}} \\ \eta_{\phi_{23}} \end{bmatrix}, \tag{4}$$

where the matrix J_{η_ϕ} is a 3×3 matrix with components

$$\begin{aligned}
J_{11} &= 2\gamma - \frac{\varepsilon}{b_1}(1+\delta) - \frac{\varepsilon}{b_2}(1+\delta),\\
J_{12} &= \gamma - \frac{\varepsilon}{b_1}(1-\delta),\\
J_{13} &= -\gamma + \frac{\varepsilon}{b_2}(1+\delta),\\
J_{21} &= \gamma - \frac{\varepsilon}{b_1}(1+\delta),\\
J_{22} &= 2\gamma - \frac{\varepsilon}{b_1}(1-\delta) - \frac{\varepsilon}{b_3}(1+\delta)\\
J_{23} &= \gamma - \frac{\varepsilon}{b_3}(1-\delta)\\
J_{31} &= -\gamma + \frac{\varepsilon}{b_2}(1-\delta)\\
J_{32} &= \gamma - \frac{\varepsilon}{b_3}(1+\delta)\\
J_{33} &= 2\gamma - \frac{\varepsilon}{b_2}(1+\delta) - \frac{\varepsilon}{b_3}(1-\delta).
\end{aligned}$$

Observe that Eq. (4) represents a linear system of differential equations. Direct calculations show that the eigenvalues of the matrix J_{η_ϕ} are

$$\sigma_1 = 0, \qquad \sigma_{23} = 3\gamma - \left(\frac{1}{b_1} + \frac{1}{b_2} + \frac{1}{b_3}\right)\varepsilon \pm \sqrt{\left(\frac{1}{b_1^2} + \frac{1}{b_2^2} + \frac{1}{b_3^2}\right) - \left(\frac{1}{b_2 b_3} + \frac{1}{b_1 b_3} + \frac{1}{b_1 b_2}\right)(1+\delta^2)}\ \varepsilon.$$

The zero eigenvalue corresponds to a neutrally stable direction, while the remaining two eigenvalues are the ones that determine the stability properties of the synchronization state $(r_j, \phi_{jk}) = (1, 0)$. Indeed, for sufficiently large δ, so that

$$\Delta = \left(\frac{1}{b_1^2} + \frac{1}{b_2^2} + \frac{1}{b_3^2}\right) - \left(\frac{1}{b_2 b_3} + \frac{1}{b_1 b_3} + \frac{1}{b_1 b_2}\right)(1+\delta^2) < 0,$$

the eigenvalues $\sigma_{2,3}$ are the complex conjugate of the form $\sigma_{23} = p \pm qi$, where $p = 3\gamma - (1/b_1 + 1/b_2 + 1/b_3)\varepsilon$ and $q = \sqrt{-\Delta}$. When $p < 0$, the perturbations of the phase differences, $\eta_{\phi_{jk}}$, decay towards zero, which imply that the zero phase differences, $\phi_{jk} = 0$, form a stable equilibrium. A stable zero phase difference, in turn, implies that the dynamics of the perturbations of the amplitude components simplifies to

$$\dot{\eta}_{r_j} = -b_j \eta_{r_j}.$$

Since $b_j > 0$, for $j = 1, 2, 3$, it follows that $\eta_{r_j} \to 0$ as $t \to \infty$. In summary, when $p < 0$, the synchronization state $(r_j, \phi_{jk}) = (1, 0)$ is locally asymptotically stable. When $p = 0$, the synchronous oscillations undergo a Hopf bifurcation at the critical point

$$\gamma_c = \left(\frac{1}{b_1} + \frac{1}{b_2} + \frac{1}{b_3}\right)\frac{\varepsilon}{3}. \tag{5}$$

To confirm these results, computer simulations of the original network Eq. (1) are carried out with $\omega = 1$. Growth rates are set to: $b_1 = 2.5$, $b_2 = 2.5$, $b_3 = 4.5$, and $\varepsilon = 1.0$ and $\delta = 0.5$ so that the damping condition $b_j \gg 2\varepsilon\delta$ is satisfied. This combination of parameter values yields $\sigma_{23} = 3\gamma - 1.022 \pm 0.223i$, with the critical Hopf bifurcation point being $\gamma_c = 0.3407$. Figure 4(left) shows simulations when γ is slightly smaller than γ_c. Since $\mathrm{Re}\{\sigma_{23} < 0\}$, the common equilibrium $r_j = 1$ is stable and, as expected, all amplitudes converge towards this common nontrivial equilibrium point, which corresponds to periodic oscillations in Cartesian coordinates, (x_j, y_j), where $x_j = r_j \cos\theta_j$ and $y_j = r_j \sin\theta_j$. The right plot shows the simulations exactly at the Hopf bifurcation point $\gamma = \gamma_c$. As it can be observed, three distinct periodic orbits in $(r_j, \theta_j - \langle\theta_j\rangle)$ phase space emerge. Now, these periodic orbits correspond to three distinct branches of quasiperiodic oscillations in Cartesian (x_j, y_j) phase space.

When all of the growth rate coefficients, b_j, are identical, i.e., $b_j = b$, for $j = 1, 2, 3$, the eigenvalues of the matrix J_{η_ϕ} become

$$\sigma_1 = 0, \quad \sigma_{2,3} = 3\left(\gamma - \frac{\varepsilon}{b}\right) \pm \varepsilon\frac{\delta}{b}\sqrt{3}i.$$

The critical value for the Hopf bifurcation of an asymmetrically coupled network with $N = 3$ homogeneous oscillators becomes

$$\gamma_c = \frac{\varepsilon}{b}, \tag{6}$$

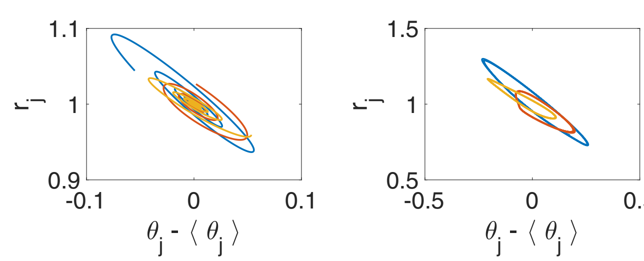

FIG. 4. Amplitude-phase portraits of an asymmetrically coupled, heterogenous, network with $N = 3$ oscillators. Parameters are: $b_1 = 2.5$, $b_2 = 2.5$, $b_3 = 4.5$, $\omega = 1.0$, $\varepsilon = 1.0$, $\delta = 0.5$. With these parameters, a Hopf bifurcation occurs at $\gamma_c = 0.34$. (Left) When $\gamma = 0.32 < \gamma_c$, solution trajectories converge towards a stable equilibrium $r_j = 1$, $\theta_j = 0$, which corresponds to stable synchronized oscillations in Cartesian space. (Right) At the Hopf bifurcation point, $\gamma = \gamma_c$, the synchronization state loses stability and three distinct periodic orbits in (r_j, θ_j) phase space emerge. These periodic orbits correspond to quasiperiodic oscillations in Cartesian phase-space,

which can also be obtained by setting $b_j = b$ in Eq. (5). It follows that when $\gamma < \gamma_c$ then $\mathrm{Re}\{\sigma_{23}\} < 0$, which indicates that synchronized oscillations $(r_j, \phi_{jk}) = (1, 0)$ are stable.

In the case of heterogenous oscillators, the region of parameters space (b_1, b_2, b_3) where $\gamma < (1/b_1 + 1/b_2 + 1/b_3)\varepsilon/3$, corresponds to the blue region of stability of the synchronization state, as it was shown in Fig. 2(b) in Ref. [43]. The shape of the blue region reflects the fact that under the dihedral $\mathbf{D}_3$ symmetry, the network equation remains invariant under cyclic rotations of state variables and parameters. In the case of homogeneous oscillators, that region reduces to the diagonal $b_1 = b_2 = b_3$ where $\gamma < \varepsilon/b$ can be satisfied.

A similar perturbation analysis for a network with $N = 4$ oscillators was performed. Since the phase dynamics, θ_j, $j = 1, \dots, 4$, are all-to-all coupled, the following phase differences can be set: $\phi_{12} = \theta_2 - \theta_1$, $\phi_{13} = \theta_3 - \theta_1$, $\phi_{14} = \theta_4 - \theta_1$, $\phi_{23} = \theta_3 - \theta_2$, $\phi_{24} = \theta_4 - \theta_2$, and $\phi_{34} = \theta_4 - \theta_3$. The procedure is essentially the same as in the case of $N = 3$ oscillators, so details are skipped, while the main results are presented. In this case, the Jacobian matrix of the phase perturbations, J_{η_ϕ}, has three zero eigenvalues, $\sigma_j = 0$, $j = 1, 2, 3$, one real-valued eigenvalue (which must be negative for stability), $\sigma_4 = 4(\gamma - \frac{\varepsilon}{b})$, and eigenvalues $\sigma_{5,6}$, which can lead to a critical Hopf bifurcation point

$$\gamma_c = \left(\frac{1}{b_1} + \frac{1}{b_2} + \frac{1}{b_3} + \frac{1}{b_4}\right)\frac{\varepsilon}{8}.$$

For a network with four homogeneous oscillators, the critical coupling strength for the Hopf bifurcation reduces to

$$\gamma_c = \frac{\varepsilon}{2b}. \tag{7}$$

Thus, if $\gamma < \gamma_c$, which also satisfies $\gamma < \varepsilon/b$, so that $\sigma_4 < 0$, then the synchronization state $(r_j, \theta_j) = (1, 0)$ is stable. Computer simulations (not shown for brevity) show results similar to those of the $N = 3$ case. For instance, consider heterogenous oscillators, with growth rates set to: $b_1 = 2.5$, $b_2 = 3.5$, $b_3 = 5.5$, $b_4 = 6.5$, and $\epsilon = 1$, $\delta = 0.5$, and $\omega = 1$. The simulations reveal that when $\gamma < \gamma_c = 0.132$ then the solution trajectories converge towards the synchronization state $(r_j, \theta_j) = (1, 0)$. At $\gamma = \gamma_c$, the synchronization state loses stability and four distinct periodic orbits emerge. The plots are very similar to those of Fig. 4.

B. Simulations with Homogeneous Oscillators

It should be emphasized that the results above imply that *synchronization can occur in spite of the growth rate parameters, b_j, being identical, i.e., in spite of the oscillators being homogeneous*. This is a critical observation because it contradicts the results published in Ref. [43]. In there, the authors note: "Along the diagonal line $b_1 = \cdots = b_n = b$ no homogeneous oscillators can be stably synchronized". Furthermore, they also write "significant differences between the oscillators are required to achieve stable synchronization." Next, computer simulations are shown in support of the existence and stability of stable synchronization states in asymmetrically coupled homogeneous oscillators.

Figure 5 shows results of computer simulations of Eq. (1) for an asymmetrically coupled network with $N = 3$ homogeneous oscillators. Growth rates are set to: $b_j = 3.0$, $j = 1, 2, 3$, and additional parameters are: $\varepsilon = 1.0$ and $\delta = 0.3$, so that the damping condition, $b_j \gg 2\varepsilon\delta$, is, again, satisfied. The Hopf bifurcation point is now $\gamma_c = 1/3$. For γ slightly smaller than γ_c, it can be seen in Fig. 5 that the amplitude and phase dynamics converge towards the synchronized oscillations $(r_j = 1, \theta_j = 0)$.

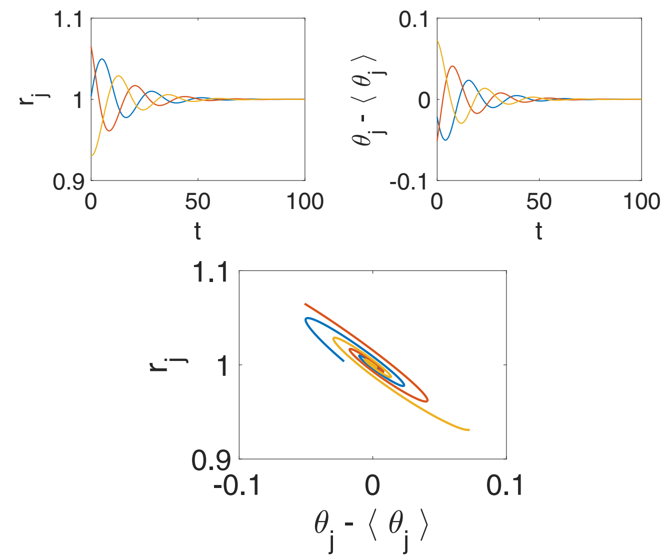

FIG. 5. Synchronization in an asymmetrically coupled network with $N = 3$ homogeneous oscillators. All solution (amplitude and phase) trajectories converge to the synchronization state $(r_j, \theta_j) = (1, 0)$. Parameters are: $b_1 = b_2 = b_3 = 3.0$, $\omega = 1.0$, $\varepsilon = 1.0$, $\delta = 0.3$, $\gamma = 0.31$.

At $\gamma = \gamma_c$, the synchronized solution loses stability and three distinct periodic orbits in (r_j, θ_j) phase space emerge, as is shown in Fig. 6. These periodic orbits correspond to quasiperiodic oscillations in cartesian (x_j, y_j) phase space.

Similar results follow for a network with $N = 4$ homogeneous oscillators. For instance, set $b_j = 4$, $\omega = 1.0$, $\varepsilon = 1.0$, and $\delta = 0.5$. Then the critical Hopf bifurcation point becomes $\gamma_c = 0.125$. For $\gamma < \gamma_c$ the synchronization state is stable, and for $\gamma > \gamma_c$, it becomes unstable. The plots are similar to those shown in Figs. 5 and 6, but they are not shown for brevity.

A few words are now in order for what can be expected for larger networks. The results presented so far suggest that as the size, N, of a network increases, the critical value of γ_c for the Hopf bifurcation that leads to stable synchronization decreases towards zero. In the limit, as $N \to \infty$, the phase-coupling strength, γ, that is required to achieve stable synchronization is zero. In fact, we can observe from the model equations that when $\gamma = 0$, the phase dynamics completely decouples from one another. Furthermore, the real part of the complex eigenvalues associated with the Hopf bifurcation become negative, i.e., $p < 0$, regardless of the actual values of the growth coefficients b_j, so long as they are positive. This means, interestingly, that the synchronization state is always stable, regardless of whether the oscillators are identical or not, i.e., independently of the actual values of the growth coefficients b_j. These observations can also be verified

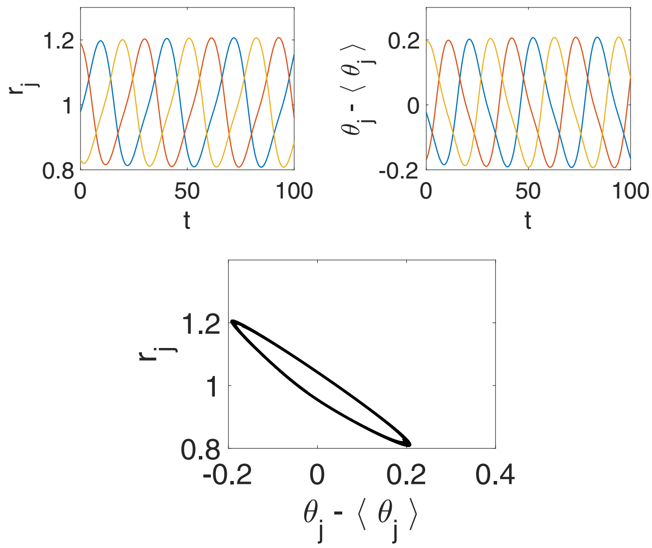

FIG. 6. Amplitude and phase dynamics of an asymmetrically coupled network with $N = 3$ homogeneous oscillators. From top to bottom (respectively): amplitude and phase dynamics, and amplitude-phase portrait. At the critical Hopf bifurcation point, both amplitude and phase oscillate. These oscillations correspond to quasiperiodic solutions in Cartesian coordinates. Parameters are: $b_1 = b_2 = b_3 = 3.0$, $\omega = 1.0$, $\varepsilon = 1.0$, $\delta = 0.3$, $\gamma = 0.33$.

with numerical simulations. In the next section, the stability of the synchronization state is studied from the geometrical point of view of symmetry. An important result is that this approach allows us to find the critical value of the coupling strength that leads to stable synchronization in networks of arbitrary size and with homogeneous oscillators.

IV. THE SYMMETRY APPROACH

In this section, the stability properties of the synchronization state in a network of asymmetrically coupled homogeneous oscillators, are studied from the equivariant, symmetry-breaking, bifurcation point of view. As a case study, the same networks that were investigated in the previous section are now considered, i.e., networks with $N = 3$ and 4 oscillators. The analysis provides additional insight into the symmetry-breaking bifurcations that lead to the existence and stability of the synchronization state. Additionally, the symmetry-based approach confirms the results from the perturbation analysis, including the findings of the critical Hopf bifurcation parameter.

A. Case Study: Three Cells

The network of $N = 3$ asymmetrically coupled oscillators, which was studied in Sec. III, is considered again. But, now, it is assumed that the network is made up of homogeneous oscillators. The model equations can be written in the following form:

$$\begin{aligned}
\dot{r}_1 &= br_1(1-r_1) + \varepsilon r_1[(1+\delta)s_{21} + (1-\delta)s_{31}],\\
\dot{\theta}_1 &= \omega + r_1 - 1 - \gamma r_1(s_{21} + s_{31}),\\
\dot{r}_2 &= br_2(1-r_2) + \varepsilon r_2[(1+\delta)s_{32} + (1-\delta)s_{12}],\\
\dot{\theta}_2 &= \omega + r_2 - 1 - r_1 - \gamma r_2(s_{32} + s_{12}),\\
\dot{r}_3 &= br_3(1-r_3) + \varepsilon r_3[(1+\delta)s_{13} + (1-\delta)s_{23}],\\
\dot{\theta}_3 &= \omega + r_3 - 1 - \gamma r_3(s_{13} + s_{23}),
\end{aligned} \tag{8}$$

where, for brevity, the notation $s_{jk} = \sin(\theta_j - \theta_k)$ is used. Let $z_j = [r_j, \theta_j]^T$, where $j = 1, \ldots, 3$, and $Z = [z_1, z_2, z_3]^T$, so that Eq. (8) can be rewritten as

$$\dot{Z} = F(Z, \mu, \alpha), \tag{9}$$

where $\mu = (\gamma, \delta)$ are the main bifurcation parameters and $\alpha = (b, \omega, \varepsilon)$, and F is of the form

$$F(z_1, z_2, z_3, \mu, \alpha) = \begin{bmatrix} f(z_1, z_2, z_3, \mu, \alpha) \\ f(z_2, z_3, z_1, \mu, \alpha) \\ f(z_3, z_1, z_2, \mu, \alpha) \end{bmatrix},$$

where

$$f(z_1, z_2, z_3, \mu, \alpha) = \begin{bmatrix} b_1 r_1(1-r_1) + \varepsilon r_1 \sum_{k=1}^N A_{1k} \sin(\theta_k - \theta_1) \\ \omega + r_1 - 1 - \gamma r_1 \sum_{k=1}^N \sin(\theta_k - \theta_1) \end{bmatrix}.$$

When $\delta = 0$, the network geometry has $\mathbf{D}_3$ symmetry, where $\mathbf{D}_3$ is the group of symmetries of a triangle. $\mathbf{D}_3$ has two generators: a cyclic rotation of the oscillators, $\beta_1 \cdot (1, 2, 3) \rightarrow (2, 3, 1)$, and a reflection across the midedge of the triangle, $\beta_2 \cdot (1, 2, 3) \rightarrow (1, 3, 2)$. This means that the model Eq. (9) is equivariant under the action of $\mathbf{D}_3$. That is, $F(\beta Z) = \beta F(Z)$, where $\beta = \{\beta_1, \beta_2\}$. When $\delta \neq 0$, however, the network is no longer $\mathbf{D}_3$ symmetric, since it only retains rotational equivariance under β_1 but not under the reflection β_2. Observe that, however, f has a combined (Z, μ, α) symmetry:

$$f(z_1, z_3, z_2, \gamma, \delta, \alpha) = f(z_1, z_2, z_3, \gamma, -\delta, \alpha).$$

This implies that $F(\beta_2 Z, \gamma, \delta, \alpha) = \beta_2 F(Z, \gamma, -\delta, \alpha)$. Now, the linearization of Eq. (9) about the synchronization state, $(r_j, \theta_k - \theta_j) = (1, 0)$, is

$$(dF)_{(1,0)} = \begin{bmatrix} A & B & D \\ D & A & B \\ B & D & A \end{bmatrix},$$

where

$$A = \begin{bmatrix} -b & -2\varepsilon \\ 1 & 2\gamma \end{bmatrix}, \quad B = \begin{bmatrix} 0 & \varepsilon(1+\delta) \\ 0 & -\gamma \end{bmatrix},$$

$$D = \begin{bmatrix} 0 & \varepsilon(1-\delta) \\ 0 & -\gamma \end{bmatrix}.$$

We choose to label the third matrix in the linearization, $(dF)_{(1,0)}$, as D, for consistency purposes with the analysis that will be conducted later on with networks of larger size.

The eigenvalues of $(dF)_{(1,0)}$ are those of

$$\begin{aligned}
&A + B + D \quad \text{with eigenvector } V_1 = [v, v, v]^T,\\
&A + \xi B + \xi^2 D \quad \text{with eigenvector } V_2 = [v, \xi v, \xi^2 v]^T,\\
&A + \xi^2 B + \xi D \quad \text{with eigenvector } V_3 = [v, \xi^2 v, \xi v]^T,
\end{aligned}$$

where $\xi = e^{2\pi i/3} = -\frac{1}{2} + \frac{\sqrt{3}}{2} i$, for some $v \in \mathbf{R}$. The matrices, $A + \xi B + \xi^2 D$ and $A + \xi^2 B + \xi D$, become

$$A + \xi B + \xi^2 D = A - \frac{1}{2}(B + D) + \frac{\sqrt{3}}{2} i(B - D),$$
$$A + \xi^2 B + \xi D = A - \frac{1}{2}(B + D) - \frac{\sqrt{3}}{2} i(B - D).$$

Since both B and D are odd in δ then symmetry-breaking can occur at $\delta = 0$. Indeed, when $\delta = 0$, it can be seen that $B = D$, and the spectrum of eigenvalues is given by

$$\sigma((dF)_{(1,0)}) = \sigma(A + 2B) \cup \sigma(A - B)(\text{twice}).$$

Generically, in order to get a two-dimensional kernel with the required $\mathbf{D}_3$ symmetry, the synchronization state, $(1, 0)$, must lose stability through a (double) zero eigenvalue of $A - B$. Now, the characteristic polynomial associated with $A - B$ is

$$\sigma^2 - (3\gamma - b)\sigma + (-3\gamma b + 3\varepsilon) = 0.$$

It follows that when

$$\gamma = \frac{\varepsilon}{b},$$

then one of the eigenvalues is exactly zero, $\sigma_1 = 0$, while the second one is $\sigma_2 = 3\gamma - b$. Actually, these eigenvalues are double since the matrix $A - B$ occurs twice in the diagonalization of $(dF)_{(1,0)}$. The condition $\gamma = \varepsilon/b$ is the same critical value, see Eq. (6), of the coupling-phase parameter that was found earlier on (through the perturbation analysis). Observe that it was not required for the zero eigenvalue to be purely imaginary and for it to cross the imaginary axis to the right, as it is commonly needed for a Hopf bifurcation to occur. The reason is that the state variable in the model Eq. (9) is in polar coordinates, (r_j, θ). Thus a steady-state symmetry-breaking bifurcation is necessary and sufficient, and it is associated with a Hopf bifurcation in Cartesian coordinates (x_j, y_j).

B. Case Study: Four Cells

The case of a network with $N = 4$ homogeneous oscillators is considered next. The model for this network is still of the form given by Eq. (9), except that now $Z = [z_1, z_2, z_3, z_4]^T$. This time, when $\delta = 0$, the network geometry has $\mathbf{D}_4$ symmetry, where $\mathbf{D}_4$ is the group of symmetries of a square. $\mathbf{D}_4$ has two generators: a midedge symmetry, represented by the permutation $\beta_1 \cdot (1, 2, 3, 4) \to (2, 1, 4, 3)$, and a diagonal symmetry, given by $\beta_2 \cdot (1, 2, 3, 4) \to (3, 2, 1, 4)$. The generators are not unique, so a cyclic rotation could have also being used. Nevertheless, the results are the same. Observe that when $\delta \neq 0$ the network loses $\mathbf{D}_4$ symmetry. Furthermore, the network topology suggests that it can be assumed that the symmetry breaking occurs with the nearest-neighbor interactions equal only on opposite pairs of edges. Under this assumption, when $\delta \neq 0$, it can be expected for the representation $\mathbf{Z}_2 \oplus \mathbf{Z}_2$ (symmetry of a rectangle) to occur generically. This representation is generated by $\langle \beta_1, \beta_2\beta_1\beta_2 \rangle$. Equivariance of F under these two generators, i.e., $F(\beta Z) = \beta F(Z)$, where $\beta = \{\beta_1, \beta_2\}$, leads to the following form for F:

$$F(z_1, z_2, z_3, z_4, \mu, \alpha) = \begin{bmatrix} f(z_1, z_2, z_3, z_4, \mu, \alpha) \\ f(z_2, z_1, z_4, z_3, \mu, \alpha) \\ f(z_3, z_4, z_1, z_2, \mu, \alpha) \\ f(z_4, z_3, z_2, z_1, \mu, \alpha) \end{bmatrix},$$

where $f(z_1, z_2, z_3, z_4, \gamma, \delta, \alpha) = f(z_1, z_4, z_3, z_2, \gamma, -\delta, \alpha)$ is defined in the same way as in the previous case of $N = 3$. The linearization of Eq. (9) about the synchronization state, $(r_j, \theta_k - \theta_j) = (1, 0)$, is

$$(dF)_{(1,0)} = \begin{bmatrix} A & B & C & D \\ D & A & B & C \\ C & D & A & B \\ B & C & D & A \end{bmatrix},$$

where

$$A = \begin{bmatrix} -b & -2\varepsilon \\ 1 & 3\gamma \end{bmatrix}, \quad B = \begin{bmatrix} 0 & \varepsilon(1 + \delta) \\ 0 & -\gamma \end{bmatrix},$$
$$C = \begin{bmatrix} 0 & 0 \\ 0 & -\gamma \end{bmatrix}, \quad D = \begin{bmatrix} 0 & \varepsilon(1 - \delta) \\ 0 & -\gamma \end{bmatrix}.$$

The eigenvalues of $(dF)_{(1,0)}$ are those of

$$A + B + C + D,$$
$$A + \xi B + \xi^2 C + \xi^3 D,$$
$$A + \xi^2 B + C + \xi^2 D,$$
$$A + \xi^3 B + \xi^2 C + \xi D,$$

with eigenvectors

$$V_1 = [v, v, v, v]^T,$$
$$V_2 = [v, \xi v, \xi^2 v, \xi^3 v]^T,$$
$$V_3 = [v, \xi^2 v, v, \xi^2 v]^T,$$
$$V_4 = [v, \xi^3 v, \xi^2 v, \xi v]^T,$$

respectively, where $\xi = e^{2\pi i/4} = i$, for some $v \in \mathbf{R}$. The last three matrices of the eigenvalues become

$$A + \xi B + \xi^2 C + \xi^3 D = A - C + (B - D)i,$$
$$A + \xi^2 B + C + \xi^2 D = A + C - (B + D),$$
$$A + \xi^3 B + \xi^2 C + \xi D = A - C - (B - D)i.$$

Since both B and D are odd in δ, then symmetry-breaking can occur at $\delta = 0$. Indeed, when $\delta = 0$, it can be seen that $B = D$, and the spectrum of eigenvalues is given by

$$\sigma((dF)_{(1,0)}) = \sigma(A + C + 2B) \cup \sigma(A + C - 2B) \cup \sigma(A - C)(\text{twice}).$$

Generically, in order to get a two-dimensional kernel with the required $\mathbf{D}_4$ symmetry, the synchronization state, $(1, 0)$, must lose stability through a (double) zero eigenvalue of $A - C$. Now, the characteristic polynomial associated with $A - C$ is

$$\sigma^2 - (4\gamma - b)\sigma + (-4\gamma b + 2\varepsilon) = 0.$$

It follows that when

$$\gamma = \frac{\varepsilon}{2b},$$

then one of the eigenvalues is exactly zero, $\sigma_1 = 0$, while the second one is $\sigma_2 = 4\gamma - b$. Again, these eigenvalues are double since the matrix $A - C$ appears twice in the diagonalization of $(dF)_{(1,0)}$. Also, the condition $\gamma = \varepsilon/(2b)$, is the same critical value of the coupling-phase parameter that was found earlier on in Eq. (7), through the perturbation analysis.

C. Generalization to Larger Networks

In this section the stability properties of the synchronization state are investigated in networks of arbitrary size N. In particular, networks of size $N > 3$ are considered. The model for the network continues to be of the form given by Eq. (9), except that now $Z = [z_1, z_2, \dots, z_N]^T$. Once again, when $\delta = 0$, the network geometry has $\mathbf{D}_N$ symmetry, where $\mathbf{D}_N$ is the group of symmetries of an N-gon.

The dihedral group, $\mathbf{D}_N$, has two generators, the cyclic permutation $\beta_1 = (1\,2\,\dots\,N)$, which is defined by

$$\beta_1 \cdot Z = \left(z_{\beta_1^{-1}(1)}, \dots, z_{\beta_1^{-1}(N)}\right),$$

and a transposition defined by

$$\beta_2 = \begin{cases} (1\,2)(3\,N)\cdots(3+j,\, N-j)\cdots \\ \left(3+\left(\lfloor \frac{N}{2}\rfloor - 1\right),\, N-\left(\lfloor \frac{N}{2}\rfloor - 1\right)\right) & N \text{ odd}, \\ (1\,2)(3\,N)\cdots(3+j,\, N-j)\cdots \\ \left(3+\lfloor \frac{N}{2}\rfloor,\, N-\lfloor \frac{N}{2}\rfloor\right) & N \text{ even}, \end{cases}$$

where $\lfloor x \rfloor$ is the floor function. Together β_1 and β_2 act on the phase space Z as

$$\beta_1 \cdot Z = \left(z_{\beta_1^{-1}(1)}, \dots, z_{\beta_1^{-1}(N)}\right),$$
$$\beta_2 \cdot Z = \left(z_{\beta_2^{-1}(1)}, \dots, z_{\beta_2^{-1}(N)}\right).$$

Since the model Eq. (9) is equivariant under this action of $\mathbf{D}_N$, then $F(\beta Z) = \beta F(Z)$, where $\beta = \{\beta_1, \beta_2\}$. As it was discussed earlier with the special cases of $N = 3$ and $N = 4$, when $\delta \neq 0$, however, the network is no longer $\mathbf{D}_N$ symmetric, since it only retains rotational equivariance under β_1 but not under the transposition β_2. Nevertheless, the model equations retain a combined (Z, μ, α) symmetry: $F(\beta_2 Z, \gamma, \delta, \alpha) = \beta_2 F(Z, \gamma, -\delta, \alpha)$. The linearization of Eq. (9) about the synchronization state, $(r_j, \theta_k - \theta_j) = (1, 0)$, is

$$(dF)_{(1,0)} = \begin{bmatrix} A & B & C & C & \dots & C & D \\ D & A & B & C & \dots & C & C \\ C & D & A & B & C & \dots & C \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ C & C & \dots & C & D & A & B \\ B & C & C & \dots & C & D & A \end{bmatrix},$$

where

$$A = \begin{bmatrix} -b & -2\varepsilon \\ 1 & (N-1)\gamma \end{bmatrix}, \quad B = \begin{bmatrix} 0 & \varepsilon(1+\delta) \\ 0 & -\gamma \end{bmatrix},$$
$$C = \begin{bmatrix} 0 & 0 \\ 0 & -\gamma \end{bmatrix}, \quad D = \begin{bmatrix} 0 & \varepsilon(1-\delta) \\ 0 & -\gamma \end{bmatrix}.$$

Observe that the case of $N = 3$ is special since it does not contain the matrix C in the linearization of the model dynamics. The case $N = 4$, and beyond, do have, however, the generalized structured of the linearization shown above. Also, observe that the matrix A is the same as before except that now the term with the coupling strength, γ, contains a scaling factor of $N - 1$.

To study the linearized matrix $(dF)_{(1,0)}$, we employ the well-known isotypic decomposition of $\mathbf{C}^N$ by $\mathbf{D}_N$, which is given by

$$\mathbf{C}^N = V_0 \oplus V_1 \oplus \cdots \oplus V_{N-1},$$

where $V_j = \mathbf{C}\{v_j\}$, with

$$v_j = (v, \xi^j v, \xi^{2j} v, \dots, \xi^{(N-1)j} v)^T,$$

$j = 0, \dots, N-1$, and $\xi = \exp(2\pi i/N)$, for some $v \in \mathbf{R}$. Using coordinates along the isotypic components, we find the eigenvalues of $(dF)_{(1,0)}$ to be those of L_j, where

$$L_j = A + \xi^j B + \xi^{2j} C + \xi^{3j} C + \dots + \xi^{(N-1)j} D,$$

where $j = 0, 1, \dots, N-1$. Since both B and D are odd in δ, then symmetry-breaking can occur when $\delta = 0$, which leads to $B = D$. Using the fact that $\xi^{n-j} = \overline{\xi^j}$, we get

$$L_j = A + (\xi^j + \overline{\xi^j})B + (\xi^{2j} + \overline{\xi^{2j}})C + (\xi^{3j} + \overline{\xi^{3j}})C + \dots + \left(\xi^{\frac{N-1}{2}j} + \overline{\xi^{\frac{N-1}{2}j}}\right)C, \quad (N > 3 \text{ odd})$$
$$L_j = A + (\xi^j + \overline{\xi^j})B + (\xi^{2j} + \overline{\xi^{2j}})C + (\xi^{3j} + \overline{\xi^{3j}})C \dots + \left(\xi^{\frac{N-2}{2}j} + \overline{\xi^{\frac{N-2}{2}j}}\right)C + \xi^{\frac{N}{2}j}C, \ (N > 2 \text{ even}).$$

These expressions can be rewritten as

$$L_j = A + p_j B + q_j C \quad (N > 3 \text{ odd}),$$
$$L_j = A + p_j B + q_j C + (-1)^j C, \quad (N > 2 \text{ even}),$$

where $p_j = 2\cos\left(\frac{2\pi}{N}j\right)$, and

$$q_j = 2\cos\left(\frac{2\pi}{N}2j\right) + \dots + 2\cos\left(\frac{2\pi}{N}\frac{N-1}{2}j\right), \quad (N > 3 \text{ odd})m$$
$$q_j = 2\cos\left(\frac{2\pi}{N}2j\right) + \dots + 2\cos\left(\frac{2\pi}{N}\frac{N-2}{2}j\right), \quad (N > 2 \text{ even}),$$

with the assumption that when $N = 4$, then $q_j = 0$. The characteristic polynomial associated with the linearized matrices, L_j, for $N > 3$ (odd), is

$$\sigma_j^2 - [(N-1-(p_j+q_j))\gamma - b]\sigma_j - b(N-1-(p_j+q_j))\gamma + (2-p_j)\varepsilon = 0, \tag{10}$$

and for $N > 2$ (even), we get

$$\sigma_j^2 - [(N-1-(-1)^j-(p_j+q_j))\gamma - b]\sigma_j - b(N-1-(-1)^j-(p_j+q_j))\gamma + (2-p_j)\varepsilon = 0. \tag{11}$$

Since the coefficients q_j can be rewritten as

$$q_j = p_{2j} + p_{3j} + \dots + p_{\frac{N-1}{2}j} \quad (N > 3 \text{ odd}),$$
$$q_j = p_{2j} + p_{3j} + \dots + p_{\frac{N-2}{2}j}, \quad (N > 2 \text{ even}),$$

then direct calculations reveal that when $N > 3$ (odd) we get

$$p_j + q_j = \begin{cases} N-1, & j=0 \\ -1, & j=1,\dots,N-1, \end{cases}$$

while in the $N > 2$ (even) case we have

$$p_j + q_j = \begin{cases} N-2, & j=0, \\ -2+2\left(\frac{N \mod 4}{2}\right), & j=\frac{N}{2}, \\ 0, & j \text{ odd}, \\ -2, & j \text{ even},\ \ j \neq 0, N/2. \end{cases}$$

Then, the full spectrum of eigenvalues can be obtained by substituting $p_i + q_i$ into Eq. (10) and Eq. (11), and then solving for σ_j. However, since $\cos\left(-\frac{2\pi}{N}j\right) = \cos\left(\frac{2\pi}{N}j\right)$, then the matrices, L_j, appear in pairs, i.e., $L_j = L_{N-j}$, except for $j = 0$, (N odd) and $j = 0, N/2$, (N even). Thus, to get the eigenvalues of multiplicity two (especially the zero eigenvalue), we need to focus only on $j = 1, \dots, \lfloor N/2 \rfloor$ for N odd, and $j = 1, \dots, \frac{N}{2} - 1$ for N even. In both cases, N odd and even, the characteristic polynomial associated with multiple eigenvalues reduces to

$$\sigma_j^2 - (N\gamma - b)\sigma_j - Nb\gamma + (2 - p_j)\varepsilon = 0. \qquad (12)$$

The condition of having a double zero eigenvalue leads to the following critical values of the coupling strength:

$$\gamma_c = \frac{2 - p_j}{Nb}\varepsilon, \qquad (13)$$

which is valid for all values of $N > 3$. The actual value of p_j depends on the representation of the dihedral group $\mathbf{D}_N$. The analysis that was presented earlier for the cases $N = 3$ and $N = 4$ assumes the standard representation in which $j = 1$, with generators (β_1, β_2), where β_1 is the cyclic rotation generator, while β_2 is the transposition generator. Thus, if we continue to assume the standard representation of $\mathbf{D}_N$, then $p_j = p_1$ and the critical value of the coupling strength associated with the Hopf bifurcation that leads to stable synchronization in asymmetrically connected networks with homogeneous oscillators becomes

$$\gamma_c = \frac{1}{Nb}\left(2 - \cos\left(\frac{2\pi}{N}\right)\right)\varepsilon. \qquad (14)$$

Observe that substituting $N = 4$ into Eq. (14) yields the previously found critical value of $\gamma_c = \varepsilon/(2b)$. For the case of $N = 7$ oscillators, which was explored (at the beginning of the manuscript in Sec. II) through numerical simulations, see Fig. 3, we get $\gamma_c \approx 0.1\varepsilon/b$. In those simulations, we considered $\varepsilon = 2.0$ and $b = 1.868$, which yield $\gamma_c \approx 0.1$. Then, $\gamma = 0.07$ was used to generate Fig. 3.

V. DISCUSSION

In this manuscript, it has been proved that networks of asymmetrically coupled homogeneous oscillators can support symmetric states, such as stable synchronization. These results contradict previous findings [43], where a network with homogeneous oscillators produced unstable synchronization, and, subsequently, used those findings to conclude that such states can only be found to be stable in asymmetrically coupled networks with heterogeneous oscillators. Perturbation analysis and equivariant bifurcation theory have been used to show that synchronization states can, and do, exist in both types of networks. The analysis also shows that the stability of this symmetric state is governed by a Hopf bifurcation. A critical parameter for the Hopf bifurcation was found for asymmetrically coupled networks, with nonhomogeneous and homogeneous oscillators. In the latter case, the results have been generalized to networks of arbitrary size. The generalization reveals that in the limit, as $N \to \infty$, the phase-coupling strength, γ_c, required for achieving stable synchronization is zero. It follows that when $\gamma = 0$ the synchronization state is stable regardless of whether the network contains homogeneous or heterogeneous oscillators.

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