Average search time bounds in cue-based searches

Vaibhav Wasnik*

Indian Institute of Technology Goa, Ponda 403401, Goa, India

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In this work we consider search problems that evaluate the probability distribution of finding the source at each step in the search. We start with a sample strategy where the movement at each time step is in the immediate neighborhood. The jump probability is taken to be proportional to the normalized difference between the probability of finding the source at the jump location with the probability of finding the source at the present location. We evaluate a lower bound on the average search time for a searcher using this strategy. We next consider the problem of evaluating the lower bound on the search time for a generic strategy which would utilize the source probability distribution to figure out the position of the source. We derive an expression for the lower bound on the search time. We present an analytic expression for this lower bound in a case in which the particles emitted by the source diffuse in a homogeneous manner. For a general probability distribution with entropy E, we find that the lower bound goes as $e^{E/2}$.

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I. INTRODUCTION

Searching for a source that emits particles is a problem that is quite ubiquitous. We see this all the way from a bacteria searching for the source of chemoattractants [1], to a robot figuring out the source of a gas leak in a room [2]. Search time is defined as the time required to find the source by a searcher. This is similar to the first passage time: the first time the searcher reaches the position occupied by the source. There is a lot of theoretical work done in this area [3]. One could classify search strategies into two broad categories: searches with cues and searches without cues. Searches without cues involve intermittent search strategies such as [4–7]. Cue-based searches can be divided into two categories. On one end we have chemotactic search strategies that measure concentration gradients where the concentrations involved are much larger than their fluctuations. The signal-to-noise ratio depends on time averaging and can be improved by increased waiting times. This is the realm of searches done by eukaryotic organisms. Robot tracking chemotactic [8–10] or plume tracking strategies [11–13] also are designed for high concentration environments. In the case in which the signal-to-noise ratio is weak, the waiting times are quite large. Moths are known to work in this regime to locate their mates through prehormones [14–17]. Robots that work in these realms where signal-to-noise ratio is weak are considered in [18–20]. In such low signal-to-noise ratio environments, the measurements involve hits from signaling molecules given out by the source, followed by time intervals without hits. A search strategy presented in literature that utilizes the information from sporadic hits detected by the searcher to guess the position of the source, is infotaxis [21,22].

A searcher moving through an environment of particles emitted by a source has a history of hits at times $t_1, \ldots, t_n$ at positions $\vec{r}(t_1), \ldots, \vec{r}(t_n)$. These make up the cues that provide all the information from the environment. This information could be utilized in deciding a future direction of motion in many ways. One important quantity that could be measured is the probability of finding the source at any location in space. One could use Bayes' theorem to evaluate this as

$$P[\vec{r}|\vec{r}(t_1), \ldots, \vec{r}(t_n)] = \frac{P[\vec{r}(t_1), \ldots, \vec{r}(t_n)|\vec{r}]}{\sum_x P[\vec{r}(t_1), \ldots, \vec{r}(t_n)|\vec{x}]}. \quad (1)$$

Here $P[\vec{r}|\vec{r}(t_1), \ldots, \vec{r}(t_n)]$ is the probability of finding the source at position $\vec{r}$ given hits at positions $\vec{r}(t_1), \ldots, \vec{r}(t_n)$ and $P[\vec{r}(t_1), \ldots, \vec{r}(t_n)|\vec{x}]$ is the probability of hits happening at positions $\vec{r}(t_1), \ldots, \vec{r}(t_n)$ given the source is at position $\vec{x}$. Infotaxis [21] utilizes this probability to evaluate the entropy of the source. The motion of the searcher at each step isin a direction in which the expected information gain is a maximum. Given the complexity of the search algorithm, evaluating the search time analytically for a searcher undergoing infotaxis is difficult. Given this issue, the question arises whether it would be possible to evaluate the search times for a class of cue-based searches, and any statement be made about certain universal features such as lower bound on these search times.

Evaluation of lower bounds are ubiquitously important in any problem related to measurement. Take the case of theoretically evaluated lower bounds on errors in measurement of chemoattractant concentration [23], chemoattractant gradients [24] by living cells, which show that cells approach an optimum design for these measurements. Similarly, lower bounds on search times are an important aspect of study in the sciences. For example, a lower bound for query time in nearest neighbor searches was investigated in [25,26]. Lower bound on search time during randomized searching on m rays was studied in [27]. Lower bounds on search times in search strategies involving a searcher figuring out the searchers position based on particles emitted by the source are useful, because they could access the effectiveness of machines constructed to do these searches. In past works the only lower bounds evaluation using the probability of finding the source by the

*wasnik@iitgoa.ac.in

searcher was done in [21]. In that work the search time was bounded by $e^{|S|}$, where S is the entropy of the probability distribution of the source position. In this work we begin with a strategy that utilizes past cues to evaluate the probability distribution of finding the source at each step. The searcher at each step moves in an immediate neighborhood location, with a probability proportional to the normalized difference in the probabilities (of finding the source), evaluated at the present and the immediate neighborhood location. We then evaluate a lower bound on the search times in the case of homogeneous diffusion of particles emitted by a source, as a function of distance of the searcher from the source. We next consider the problem of evaluating the lower bound on the search time for a generic strategy which would utilize information in the source probability distribution to figure out the position of the source. We derive an expression for the lower bound on the search time. We evaluate an analytical expression for this lower bound in the case of homogeneous diffusion of particles emitted by the source.

II. NARROWING THE SOURCE

Let us assume that the source emitting particles is located at the origin. Assume the searcher has traversed a particular trajectory, such that the position of the searcher as a function of time t' for this trajectory is $r(t')$. The searcher travels for a total time t, with $\vec{r}(t)=\vec{r}_f$ and $\vec{r}(0)=\vec{r}_i$, respectively being the initial and final positions of the searcher in its trajectory. A searcher moving through an environment of particles emitted by a source has a history of hits at times $t_1,\dots,t_n$, occurring at positions $\vec{r}(t_1),\dots,\vec{r}(t_n)$. We have

$$P[\vec{r}|\vec{r}(t_1),\dots,\vec{r}(t_n)] = \frac{\exp\left\{-\int_0^t P[\vec{r}(t')|\vec{r}]\right\}dt' P[\vec{r}(t_1),\dots,\vec{r}(t_n)|\vec{r}]}{\sum_x \exp\left\{-\int_0^t P[\vec{r}(t')|\vec{x}]\right\}dt' P[\vec{r}(t_1),\dots,\vec{r}(t_n)|\vec{x}]}. \quad (2)$$

Here, $P[\vec{r}(t_1),\dots,\vec{r}(t_n)|\vec{r}]$ is the probability of having hits at positions $\vec{r}(t_1),\dots,\vec{r}(t_n)$, given the source is at position $\vec{r}$. The exponentials correspond to the probability of no hits happening at the other locations along the trajectory. Because the hits are independent of each other and can happen at any time, we have

$$P[\vec{r}(t_1),\dots,\vec{r}(t_n)|\vec{r}] = S(\vec{r}_1|\vec{r})\cdots S(\vec{r}_n|\vec{r}). \quad (3)$$

Here $S(\vec{r}_1|\vec{x})$ is the normalized concentration of particles at location $\vec{r}_1$, assuming the source is at $\vec{x}$. This is simply because the probability of having hits at location $\vec{r}_1$ should be equal to the normalized concentration of particles at location $\vec{r}_1$. We will be using the normalized concentration of particles at a particular location interchangeably with the probability of having hits at the location in this paper. Hence,

$$P(\vec{r}|\vec{r}_1,\dots,\vec{r}_n) = \frac{\exp\left\{-\int_0^t S[\vec{r}(t')|\vec{r}]dt'\right\} S(\vec{r}_1|\vec{r})\cdots S(\vec{r}_n|\vec{r})}{\sum_x \exp\left\{-\int_0^t S[\vec{r}(t')|\vec{x}]dt'\right\} S(\vec{r}_1|\vec{x})\cdots S(\vec{r}_n|\vec{x})}. \quad (4)$$

We assume that the searcher has an analytical expression for the distribution of particles emitted by the source, given the source location.

The probability that the hits occurred at these positions is simply

$$e^{-\int_0^t \{S[\vec{r}(t')|0]\}dt'} S(\vec{r}_1|\vec{0})\cdots S(\vec{r}_n|\vec{0}). \quad (5)$$

If we were to construct an average measure of the probability of finding the source at $\vec{r}$, the way forward would be to convolute Eq. (4) with the probability of realizing a single trajectory and summing over all trajectories. We hence get the average probability of finding the source at $\vec{r}$ to be

$$P(\vec{r}) = \int_{\vec{r}(0)=\vec{r}_i}^{\vec{r}(t)=\vec{r}_f} D\vec{r}(\tau) \sum_{n=1,\infty} \frac{1}{n!} \int_{\text{path}} dr_1 \int_{\text{path}} dr_2 \cdots \int_{\text{path}} dr_n e^{-\int_0^t \{S[\vec{r}(t')|\vec{r}]+S[\vec{r}(t')|0]\}dt'} S(\vec{r}_1|\vec{0})\cdots S(\vec{r}_n|\vec{0}) \times \frac{S(\vec{r}_1|\vec{r})\cdots S(\vec{r}_n|\vec{r})}{\sum_x e^{-\int_0^t S[\vec{r}(t')|\vec{x}]dt'} S(\vec{r}_1|\vec{x})\cdots S(\vec{r}_n|\vec{x})}. \quad (6)$$

Here, $\int_{\vec{r}(0)=\vec{r}_i}^{\vec{r}(t)=\vec{r}_f} D\vec{r}(\tau)$ is a path integral summing over all possible trajectories starting from $\vec{r}(0)=\vec{r}_i$ and ending at $\vec{r}(t)=\vec{r}_f$. $\int_{\text{path}}$ refers to an integral over a particular path in the path integral. $\int_{\text{path}} dr_1 \int_{\text{path}} dr_2 \cdots \int_{\text{path}} dr_n$ considers all possible ways of realizing n hits on the path and $\frac{1}{n!}$ exists to prevent overcounting. It is obvious that if our trajectory took an infinite time we would have the best narrowing of the source location. Hence, the best possible average probability distribution possible is

$$P_\infty(\vec{r}) = \int_{\vec{r}(0)=\vec{r}_i}^{\vec{r}(t)=\vec{r}_f} D\vec{r}(\tau) \sum_{n=1,\infty} \frac{1}{n!} \int_{\text{path}} dr_1 \int_{\text{path}} dr_2 \cdots \int_{\text{path}} dr_n e^{-\int_0^\infty \{S[\vec{r}(t')|\vec{r}]+S[\vec{r}(t')|0]\}dt'} S(\vec{r}_1|\vec{0})\cdots S(\vec{r}_n|\vec{0}) \times \frac{S(\vec{r}_1|\vec{r})\cdots S(\vec{r}_n|\vec{r})}{\sum_x e^{-\int_0^\infty S[\vec{r}(t')|\vec{x}]dt'} S(\vec{r}_1|\vec{x})\cdots S(\vec{r}_n|\vec{x})}. \quad (7)$$

Let us assume for illustrative purposes that $S(\vec{r}_1|\vec{x}) = S(|\vec{r}_1-\vec{x}|)$. Also, let us assume that S is appreciable only up to a distance L away from the source. Because of the presence of terms such as $S(\vec{r}_1|\vec{0})S(\vec{r}_1|\vec{r})$, the average probability distribution of finding the source evaluated above is appreciable over a distance $2L$, as long as we are considering trajectories of lengths a few orders larger than L. This implies that the probability distribution measured by the searcher will not narrow the source better compared to the probability distribution of particles emitted by the source S. If we consider the limit in which $t \to 0$ in Eq. (6), we can see that the probability distribution measured by the searcher is centered at the searcher position. The measured probability distribution is similarly in general not centered at the location of the source for other values of t. This implies that the measured probability distribution by the searcher cannot narrow the source better than $S(\vec{x})$.

III. EXAMPLE STRATEGY

Let us consider a search strategy in which the probability to jump to a neighboring location is proportional to the normalized difference in the probability of finding the source from its present location. The probability for the searcher to jump to the nearest neighbor $(x + dx, y)$ on an average would go as $\Pi(x + dx, y) = \beta\Theta[P(x + dx, y) - P(x, y)]\frac{[P(x+dx,y)-P(x,y)]}{P(x,y)}$. Let us also use α to label the jump probability, in a case in which the probability of finding the source as the present site is the same as the neighboring site. $P(x, y)$ is the average probability of finding the source at position (x, y), which is evaluated by the searcher using the Bayes theorem as described by Eq. (6). $\Theta(x)$ is defined as

$$\Theta(x) = \begin{cases} 1, & \text{if } x > 0 \\ 0, & \text{otherwise.} \end{cases} \quad (8)$$

Let the average time to reach the source from position (x, y) be $T(x, y)$. As derived in the Appendix,

$$0 = -P(x, y) - 2\beta\nabla T(x, y) \cdot \nabla P(x, y) - \beta T(x, y)\nabla^2 P(x, y) + \alpha\beta\nabla^2 T(x, y). \quad (9)$$

Consider a simple case where the probability distribution has a radial symmetry with the source located at $r = 0$. The above equation then becomes

$$0 = -P(r) - \beta\frac{\partial T(\bar{r})}{\partial r}\frac{\partial P(\bar{r})}{\partial r} - \beta T(\bar{r})\left[\frac{\partial^2 P(\bar{r})}{\partial r^2} + \frac{1}{r}\frac{\partial P(\bar{r})}{\partial r}\right] + \alpha\beta P(r)\left[\frac{\partial^2 T(\bar{r})}{\partial r^2} + \frac{1}{r}\frac{\partial T(\bar{r})}{\partial r}\right]. \quad (10)$$

For $\alpha = 0$ and the boundary condition $T(r = 0) = 0$, we have

$$T(r) = -\frac{1}{\beta r P'(r)}\int_0^r xP(x)dx. \quad (11)$$

As mentioned above, the probability distribution $P(x)$ is more poorly localized near the source in comparison to $S(x)$. In the case of a homogeneous diffusion of particles emitted by a source located at the origin in two dimensions, the equilibrium particle concentration at r goes as $K_0(r/L)$. Hence the lower bound on search time is

$$T(r) > \mathrm{LB}(r) = -\frac{1}{\beta r K_0'(r/L)}\int_0^r xK_0(x/L)dx. \quad (12)$$

This is plotted in Fig. 1. One can see that for $r \gg L$, LB(r) increases exponentially with r. LB(r) would be the lower bound even if $\alpha \neq 0$. This is because α adds randomness to the search and $\alpha \neq 0$ would hence increase the search time.

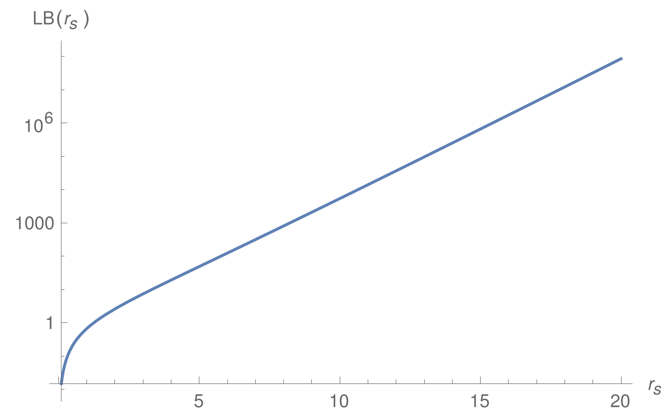

FIG. 1. LB(r) plotted against r for $L = 1$. We see that the lower bound increases exponentially with r, for larger values of r.

IV. GENERIC LOWER BOUND

We can use the fact that the probability distribution evaluated by Bayes' theorem is not as concentrated near the source as $S(x, y)$, to evaluate a lower bound on search time as follows. First let us assume that the searcher knows that the source is located at the origin with a probability 1. Then, the least amount of time taken by the searcher to reach the source goes as r: the distance between the source and the searcher. In the case in which the searcher instead has knowledge that the source is located at one of the two locations $\vec{x}_1$ and $\vec{x}_2$, with probabilities of occurrence at these locations p_1 and p_2, respectively, the smallest possible search time would then go as $p_1|\vec{x}_1 - \vec{x}_s| + p_2|\vec{x}_2 - \vec{x}_s|$, where $\vec{x}_s$ is the searcher's position. This is obvious because out of N possible measurements, the source will be detected Np_1 times at $\vec{x}_1$ and Np_2 times at $\vec{x}_2$, for $N \gg 1$. One could extend this argument to conclude that for a source probability distribution $P(\vec{x})$ as evaluated by the searcher, the shortest time to reach the source on an average

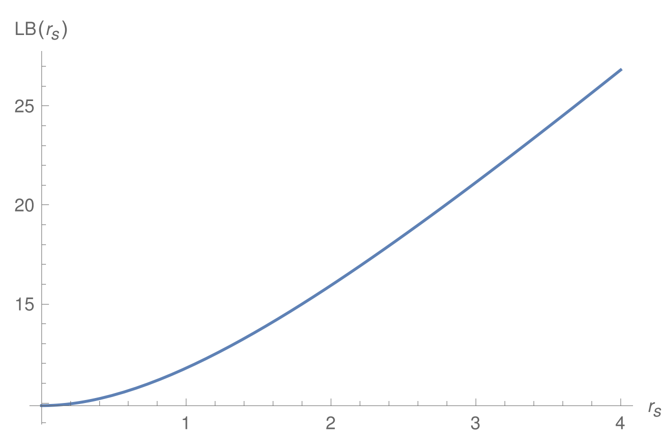

FIG. 2. LB(r_s) obtained by solving Eq. (13), is plotted as a function of r_s with $L = 1$. As can be seen for $r_s \gg L$, the expected lower bound goes as r_s. Also note that for $r_s = 0$ the lower bound on the search time is not zero.

should go as $\int d\vec{x}|\vec{x}_s - \vec{x}|P(\vec{x})$. Since the probability distribution evaluated using Bayes' theorem is not as concentrated near the source as $S(\vec{x})$, the shortest search time on an average will not be larger than $\frac{1}{v_s}\int d\vec{x}|\vec{x}_s - \vec{x}|S(\vec{x})$ (v_s is the speed of the searcher, which we take to be equal to 1 below), which for $S \sim K_0(r/L)$ is

$$\mathrm{LB}(r_s) \sim \int r\,d\theta\,dr\,K_0(r/L)\sqrt{(r_s - r\cos\theta)^2 + r^2\sin^2\theta} = \int r\,d\theta\,dr\,K_0(r/L)\sqrt{r_s^2 + r^2 - 2rr_s\cos\theta}. \tag{13}$$

Substituting $x = r/L$ and considering the case $r_s \gg L$, we get

$$\begin{aligned}\mathrm{LB}(r_s) &\sim r_s L^2 \int_0^\infty dx \int_0^{2\pi} d\theta\, xK_0(x)\sqrt{1 + x^2\frac{L^2}{r_s^2} - 2x\frac{L}{r_s}\cos\theta}\\ &= r_s L^2 \int_0^\infty dx \int_0^{2\pi} d\theta\, xK_0(x)\left\{1 - x\frac{L}{r_s}\cos\theta + \text{terms of order } O\left[\left(\frac{L}{r_s}\right)^2\right] \text{ and higher}\right\}\\ &= 2\pi r_s L^2 + \text{terms of order } O\left[\left(\frac{L}{r_s}\right)^2\right] \text{ and higher.}\end{aligned} \tag{14}$$

In going from the first to the second equation we have used the fact that $K_0(x)$ is of a substantial magnitude only for small values of x and hence majoritarian contributions to the integral come only from small values of x. We have also used $\int_0^\infty r\,dr\,K_0(r) = 1$. As r_s is made smaller, terms of order $O[(\frac{L}{r_s})^2]$ and higher start appearing. However, we note that as r_s becomes larger and larger, the lower bound on search time goes as r_s. This behavior is seen by solving Eq. (13) for $L = 1$ as plotted in Fig. 2.

Now

$$\begin{aligned}\mathrm{LB}(r_s) &\sim \int r\,d\theta\,dr\,K_0(r/L)\sqrt{(r_s - r\cos\theta)^2 + r^2\sin^2\theta}\\ &= L^3 \int \frac{r}{L}d\theta\, d\frac{r}{L}K_0(r/L)\sqrt{\left(\frac{r_s}{L}\right)^2 + \left(\frac{r}{L}\right)^2 - 2\frac{r}{L}\frac{r_s}{L}\cos\theta}\\ &= L^3 \int x\,d\theta\,dx\,K_0(x)\sqrt{\left(\frac{r_s}{L}\right)^2 + x^2 - 2x\frac{r_s}{L}\cos\theta}.\end{aligned} \tag{15}$$

If $\frac{r_s}{L} \ll 1$, we can Taylor expand in powers of $\frac{r_s}{L}$

$$\begin{aligned}\mathrm{LB}(r_s : r_s \ll L) &\sim L^3 \int x^2 d\theta\,dx\,K_0(x) + L^3 \int d\theta\,dx\,xK_0(x)\frac{\frac{r_s}{L} - x\cos\theta}{\sqrt{\left(\frac{r_s}{L}\right)^2 + x^2 - 2x\frac{r_s}{L}\cos\theta}}\Bigg|_{r_s/L=0}\left(\frac{r_s}{L}\right)\\ &\quad + \text{terms of order } O\left[\left(\frac{r_s}{L}\right)^2\right] \text{ and higher}\\ &= \pi^2 L^3 - r_s L^2 \int d\theta\,dx\,xK_0(x)\cos\theta + \text{terms of order } O\left[\left(\frac{r_s}{L}\right)^2\right] \text{ and higher}\\ &= \pi^2 L^3 + \text{terms of order } O\left[\left(\frac{r_s}{L}\right)^2\right] \text{ and higher.}\end{aligned} \tag{16}$$

From Fig. 2 we can see that $\mathrm{LB}(r_s)$ is linear for $r_s > 2L$. From Eq. (14) we can see that the slope of this line is $2\pi L^2$. One can hence say that

$$\mathrm{LB}(r_s) > \frac{\pi^2 L^3}{2} + 2\pi L^2 r_s. \tag{17}$$

The Shannon entropy of the probability distribution is

$$E = -\int 2\pi r\,dr\,d\theta\,S(r,\theta)\ln S(r,\theta). \tag{18}$$

$S(r,\theta)$ is the normalized concentration of particles at (r,θ) assuming the source is at the origin. In the language of the text below Eq. (3), $S(r,\theta|\vec{0}) = S(r,\theta)$. If we were to evaluate the lower bound on the search time, with the constraint that the probability distributions could be of any kind but should have a fixed entropy E, then we would have to include a Langrange multiplier λ that sets E equal to the entropy of the probability distribution $S(r,\theta)$. We have the following:

$$\mathrm{LB} = \int 2\pi r\,dr\,d\theta\,rS(r,\theta) - \lambda\left[\int 2\pi r\,dr\,d\theta\,S(r,\theta)\ln S(r,\theta) + E\right] - \beta\left[\int 2\pi r\,dr\,d\theta\,S(r,\theta) - 1\right]. \tag{19}$$

We have assumed the searcher is located at $r = 0$. Minimizing with respect to $S(r,\theta)$ gives

$$r - \lambda[\ln S(r,\theta) + 1] - \beta = 0, \tag{20}$$

which solves to

$$S(r,\theta) = e^{r/\lambda - \beta/\lambda - 1}. \tag{21}$$

$\lambda < 0$ for consistency. Requiring that

$$\begin{aligned}&\int 2\pi r\, dr\, d\theta [S(r,\theta)\ln S(r,\theta)]\\ &= -E, \quad \rightarrow (2\pi)^2 \lambda e^{\beta/\lambda - 1}(\beta - 3\lambda) = -E,\\ &\int 2\pi r\, dr\, d\theta\, S(r,\theta)\\ &= 1 \rightarrow e^{\beta/\lambda-1}(2\pi)^2\lambda^2 = 1,\end{aligned} \tag{22}$$

which implies $(\beta - 3\lambda) = -E\lambda \rightarrow (\frac{\beta}{\lambda}) = 3 - E$ and $\lambda = -\frac{1}{2\pi}e^{E/2-1}$.

Hence the lower bound is

$$\begin{aligned}\text{LB} &= \int 2\pi r\, dr\, d\theta\, rS(r,\theta)\\ &= -2(2\pi)^2\lambda^3 e^{\beta/\lambda - 1} = -2\lambda = \frac{e^{E/2-1}}{\pi}.\end{aligned} \tag{23}$$

V. CONCLUSION

In [21] the difficulty in evaluating the search time for infotaxis was highlighted, and instead a calculation for a different search strategy which does not utilize information about past hits was presented. They evaluated the lower limit for search time for this strategy in certain limits as $\sim e^E$, where E is the entropy of the probability distribution of finding the source. In this work we first evaluated a lower bound on the average search time in a search strategy that evaluates the probability distribution of finding the source at each step given the information of past hits. The rate of jumps to a neighboring site is proportional to the normalized difference of evaluated probability of finding the source with the present site of the searcher. This lower bound goes as the exponential of distance from the source for large distances. We then provided an expression for the lower bound for the search time for a generic cues-based search strategy. For a general probability distribution with entropy E, we showed that the lower bound goes as $e^{E/2}$, which is similar to e^E in [21], which was evaluated for a non-cue-based search strategy in the limit in which the search time as well as entropy are much larger than 1.

APPENDIX

To simplify things, let us consider the system in one dimension. The final result can be easily generalized to higher dimensions. We have

$$\begin{aligned}T(x) = &-dt + T(x+dx)[\beta\Pi(x+dx) + \alpha\Delta(x+dx)]\\ &+T(x-dx)[\beta\Pi(x-dx) + \alpha\Delta(x-dx)]\\ &+T(x)\{1 - \beta[\Pi(x+dx)^-\\ &+\Pi(x-dx)^- + 2\alpha\Delta(x)]\},\end{aligned} \tag{A1}$$

where

$$\begin{aligned}\Pi(i) &= \Theta[P(x) - P(i)]\frac{[P(x) - P(i)]}{P(x)},\\ \Pi(i)^- &= \Theta[-P(x) + P(i)]\frac{[-P(x) + P(i)]}{P(x)},\\ \Delta(i) &= 1, \quad P(x) = P(i),\\ &= 0, \quad P(x) \neq P(i).\end{aligned} \tag{A2}$$

Equation (A1) states that we can reach the point x from its neighbors $x + dx$ and $x - dx$, which subtracts time dt from times $T(x+dx)$, $T(x-dx)$ to reach the source from these sites. Each of the times $T(x+dx)$, $T(x-dx)$ are multiplied by the probabilities to make the jump from $x + dx$ and $x - dx$ to x, respectively. The term multiplying $T(x)$ on the right-hand side is the probability of not making a jump to the neighbors $x + dx$, $x - dx$. α is the probability of making a jump randomly in the case in which the neighboring site has the same probability of finding the source as the present site.

Equation (A1) becomes

$$\begin{aligned}0 = &-dt + \beta[T(x+dx)\Pi(x+dx) + T(x-dx)\Pi(x-dx)]\\ &-T(x)\beta\{[\Pi(x+dx)^- + \Pi(x-dx)^-]\}\\ &+\beta\alpha\, dx^2\nabla^2 T(x)\end{aligned} \tag{A3}$$

or

$$\begin{aligned}0 = &-dt + \beta\{[T(x) + dx\,\partial_x T(x)]\Pi(x+dx)\\ &+[T(x) - dx\,\partial_x T(x)]\Pi(x-dx)\}\\ &-T(x)\beta\{[\Pi(x+dx)^- + \Pi(x-dx)^-]\}\\ &+\beta\alpha\, dx^2\nabla^2 T(x)\end{aligned} \tag{A4}$$

or

$$\begin{aligned}0 = &-dt - \beta\, dx^2\partial_x T(x)\left\{\frac{\partial_x P(x)}{P(x)}\Theta[P(x) - P(x+dx)]\right.\\ &\left.+\frac{\partial_x P(x)}{P(x)}\Theta[P(x) - P(x-dx)]\right\}\\ &+T(x)\beta\{[\Pi(x+dx) + \Pi(x-dx)]\}\\ &-T(x)\beta\{[\Pi(x+dx)^- + \Pi(x-dx)^-]\}\\ &+\beta\alpha\, dx^2\nabla^2 T(x).\end{aligned} \tag{A5}$$

Now

$$\begin{aligned}\Pi(i) - \Pi(i)^- &= \{\Theta[P(x) - P(i)] + \Theta[-P(x) + P(i)]\}\\ &\times\frac{[P(x) - P(i)]}{P(x)} = \frac{[P(x) - P(i)]}{P(x)}.\end{aligned} \tag{A6}$$

Hence

$$\begin{aligned}0 = &-dt - \beta\, dx^2\partial_x T(x)\frac{\partial_x P(x)}{P(x)}\\ &-T(x)\beta\left[\frac{[P(x+dx) + P(x-dx) - 2P(x)]}{P(x)}\right]\\ &+\beta\alpha\, dx^2\nabla^2 T(x)\end{aligned} \tag{A7}$$

or

$$0 = -P(x)dt - \beta\, dx^2 \partial_x T(x) \partial_x P(x) - T(x)\beta\, dx^2 \nabla^2 P(x) + \beta\alpha\, dx^2 P(x) \nabla^2 T(x), \quad \text{(A8)}$$

which becomes in two dimensions

$$0 = -P(x, y) - \beta \nabla T(x, y) \cdot \nabla P(x, y) - \beta T(x, y) \nabla^2 P(x, y) + \beta\alpha P(x, y) \nabla^2 T(x, y). \quad \text{(A9)}$$

We have redefined $\frac{\beta dx^2}{dt} \rightarrow \beta$ above.

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