Emergence of logarithmic-periodic oscillations in contact process with topological disorder

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We present a model of contact process on Domany-Kinzel cellular automata with a geometrical disorder. In the 1D model, each site is connected to two nearest neighbors which are either on the left or the right. The system is always attracted to an absorbing state with algebraic decay of average density with a continuously varying complex exponent. The log-periodic oscillations are imposed over and above the usual power law and are clearly evident as $p \to 1$. This effect is purely due to an underlying topology because all sites have the same infection probability p and there is no disorder in the infection rate. An extension of this model to two and three dimensions leads to similar results. This may be a common feature in systems where quenched disorder leads to effective fragmentation of the lattice.

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I. INTRODUCTION

The dynamic phase transition to a fully absorbing vacuum state is the most studied phase transition in nonequilibrium statistical physics. Several universality classes have been proposed for this transition. In simulations, the directed percolation (DP) is the most observed universality class [1]. However, experimental verification of this universality class is obtained in very few cases [2,3]. The reasons could be the inevitable presence of noise and disorder in the experimental systems [4,5]. The quenched disorder is a relevant perturbation if the spatial correlation length critical exponent $\nu_\perp$ fulfills the condition $\nu_\perp d > 2$ where d is dimensionality and $\nu_\perp$ is the correlation length exponent in the spatial direction of the pure system [6]. This is known as the Harris criterion.

In some cases, quenched disorder leads to a whole parameter range of very slow dynamics instead of a clean critical point. In this phase, the exponent of the power law is continuously changing due to the formation of rare region. This phase is known as Griffiths phase [7,8]. This is in contrast with continuous phase transitions where the power law associated is observable only at the critical point.

Power law in Griffiths phase has a real exponent usually. A complex exponent will lead to log-periodic oscillatory corrections to the power law [9]. Complex exponent has been obtained and studied in systems embedded with geometrical hierarchy, growth process, rupture [10,11]. They have also been identified in complex network [12,13]. (This model does undergo fragmentation and the underlying mechanism maybe similar to one proposed in this work.) Recently, we have observed complex persistence exponent in a 1D model where half of the sites obey rules leading to DP class and the rest evolve according to rules leading to compact directed percolation (CDP) class [14]. It demonstrates that such exponent may appear spontaneously in systems without pre-existing hierarchy. Even for a random walk, the discrete scale invariance hierarchy is dynamically constructed due to intermittent encounter with the slow region [15].

In this work, we study the contact process on a d-dimensional lattice with directed asymmetric coupling. We observe log-periodic oscillations in the decay of the fraction of infected sites $\rho(t)$. This can be an outcome of the quenched disorder leading to the effective fragmentation of lattice. The model does not have a self-similar structure in the defects or the lattice. This is a topological disorder. Griffiths phase has been observed for complex networks that have inherent topological disorder [16]. The topological disorder may lead to the disappearance of an active phase transition for the model of the resilience of the internet against breakdown [17,18], or disease spread for sufficiently small infection rate [19]. The model studied in this work does not show an active phase either. Throughout the phase diagram, we have an absorbing phase. However, the dynamical approach to the vacuum state changes with the parameter values.

Janssen-Grassberger conjecture [20,21] stated the conditions for DP transition. It can be stated as [22] “the universality class of DP contains all continuous transitions from a ‘dead’ or ‘absorbing’ state to an ‘active’ one with a single scalar order parameter, provided the dead state is not degenerate [and provided some technical points are fulfilled: Short-range interactions both in space and time, nonvanishing probability for any active state to die locally, translational invariance (absence of ‘frozen’ randomness), and absence of multicritical points].” We relax these conditions. In a 1D model, every site is randomly labeled as R or L. The site labeled R (L) is coupled to two nearest neighbors on the right (left) side. This topological disorder results in effectively partitioning the cluster into several disconnected pieces as would be explained in the next section. We extend the study in 2D and 3D and obtain similar results.

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II. MODEL AND SIMULATION

We consider the cellular automata model of contact process proposed by Domany-Kinzel [23]. The state of ith site of a 1D lattice ($v_i = 0$ or 1) is specified at time t by $v_i(t)$. In the models of DP, sites marked as "1" could be interpreted as wet or infected or chaotic while sites marked as "0" correspond to dry or healthy or close to the fixed point.

Consider a 1D lattice of length N updated synchronously. We introduce quenched disorder in the system by choosing a randomly chosen fraction r of sites as type L and rest are chosen as type R. We start with random initial condition with half of the sites chosen as "active." The sites L and R evolve according to conditional probability P_L and P_R. We define $P_L[v_i(t+1)|v_{i-1}(t)+v_{i-2}(t)]$ as $P_L(1|0) = 0$, $P_L(1|2) = P_L(1|1) = p$ and $P_R[v_i(t+1)|v_{i+1}(t)+v_{i+2}(t)]$ is defined as $P_R(1|0) = 0$, $P_R(1|2) = P_L(1|1) = p$ where $p \neq 0$. The boundary condition are periodic: If $i-1 < 0, i-2 < 0$, then $i-1 \equiv N+(i-1), i-2 \equiv N+(i-2)$ if $i+1 > N, i+2 > N$, then $i+1 \equiv i+1-N$, $i+2 \equiv i+2-N$. The order parameter, fraction of active sites is given by $\rho(t) = \frac{1}{N}\sum_{i=1}^{N} v_i(t)$. The quantity $\rho(\infty)$ approaches zero asymptotically for $p < 1$. The updating scheme is synchronous. It is noted for completeness.

We simulate the 1D lattice of size $N = 2 \times 10^6$ for time up to 5×10^6 and average over approximately 1000 configurations. All the averages are done over different disorder configurations and different initial conditions. We present the results for $r = 0.5$, i.e., half the sites are of type L and the other half are of type R. This is a case with the maximum disorder. At the same time, there is no anisotropy on average. For small values of infection probability p, the fraction of active sites $\rho(t)$ undergoes exponential or stretched exponential decay for small values of infection probability p [see Fig. 1(a)]. Nevertheless, as $p \to 1$, we observe a regime where $\rho(t)$ decays as a power law. The power law is given by $\rho(t) \sim t^{-\delta}$, where the exponent is complex and real part of δ is continuously decreasing as $p \to 1$. The region of continuously varying power law is known as the Griffiths phase. Thus, the above phase can be named a complex Griffiths phase. The Griffiths phase usually results due to the rare region effect.

The Griffiths phase observed in the above connections has origin in effective fragmentation of lattice in disconnected parts of different sizes. Consider the sequence *RRLL*. The first two sites evolve according to two sites on the right side and the next two sites evolve according to two on the left. Thus, the evolution in these four sites is practically independent of the rest of the lattice (independent of the nature of update). Thus, if such a group or any group which starts with *RR* and ends with *LL* or *LLL* or *LL* . . . *L* (two or more consecutive *L*'s) reaches an absorbing state, it cannot come out of such a state. We call such groups clusters of type 1. Clusters will decompose the entire lattice into several independent sections when they reach an absorbing state. Now the decay of the number of active sites will be dictated by the sum of active sites in several such independent sections. Consider any sequence which is sandwiched between two consecutive type 1 clusters. This sequence is essentially driven by type 1 cluster on the right side as well as the left side. This sandwiched sequence of sites will evolve independently of the rest of the lattice when clusters on either side become inactive. This group of sites is also a finite cluster. We call it a type 2 cluster. Being finite size, it will eventually reach an absorbing state when the clusters of type 1 on either side become inactive. We have two types of clusters. First is type 1 clusters which evolve essentially independently of the rest of the lattice. Type 2 clusters are sandwiched group of sites between two type 1 clusters. When both of them reach an absorbing state, this cluster will eventually reach an absorbing state. These sections of type 1 and type 2 have different lengths and the expected time by which they reach an absorbing state is different as well. The absence of a single length-scale (or timescale) in evolution could lead to non-exponential relaxation. We indeed obtain power-law relaxation over a large range of parameters in this model.

The lattice decomposes into several finite-size clusters. The activity of these some large rare clusters is disassociated from the bulk, i.e., although the bulk lattice is in the absorbing phase, the rare regions are locally in the fluctuating phase. This leads to slow dynamics in the Griffiths phase [4]. Due to the finite size of these disconnected sets of sites, the system always collapses to the absorbing phase [see Fig. 1(a)] for any value of $p < 1$. Thus, the model has only an absorbing phase and no fluctuating phase.

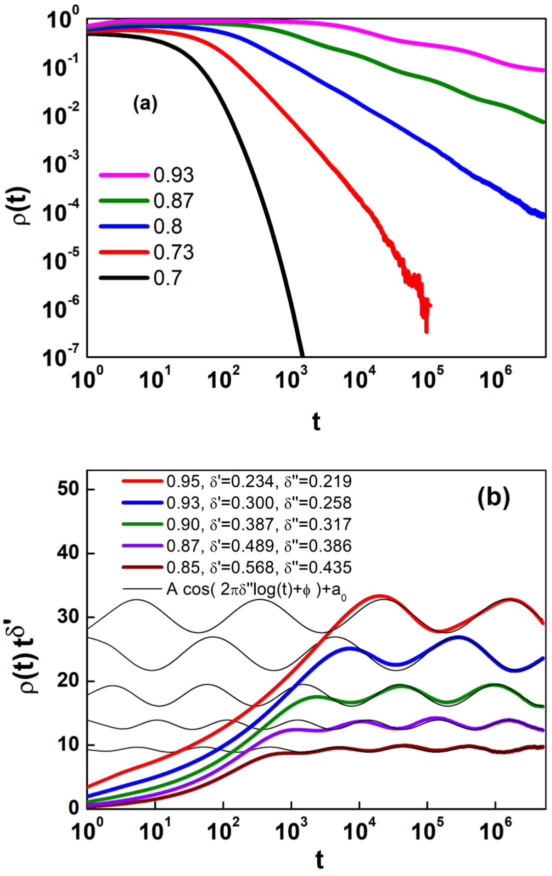

FIG. 1. (a) The time evolution of $\rho(t)$ vs t in 1D model on log-log scale, with $r = 0.5$ and values of p ranging for 0.7 to 0.93 (from bottom to top). Clearly, there is a continuous change in the exponent of power law. (b) We plot $\rho(t) \times t^{\delta'}$ vs $\log(t)$ for p ranging from 0.85 to 0.95 (from bottom to top). The log-periodic oscillations are clearly evident ($\omega = 2\pi\delta''$. The y axis is multiplied by an arbitrary constant for better visualization.

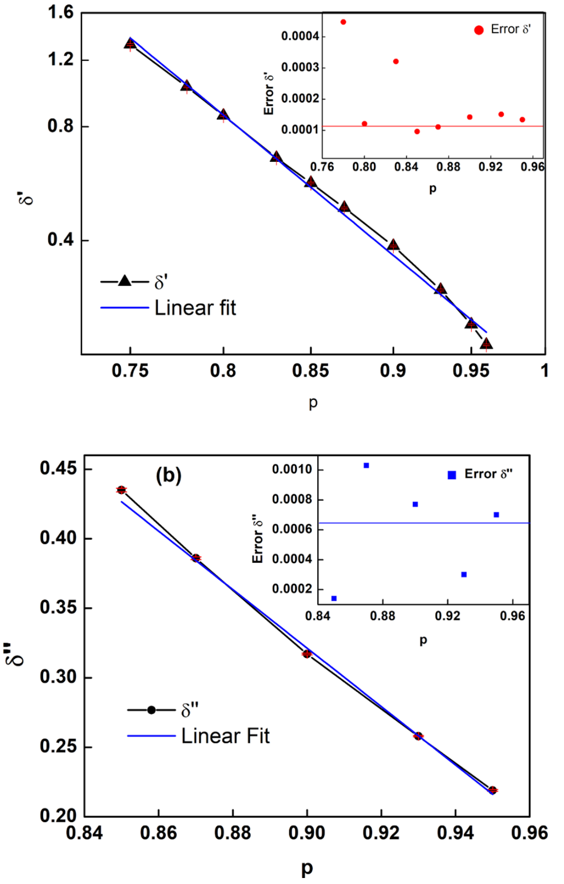

FIG. 2. (a) The plot of Re(δ) as a function of p ranging for 0.76 to 0.95. The error bar are also shown in figure (red marks). (b) The plot of Img(δ)/2π as a function of p ranging for 0.83 to 0.95. The error bar are also shown in figure (blue marks). In both cases, the error in the parameter is found using the least-squares method and found to be smaller than the size of symbols.

With no effective fragmentation of lattice, DP universality class is restored even with the quenched disorder in topology. With probability r, we connect a given site i to both nearest neighbors and with probability $1-r$ we connect them to both next-nearest neighbors. There is no disorder in infection probability. The disorder is only in the sense that some sites are connected to nearest neighbors only and some sites are connected to next-nearest neighbors only. The system undergoes DP transition with a clean critical point.

The exponent of power law is complex in nature. A complex exponent can be written as $\delta = \delta' + i2\pi\delta''$. Thus, $\rho(t) \sim Re(At^{-\delta'-i2\pi\delta''}) \sim At^{-\delta'}\cos[2\pi\delta''\log(t)]$ and $\rho(t)t^{\delta'} \sim A\cos[2\pi\delta''\log(t)]$. As the function is log-periodic, it is very difficult to extract the exact periodicity. The amplitude of these oscillation increases as $p \to 1$. These oscillations become more evident if we plot the quantity $\rho(t) \times t^{\delta'}$ with time [Fig. 1(b)]. This behavior can be fitted by a constant superposed by log-periodic oscillations. The amplitude and wavelength of these oscillations grow as $p \to 1$. For small values of p the amplitude is very small (if any) and it makes it difficult to determine if $\delta'' \neq 0$. The value of δ' decreases as $p \to 1$ [see Fig. 1(b)]. Figures 2(a) and 2(b) show the linear fit of Re(δ) and Img(δ)/(2π). The error bars are shown as well.

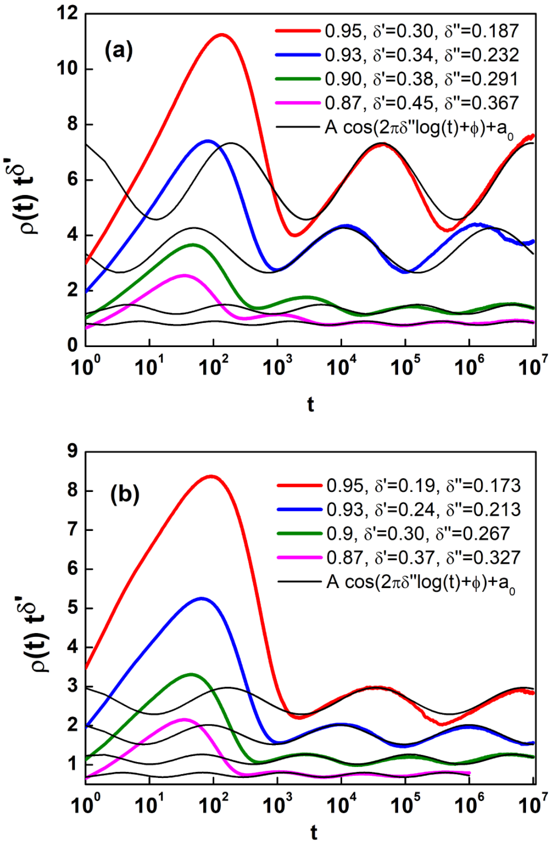

FIG. 3. (a) We plot time evolution of $\rho(t) \times t^{\delta'}$ for various values of p in the range 0.87 to 0.95 (bottom to top) in 2D model. (b) We plot $\rho(t) \times t^{\delta'}$ for values of p in the range 0.87 to 0.95 (bottom to top) in 3D model. The log-periodic oscillations are evident. The y axis is multiplied by an arbitrary constant for better visualization in both figures.

The error in case of δ' is calculated from the goodness of linear fit of $\rho(t)$ versus t for various values of p. In the case of δ'', we use this value of δ' and plot $\rho(t)t^{\delta'}$ as a function of t. We have fitted a function $A\cos(2\pi\delta''\log(t)+\phi)+a_0$ and found error in δ'' using nonlinear least squares. A "fit" function in gnuplot which uses an implementation of the nonlinear least-squares (NLLS) Marquardt-Levenberg algorithm was used [24]. This procedure has been followed for all fits to complex exponents in this work. (Of course, the fit is carried out over the relevant range. For larger values of p, the fragmentation of lattice occurs late and the onset of logarithmic oscillations is delayed.) The error in δ'' is found to be very small and less than 1% in all cases. It is less than 0.1% for the 1D case.

We extend this study to 2D and 3D. Let us consider the state of a site at time t, $v_{i,j}(t)$ on lattice of size N^2. Let the sites be of four types: L, R, U, and D. For all four types, the evolution occurs according to the value of the site itself and one of its nearest neighbors (whichever is not blocked by the quenched defects). The conditional probabilities $P_L[v_{i,j}(t+1)|v_{i,j}(t)+v_{i,j-1}(t)]$ and $P_R[v_{i,j}(t+1)|v_{i,j}(t)+v_{i,j+1}(t)]$ and $P_U[v_{i,j}(t+1)|v_{i,j}(t)+v_{i-1,j}(t)]$ and $P_D[v_{i,j}(t+1)|v_{i,j}(t)+v_{i+1,j}(t)]$ are defined as follows. $P_L(1|k) = P_R(1|k) = P_U(1|k) = P_D(1|k) = p$ for $k \neq 0$ and 0 for $k = 0$. We introduce a defect in the lattice by assigning type L, U, R, and D to each site. This results in the coupling of a given site in a certain randomly chosen direction. At time $t = 0$ almost half of the $N \times N$ sites are randomly chosen as "active." Boundary condition are analogous to 1D case. In 2D, we define boundary condition as follows: If $i - 1 < 1$, $j - 1 < 1$, then $i - 1 \equiv N + i - 1$, $j - 1 \equiv N + j - 1$, and if $i + 1 > N$, $j + 1 > N$, then $i + 1 \equiv i + 1 - N$, $j + 1 \equiv j + 1 - N$. In the case of 3D, we have six possible directions. Each site is coupled with neighbors only in one direction and the conditional probabilities for their evolution can be defined on similar lines. In all these cases $\rho(t) \to 0$ as $t \to \infty$ for $p < 1$.

In the 2D model, we simulate a 2D lattice of size N^2, where $N = 600$ for a very long time $t = 1 \times 10^7$ and average over more than 600 configurations. In 3D we consider a lattice of size N^3 where $N = 50$ for time up to 5×10^6 and average over more than 10^3 configurations. In both cases, $\rho(t)$ decays to an absorbing phase for $p \neq 0$. For small values of p, $\rho(t)$ decays exponentially or stretched exponentially to an inactive state. As in 1D, a regime of continuously changing power-law decay with complex exponent $\delta = \delta' + i2\pi\delta''$ is observed for p close to 1. The fact that the exponent is complex is reflected in logarithmic oscillations in the decay of $\rho(t)$ particularly as $p \to 1$ in both 2D and 3D [see Figs. 3(a) and 3(b)].

In 2D and 3D, we connect each site with only one neighbor. This neighbor is affected by only one neighbor and so on. This can be viewed as a random walk. Thus, if we start this walk from a site that does not affect any other site and continue till the walker backtracks or gets connected to a site visited previously, then the walk ends there since each site couples to only one site. A few examples of such walks are shown in Figs. 4(a), 4(b), and 4(c). The fraction of such surviving walks go down exponentially in 2D as well as 3D, though the exponent is smaller in 3D [see Fig. 4(d)]. Now if all the sites

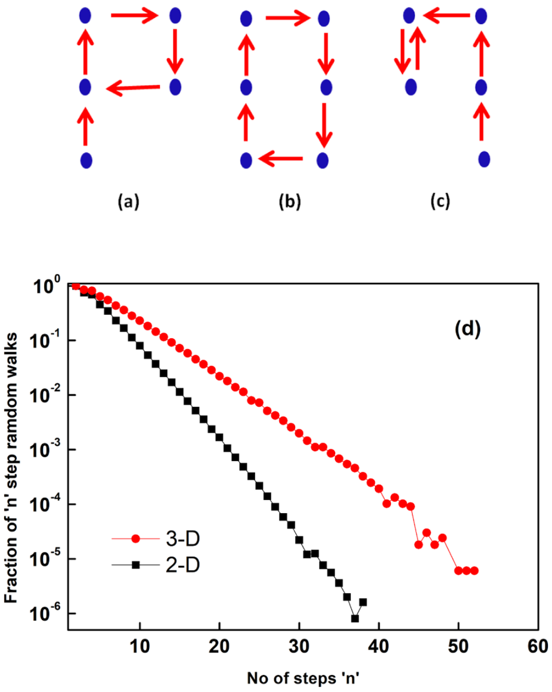

FIG. 4. (a), (b), (c) A few examples of random walk in the 2D model which ends up in an absorbing state. If the evolution of site A is affected by site B, then we draw an arrow from site A to site B. Panel (d) shows the fraction of walks surviving till step n in 2D and 3D.

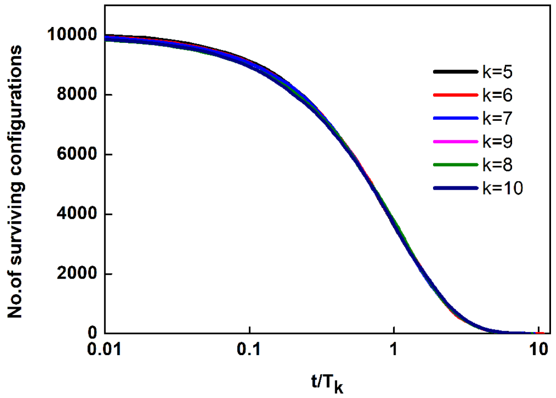

FIG. 5. We plot the number of surviving configuration vs t/T_k for group of size $k = 5, 6, 7, 8, 9, 10$. The initial number of configurations are 10^4 and $p = 0.94$.

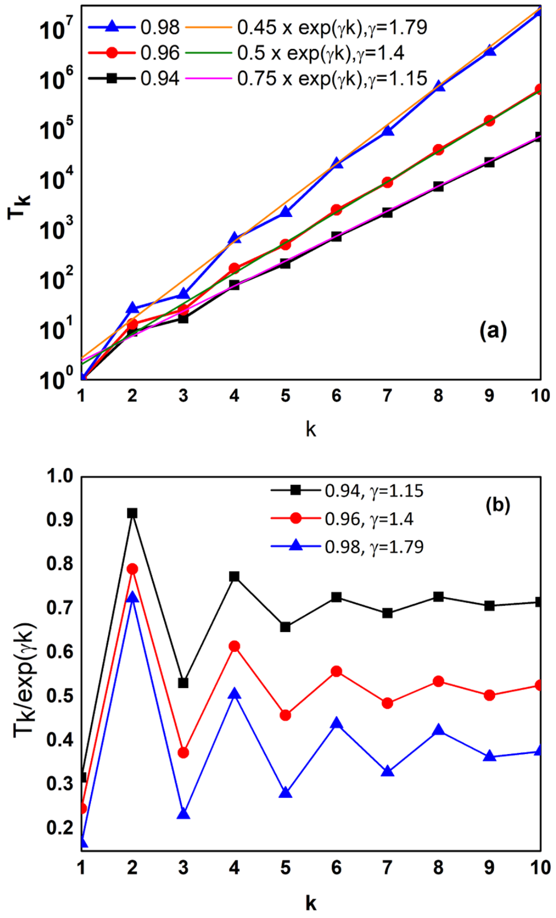

FIG. 6. (a) We plot T_k vs k, where T_k is the average time taken by cluster of k sites to become inactive starting with 10^4 configurations. We consider $k = 1$–10. T_k can be fitted as $\exp(\gamma_p k)$, where $\gamma_p = 1.15, 1.4, 1.79$ for $p = 0.94, 0.96, 0.98$. (b) We plot the relation $T_k/\exp(\gamma_p k)$ with k for $p = 0.94, 0.96, 0.98$. The oscillations are pronounced for larger values of p.

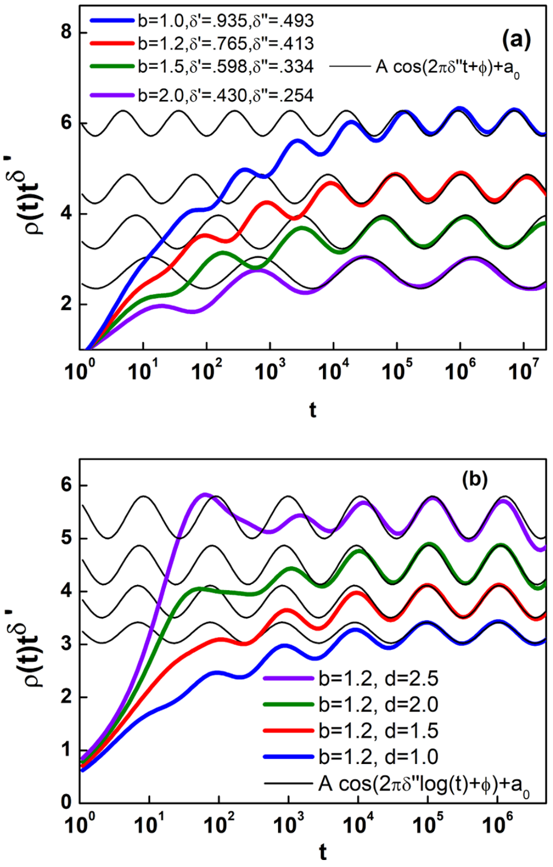

FIG. 7. Panel (a) shows the plot of $\rho(t)t^{\delta'}$ with time for various values of b (from top to bottom in increasing order of b. We assume functional form $\tau(x)=\exp[bx-d\frac{\cos(\pi x)}{\sqrt{x}}]$ for relaxation times. The log-periodic oscillations are evident in the picture. However δ' and δ'' keep changing with b. Onset of oscillations is delayed for higher wavelength oscillations. Panel (b) shows the plot of $\rho(t)t^{\delta'}$ with time for various values of d (from top to bottom in decreasing order of d), while $b=1.2$, $c=1$, $\delta'=0.765$, $\delta''=0.413$. Both δ' and δ'' remain unchanged. But the onset of logarithmic oscillations is delayed for small $|d|$. The y axis is multiplied by an arbitrary constant for better visualization and $\omega=2\pi\delta''$ in both figures.

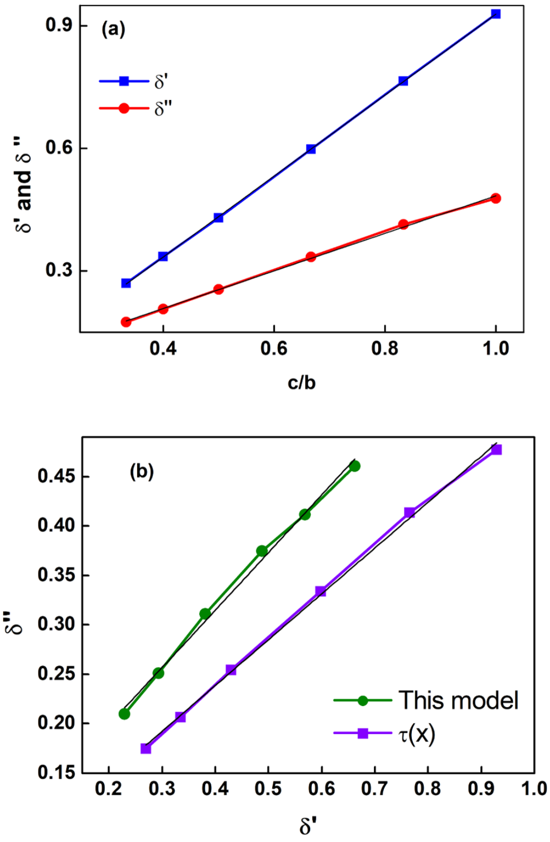

FIG. 8. Panel (a) shows the δ' and δ'' as a function of c/b where $c=1$ for simulation of $\tau(x)$ defined in the text. (b) We plot δ'' as a function of δ' for1D model and our choice of $\tau(x)$. We observe a linear relationship between two exponents in both cases.

covered by this walk go to zero, then these sites will be in an absorbing state forever since all the sites that affect them are in an absorbing state. These sites form a 1D lattice of finite size in all practical senses. This leads to fragmentation of lattice and the mechanism in higher dimensions could be the same as in one dimension.

Now the question is what is the origin of the complex exponents? In the case of the 1D model, we simulate systems of $k+2$ sites such that the first and the $k+2$th are set as inactive and the rest of the sites are kept active. Thus, we have a group of k sites of evolving according to rules of the 1D model as described in the above section. For uncorrelated disorder, we expect the probability of occurrence of a group of a given size to decay exponentially. We denote the average time taken by this group to become inactive by T_k. We note that the fraction of groups of size k surviving till time t seems to be a function of t/T_k (see Fig. 5) Thus, T_k is a characteristic time for system size k. This average time T_k increases exponentially with relation $T_k\sim\exp(\gamma k)$ with odd-even oscillations imposed over and above the exponential [see Fig. 6(a)]. [In Fig. 6(b), we plot $T_k/\exp(\gamma_p k)$ versus k for various values of p.] We argue that the combination of exponentially rare regions that survive for exponentially long times leads to a power law and the oscillations over and above this exponential leads to log-periodicity.

The standard explanation of Griffiths phase goes as follows. The probability of finding rare regions is exponentially small $P(x)\propto\exp(-cx)$ but can exhibit exponentially large lifetime $\tau(x)=\tau_0(x)=\exp(bx)$. The fraction of active sites can be approximated by $\rho(t)\sim\int\ xP(x)\exp{[-t/\tau(x)]}dx$. Using saddle-point approximation, it can be shown that this leads to algebraic decay of $\rho(t)$, i.e., $\rho(t)\sim t^{-c/b}$ with continuously varying exponent. We do not exactly know the functional form of relaxation times for different sizes. However, in this model, the oscillations get damped, and we propose a functional form as $\tau(x)\sim\tau_0(x)\tau_p(x)$, where $\tau_p(x)=\exp[-d\frac{\cos(\pi x)}{\sqrt{x}}]$, because the odd-even oscillations get damped quickly. (Sign of d in the expression of $\tau_p(x)$ does not matter because cos takes either sign.) We numerically compute the above sum $\rho(t)=\sum_{x=1}^{N}x\exp(-cx)\exp[-t\exp(-bx+d\frac{\cos(\pi x)}{\sqrt{x}})]$ for large N, i.e., $N=10^7$. The plot of $\rho(t)t^{\delta'}$ versus time clearly shows log-periodic oscillations as shown in Fig. 7(a). The values of δ' and δ'' do not depend on d as shown in Fig. 7(b). The obtained values of δ' are close to c/b as expected and it does not depend on d at all. Similarly δ'' does not change for any $|d| \neq 0$. However, for smaller values of $|d|$, the amplitude of oscillations is reduced and the onset of oscillations is delayed. Both exponents vary linearly with $\frac{c}{b}$ and are expected to vary linearly with each other. Although $\tau(x)$ is not a perfect analogy of the model presented here, the above simulations show that the lifetime of a rare region imposed with the periodic term can generate log-periodic oscillations. The exponents δ' and δ'' have a linear relationship in exponents computed using this ansatz as well as in our model. This linear variation is shown in Fig. 8. The onset of oscillations is delayed when oscillations have longer periodicity in our model. We observe it in our ansatz as well. We have studied a few different functional forms of $\tau_p(x)$ and they lead to log-periodic oscillations as well. Thus we believe that odd-even oscillations over and above the exponential in relaxation times are the likely reason for log-periodic oscillations.

III. SUMMARY

We have studied a contact process with random asymmetric couplings in one to three dimensions on Domnay-Kinzel automaton. In one dimension, we study a system in which each lattice site is coupled to two neighbors either on left or on right. In two and three dimensions each site is coupled to a neighbor chosen randomly. This is a quenched disorder. (There is no disorder in the infection probability, only in connections.) For low values of p the fraction of active sites $\rho(t)$ decays exponentially or stretched exponentially. But for $p \to 1$, $\rho(t)$ shows a power-law decay with a complex exponent. Thus, we observe log-periodic oscillations in time over and above the power-law decay. This power-law decay of order parameter with complex exponent can be termed as a complex Griffiths phase. Such a transition is not observed when the lattice is not effectively fragmented in disjoint units. We have also given an argument that this is likely an effect of odd-even oscillations in relaxation times as a function of size.

In the complex Griffiths phase, the real part of the exponent decreases continuously while amplitude and wavelength of oscillations increases as $p \to 1$. This is a system with a parallel update of all sites. It can be of interest to study the impact of changes in updating schemes, dimensionality, and other factors in the complex Griffiths phase.

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