Microscopic quantum generalization of classical Liénard oscillators

Srijan Bhattacharyya [ ],[1] Arnab Ghosh [ ],[1,*] and Deb Shankar Ray[2]

[1]*Indian Institute of Technology Kanpur, Kanpur, Uttar Pradesh 208016, India*
[2]*Indian Association for the Cultivation of Science, Jadavpur, Kolkata 700032, India*

Based on a system-reservoir model and an appropriate choice of nonlinear coupling, we have explored the microscopic quantum generalization of classical Liénard systems. Making use of oscillator coherent states and canonical thermal distributions of the associated c numbers, we have derived the quantum Langevin equation of the reduced system which admits single or multiple limit cycles. It has been shown that detailed balance in the form of the fluctuation-dissipation relation preserves the dynamical stability of the attractors even in the case of vacuum excitation. The quantum versions of Rayleigh, van der Pol, and several other variants of Liénard oscillators are derived as special cases in our theoretical scheme within a mean-field description.

I. INTRODUCTION

Dissipation is an intriguing issue in physical sciences [1]. Although its incorporation in dynamical problems from the classical point of view is largely phenomenological, the approach turns out to be untenable because of its violation of the uncertainty principle when the system is quantized. The problem is circumvented by coupling the system to a reservoir with infinite degrees of freedom kept at a finite temperature, which allows the fluctuations of the reservoir to act on the system inducing its dissipation or decay [2]. To maintain thermal equilibrium of the system in contact with the reservoir, fluctuation, and dissipation get connected through the fluctuation-dissipation theorem [3]. The system-reservoir model describing a dissipative quantum system lies at the heart of the problems related to macroscopic quantum coherence [4], quantized tunneling in condensed matter physics [5], relaxation processes in quantum optics [6], and magnetic resonance spectroscopy [7], to name a few. An important note in this context is that the dissipative term appearing in the reduced stochastic equation of motion for the system is by and large linear, arising out of linear coupling between the system and the reservoir [8–10]. Introduction of nonlinearity in the dissipative term in a classical system without any noise, on the other hand, may lead to a complete modification of the dynamics. Such nonlinearity in dissipation may result in a force which acts on the system as an intrinsic source of creation of the limit cycle [11], an asymptotically isolated trajectory in phase space. The Liénard equation represents a prototypical paradigm for such a classical nonlinear dissipative system of which two special cases are the van der Pol [12] and Rayleigh oscillators [13]. The object of the present paper is the search for a quantum analog of such classical Liénard oscillators within the framework of system-reservoir theory.

The Liénard system is widely used in describing many oscillating circuits in the development of radio and vacuum tube technologies [11–13]. The second order differential equation describing the classical Liénard system is

$$\ddot{x} + f(x)\dot{x} + g(x) = 0. \tag{1}$$

If $f(x)$ and $g(x)$ satisfy the following conditions of Liénard's theorem: (i) $f(x)$ and $g(x)$ are continuously differentiable functions for all x. (ii) $g(-x) = -g(x)$ for all x. (iii) $g(x) > 0$ for all $x > 0$. (iv) $f(-x) = f(x)$ for all x. (v) The odd function $F(x) = \int_0^x f(s)ds$ has exactly one positive zero at $x = x_s$; is negative for $0 < x < x_s$; is positive and nondecreasing for $x > x_s$; and $f(x) \to \infty$ as $x \to \infty$, then the system has, at least, one unique stable limit cycle around its origin in the phase plane [11]. The theorem assures that the *odd function $g(x)$* acts as a restoring force that tries to reduce any displacement. Although assumptions on the *even function $f(x)$* indicate that it acts as a negative damping at small displacement and a positive damping on large displacement. As a result, small oscillations are forced up whereas large ones are damped down. Thus, it is not difficult to anticipate that the system will settle into a self-sustained oscillation of some intermediate amplitude. Apart from monorhythmic models, such as van der Pol or Rayleigh oscillators, the Liénard equation depending on the form of the polynomial $f(x)$, may admit birhythmic solutions. The equation has been generalized further to the Liénard-Smith-Levinson oscillator form that includes the multiple limit cycles [14–16].

The aim of our present paper is to propose a microscopic quantum description of the classical Liénard oscillator in the system-reservoir model and subsequent generalization to its several variants. To capture the nature of the Liénard system, we have defined the interaction Hamiltonian containing appropriate nonlinear coupling terms. The time evolution of the dynamics is followed by the Heisenberg equation of motion to obtain the operator Langevin equation for the reduced system. Our approach is based on a c-number Langevin equation using harmonic oscillator coherent states and canonical

*arnab@iitk.ac.in

thermal distribution of the associated c numbers for the reservoirs. The c-number Langevin equation plays a key role in describing the noisy quantum Liénard system. van der Pol and Rayleigh oscillators are also depicted with a quantum noise term as model examples. Finally our proposal is generalized for several model cases of Eq. (1). The noise and the nonlinear dissipation originating from the system-reservoir coupling are shown to follow a detailed balance in the form of fluctuation-dissipation relation. This ensures that the *overall system* (system + reservoir) is thermodynamically closed as emphasized by Lindenberg and West [1]. If the noises are of external nature due to the distinct origins of noise and dissipation, they do not obey fluctuation-dissipation relation. Systems lacking such detailed balance are shown to destroy limit cycle oscillations.

The present paper is organized as follows: In Sec. II we introduce the model Hamiltonian to derive reduced dynamics for the system in terms of the operator Langevin equation. In the next section oscillator coherent states are used to construct quantum Langevin equation with c-number noise. Microscopic realizations of the classical van der Pol and Rayleigh oscillators are discussed in Sec. IV. Later on further generalizations of the theoretical scheme for the arbitrary form of classical Liénard systems are presented. Finally we conclude in Sec. V.

II. QUANTUM ANALOG OF THE CLASSICAL LIÉNARD OSCILLATOR

Our search for the microscopic description of the classical Liénard oscillation is based on a quantum harmonic oscillator, whose Hilbert space is given by Fock states $|n\rangle$, where n is the number of quanta in that state. The form of the total system-reservoir Hamiltonian is given by

$$\hat{H} = \hbar\omega_0\hat{a}^\dagger\hat{a} + \sum_k \hbar\omega_k\hat{n}_k + i\hbar\sum_k g_k[(\hat{a}^\dagger)^{n+1}\hat{b}_k - \hat{a}^{n+1}\hat{b}_k^\dagger],$$
$$(2)$$

where the first term is the unperturbed system Hamiltonian $\hat{H}_S = \hbar\omega_0\hat{a}^\dagger\hat{a}$, the second term represents the free reservoir (bath) Hamiltonian $\hat{H}_R = \sum_j \hbar\omega_j\hat{n}_j \equiv \sum_j \hbar\omega_j\hat{b}_j^\dagger\hat{b}_j$, consisting of a large number of harmonic oscillators, and the last term describes the interaction Hamiltonian for $n \in \mathbb{Z}^+$. Here $\hat{a}$ and $\hat{b}_k$ are the annihilation operators for the system and reservoir, respectively, which fulfill the following commutation relations:

$$[\hat{a}, \hat{a}^\dagger] = 1, \qquad [\hat{b}_m, \hat{b}_n^\dagger] = \delta_{mn}, \qquad [\hat{a}, \hat{b}_k] = 0. \quad (3)$$

The elementary exchange of energy between the system and the reservoir consists of single quantum absorption from the kth bath mode and the simultaneous creation of the $(n + 1)$ quanta of excitation in the system mode and vice versa.

Since our object here is to recover the dissipative dynamics of the Liénard oscillation from the system-reservoir Hamiltonian Eq. (2) for which dissipation is always accompanied by an internal quantum noise, we expect a modification for Eq. (1) in the following form in a c-number description:

$$\ddot{x} + f(x)\dot{x} + \omega_0^2 x = \eta(t). \quad (4)$$

Here $\eta(t)$ is a Gaussian white δ-correlated quantum noise with zero mean. In other words, we look for a connection between the nonlinear dissipation $f(x)\dot{x}$ and the stochastic noise term $\eta(t)$ which allows the dynamical system to admit stable but noisy limit cycle oscillations.

From Eq. (2), using the commutation relations [Eq. (3)], we can easily evaluate the Heisenberg operator equations for the system and bath degrees of freedom as

$$\dot{\hat{a}}(t) = -i\omega_0\hat{a}(t) + (n+1)\sum_k g_k(\hat{a}^\dagger)^n\hat{b}_k(t), \quad (5)$$

and

$$\dot{\hat{b}}_j(t) = -i\omega_j\hat{b}_j(t) - g_j\hat{a}^{n+1}(t), \quad (6)$$

respectively. Formally integrating Eq. (6), we get

$$\hat{b}_j(t) = \hat{b}_j e^{-i\omega_j t} - g_j\int_0^t \hat{a}^{n+1}(t')e^{-i\omega_j(t-t')}dt', \quad (7)$$

where the first term is the free evolution of the bath operator, whereas the second term is arising due to the interaction with the system. Inserting Eq. (7) into Eq. (5), we find

$$\dot{\hat{a}}(t) = -i\omega_0\hat{a}(t) + (n+1)\sum_k g_k\hat{b}_k(\hat{a}^\dagger)^n(t)e^{-i\omega_k t}$$

$$- (n+1)\sum_k g_k^2(\hat{a}^\dagger)^n(t)\int_0^t \hat{a}^{n+1}(t')e^{-i\omega_k(t-t')}dt'.$$
$$(8)$$

Now introducing the slowly varying operator $\hat{A}(t) = \hat{a}(t)e^{i\omega_0 t}$ in Eq. (8), which varies little over the inverse reservoir bandwidth, we can take the system operator out of the integral by substituting $\hat{A}(t') \simeq \hat{A}(t)$ under the Markov approximation [2]. Then replacing the remaining integral of t' by the usual δ function [6], we finally arrive at the reduced operator equation for the system which is given by

$$\dot{\hat{A}}(t) = -\gamma_{n+1}(\hat{A}^\dagger)^n(t)\hat{A}^{n+1}(t) + \hat{F}_{n+1}(t)(\hat{A}^\dagger)^n(t), \quad (9)$$

and its Hermitian adjoint. The reduced dynamics described by the operator Langevin Eq. (9) contains the usual dissipative term,

$$\gamma_{n+1} = (n+1)\sum_k g_k^2\pi\delta[\omega_k - (n+1)\omega_0], \quad (10)$$

as well as the noise term,

$$\hat{F}_{n+1}(t) = (n+1)\sum_k g_k\hat{b}_k\exp[-i\{\omega_k - (n+1)\omega_0\}t], \quad (11)$$

which is multiplicative in nature. The noise operator in Eq. (9) appears as a natural consequence of the system-reservoir coupling. It is, thus, imperative that the Liénard system with a microscopic basis must be internally noisy. Combining Eq. (9) and its Hermitian adjoint we may further define a noise operator $\hat{G}_{n+1}(t) = \hat{F}_{n+1}(t) + \hat{F}_{n+1}^\dagger(t)$.

The properties of the reservoir can then be calculated by thermal averaging over appropriately ordered noise operators. To this end we define the quantum statistical average of any reservoir operator $\hat{O}$ as

$$\langle\hat{O}(t)\rangle_{\text{qs}} = \frac{\text{Tr}[\hat{O}\exp(-\hat{H}_R/KT)]}{\text{Tr}[\exp(-\hat{H}_R/KT)]}, \quad (12)$$

where $\hat{H}_R = \sum_j \hbar\omega_j \hat{n}_j$ at $t = 0$ and $\hat{n}_j$ denotes the number operator for the jth bath mode.

Based on the above considerations, the noise properties of the operator may be calculated by using the canonical thermal distribution Eq. (12). The noise operator $\hat{G}_{n+1}(t)$ depends on bath operators $\hat{b}_k$ and $\hat{b}_k^\dagger$ whose standard quantum mechanical average over Fock states $|n\rangle$ is identically zero ($\langle n|\hat{b}_k|n\rangle = 0 = \langle n|\hat{b}_k^\dagger|n\rangle$). Equation (12), therefore, immediately gives

$$\langle \hat{G}_{n+1}(t) \rangle_{qs} = 0, \tag{13}$$

and satisfies the following relation for the noise operators:

$$\Re[\langle \hat{F}_{n+1}^\dagger(t)\hat{F}_{n+1}(t') + \hat{F}_{n+1}(t)\hat{F}_{n+1}^\dagger(t')\rangle_{qs}]$$
$$= (n+1)^2 \sum_k g_k^2 \langle (2\hat{n}_k^B + 1) \rangle_{qs}$$
$$\times \cos\{[\omega_k - (n+1)\omega_0](t - t')\}$$
$$= (n+1)^2 \sum_k g_k^2 \coth\left(\frac{\hbar\omega_k}{2KT}\right)$$
$$\times \cos\{[\omega_k - (n+1)\omega_0](t - t')\}, \tag{14}$$

where the cotangent hyperbolic factor in Eq. (14) can be identified with the Bose-Einstein distribution,

$$\langle \hat{n}_k^B \rangle_{qs} = \frac{1}{e^{\hbar\omega_k/KT} - 1} = \bar{n}_B(\omega_k), \tag{15}$$

using the following relation:

$$\langle (2\hat{n}_k^B + 1) \rangle_{qs} = 2\bar{n}_B(\omega_k) + 1 = \coth\left(\frac{\hbar\omega_k}{2KT}\right). \tag{16}$$

Equation (14) refers to the fluctuation-dissipation relation for a bosonic bath, which guarantees that the overall system is thermodynamically closed. In the subsequent sections we show that the above detailed balance helps us to preserve the dynamical stability of the limit cycle even in the presence of noise. An external noise of even very weak intensity on the other hand destroys the attractors. The plus one factor in Eq. (16) is responsible for vacuum fluctuation which is always present on the quantum scale even at absolute zero. Its implication on the microscopic realization of limit cycles will be analyzed in the following sections.

III. *C*-NUMBER DESCRIPTION OF THE QUANTUM LANGEVIN EQUATION

The main purpose of this section is to construct a quantum Langevin equation with c-number noise. As a first step, we return to Eq. (9) and carry out the quantum mechanical average $\langle \cdots \rangle$ over the initial product separable quantum states of the system oscillator and the bath oscillators at $t = 0$, $|\alpha\rangle |\mu_1\rangle |\mu_2\rangle \cdots |\mu_k\rangle \cdots |\mu_N\rangle$ to obtain

$$\langle \dot{\hat{A}}(t) \rangle = -\gamma_{n+1} \langle (\hat{A}^\dagger)^n(t)\hat{A}^{n+1}(t) \rangle + \langle \hat{F}_{n+1}(t) \rangle \langle (\hat{A}^\dagger)^n(t) \rangle. \tag{17}$$

Here $|\alpha\rangle$ refers to the initial coherent state of the system, and $\{|\mu_k\rangle\}$ corresponds to initial coherent states of the bath operators. The rationale behind using harmonic oscillator coherent states for the system and bath operators is to recast Eq. (17) as

a classical-looking Langevin equation for the reduced system oscillator interacting with a bosonic bath.

Defining the following quantum mechanical averages for the system and noise operators,

$$\langle \hat{A}(t) \rangle = \alpha(t), \quad \langle \hat{A}^\dagger(t) \rangle = \alpha^*(t), \quad \langle \hat{G}_{n+1}(t) \rangle = \xi_{n+1}(t), \tag{18}$$

Eq. (17) and its adjoint can be written as (see the Appendix)

$$\dot{\alpha}(t) = -\gamma_{n+1}|\alpha|^{2n}\alpha + f_{n+1}(t)(\alpha^*)^n,$$
$$\dot{\alpha}^*(t) = -\gamma_{n+1}|\alpha|^{2n}\alpha^* + f_{n+1}^*(t)\alpha^n. \tag{19}$$

Here the c-number quantum noise $\xi_{n+1}(t)$ is given by

$$\xi_{n+1}(t) = f_{n+1}(t) + f_{n+1}^*(t),$$
$$= (n+1)\sum_k g_k[\mu_k(0)\exp\{-i[\omega_k - (n+1)\omega_0]t\}$$
$$+ \mu_k^*(0)\exp\{i[\omega_k - (n+1)\omega_0]t\}], \tag{20}$$

where μ_k and μ_k^* are the associated c numbers for the bath operators. In deriving Eqs. (19) from Eq. (17), we only consider normal ordering for the system operators. Instead for use of a different ordering of the operators, we may end up with altogether different (but equivalent [2]) forms for the nonlinear damping. We will return to this point in the next section when we discuss the quantum-classical correspondence for the limit cycles within the framework of the same interaction Hamiltonian under mean-field approximation [17–19].

Now to realize $\xi_{n+1}(t)$ as an effective c-number noise, we introduce the ansatz that $\mu_k(0)$ and $\mu_k^*(0)$ in Eq. (20) are distributed according to Wigner thermal canonical distribution of Gaussian form [20] as follows:

$$W_k^B[\mu_k(0), \mu_k^*(0)] = N_B \exp\left\{-\frac{|\mu_k(0)|^2}{2\coth\left(\frac{\hbar\omega_k}{2KT}\right)}\right\}. \tag{21}$$

Here N_B is the normalization constant and $\coth(\frac{\hbar\omega_k}{2KT})$ is the width of the distribution. For any arbitrary quantum mechanical mean value of a bath operator $\langle \hat{B}_k \rangle$, which is a function of $\mu_k(0)$ and $\mu_k^*(0)$, its statistical average can then be calculated as

$$\langle\langle \hat{B}_k \rangle\rangle_s = \int \langle \hat{B}_k \rangle W_k^B[\mu_k(0), \mu_k^*(0)]d\mu_k(0)d\mu_k^*(0). \tag{22}$$

Using the ansatz of Eq. (21) and the definition of statistical average Eq. (22), one can show that the c-number noise $\xi_{n+1}(t)$ satisfies the following relations, respectively:

$$\langle \xi_{n+1}(t) \rangle_s = 0, \tag{23}$$

and

$$\langle \xi_{n+1}(t)\xi_{n+1}^*(t') \rangle_s = (n+1)^2 \sum_k g_k^2 \coth\left(\frac{\hbar\omega_k}{2KT}\right)$$
$$\times \cos\{[\omega_k - (n+1)\omega_0](t - t')\}. \tag{24}$$

Equations (23) and (24) imply that the c-number noise $\xi_{n+1}(t)$ is zero centered and follows the fluctuation-dissipation relation as expressed in Eq. (14). Therefore, Eqs. (24) and (14) are equivalent. In order to identify the connection between

Eqs. (14) and (24), it is sometime convenient to write down the fluctuation-dissipation relation in the following form:

$$\langle \xi_{n+1}(t)\xi_{n+1}^*(t')\rangle_s = \Re[\langle \hat{F}_{n+1}^\dagger(t)\hat{F}_{n+1}(t') + \hat{F}_{n+1}(t)\hat{F}_{n+1}^\dagger(t')\rangle_{qs}],$$

$$= 2(n+1)\gamma_{n+1} \coth\left[\frac{(n+1)\hbar\omega_0}{2KT}\right]\delta(t-t'). \tag{25}$$

In deriving the above expression we have assumed that the bath modes are closely spaced in frequency so that one may replace the summation over k in Eq. (24) by an integral over ω using the density of states $\rho(\omega)$ [6] which yields $\gamma_{n+1} = (n+1)\pi g^2[(n+1)\omega_0]\rho[(n+1)\omega_0]$. Second, Eq. (25) depends on the noise operator ordering but not on time ordering. This indicates a clear-cut advantage of the c-number formalism which allows us to bypass the operator ordering prescription for the derivation of noise properties [21]. The c-number noise $\xi_{n+1}(t)$ as characterized by Eqs. (23) and (24) is classical looking in form but essentially quantum mechanical in nature. Therefore, the essence of the microscopic origin of quantum limit cycles can be captured in the present formalism quite effectively in terms of the c-number description by simply implementing the techniques of classical nonequilibrium statistical mechanics. The above formalism has been extensively used in several earlier occasions in connection with quantum optics, chemical dynamics, and multidimensional transition state theory in the context of spin and bosonic baths [21–23].

IV. CONSTRUCTION OF THE QUANTUM LIÉNARD SYSTEM; APPLICATION OF THE PROPOSED MODEL

We are now in a position to apply our method to various nonlinear systems that produce limit cycle oscillations. Two such classic examples are van der Pol and Rayleigh oscillators, whose basic equation of motions are

$$\ddot{x} + \omega_0^2 x - \epsilon(1-x^2)\dot{x} = 0, \tag{26}$$

and

$$\ddot{x} + \omega_0^2 x - \epsilon(1-\dot{x}^2)\dot{x} = 0, \tag{27}$$

respectively, where we assume $\epsilon > 0$. A broad class of biological and chemical oscillations are modeled either in terms of the Rayleigh or van der Pol oscillator or in terms of their generalization [24–26]. According to Eq. (2), both above models are subject to internal quantum noise satisfying the fluctuation-dissipation theorem when derived from their respective microscopic Hamiltonians. The form of the interaction for these specific examples simplifies to

$$\hat{H}_I = i\hbar \sum_k g_k[(\hat{a}^\dagger)^2\hat{b}_k - (\hat{a})^2\hat{b}_k^\dagger], \tag{28}$$

which is a special case of Eq. (2). From this interaction Hamiltonian [Eq. (28)], we proceed as in Sec. II and finally arrive at the following operator Langevin equations for the reduced system:

$$\dot{\hat{A}}(t) = -\gamma_2\hat{A}^\dagger(t)\hat{A}^2(t) + \hat{F}_2(t)\hat{A}^\dagger(t), \tag{29}$$

which is the generalized operator Eq. (9) for $n = 1$. If we follow the usual normal ordering prescription discussed in

Sec. III, the resulting quantum dynamics becomes identical to the generalized c-number Eq. (19) with $n = 1$:

$$\dot{\alpha}(t) = -\gamma_2|\alpha|^2\alpha + f_2(t)\alpha^*. \tag{30}$$

Substituting α and α^* by

$$\alpha = \frac{1}{2}\left(x - \frac{i\dot{x}}{\omega_0}\right)\exp[-i\omega_0 t],$$

$$\alpha^* = \frac{1}{2}\left(x + \frac{i\dot{x}}{\omega_0}\right)\exp[i\omega_0 t], \tag{31}$$

and collecting the terms of right order from Eq. (30), we have the following differential equation for the system:

$$\ddot{x} + \gamma_2 x^2\dot{x} + \omega_0^2 x = \eta_2(x,\dot{x}), \tag{32}$$

where

$$\eta_2(x,\dot{x}) = \frac{i\omega_0}{2}\left[f_2\left(x + \frac{i\dot{x}}{\omega_0}\right)e^{2i\omega_0 t} - f_2^*\left(x - \frac{i\dot{x}}{\omega_0}\right)e^{-2i\omega_0 t}\right] \tag{33}$$

is the quantum noise term arising out of the system-reservoir interaction. Equation (32) may be regarded as the simplest representation of the quantum Liénard system [Eq. (4)] where $f(x) = \epsilon x^2$ and $g(x) = \omega_0^2 x$. The typical phase portraits of the system in the absence and presence of internal noise are shown in Fig. 1. In general parameters γ_2 and ϵ are proportional to each other as we will establish shortly. For this simplest possible case they turn out to be equal with proportionality constant unity. We use the PYTHON code for random number generators [27] to simulate white noise $\eta_2(x,\dot{x})$ [28] satisfying Gaussian statistics. Throughout our paper we choose $\omega_0 = 1$ for numerical calculations. In the presence of noise, the stochastic trajectories neither collapse to a steady state nor diverge. This implies that the detailed balance in the form of fluctuation-dissipation [Eq. (24)] guarantees the dynamical stability of motion [Fig. 1(b)]. In the numerical simulation whereas the noise term maintains the fluctuation-dissipation relation, the limit cycle retains its shape but becomes fuzzy [Fig. 1(b)] due to noise. An average over the trajectories of such a noisy limit cycle yields the identical stable cycle inferred from the classical picture [blue lines in Figs. 1(b)–3(b)]. Numerical errors in computing the statistical average with the steepest slope for the van der Pol oscillator is responsible for slight mismatch of the averaged trajectory at corners of Fig. 2(b). If the detailed balance in the form of Eq. (25) is not maintained, i.e., when the noise is of external origin, the limit cycle is destroyed in every case. This is illustrated in all three examples [Figs. 1(c)–3(c)] when the noise is of external origin.

Equations (29) and (30) are the quantum Liénard equations in operator (and c-number) form. In order to recover the two special cases of this system we proceed as follows: We first note that the use of commutation relation $[\hat{A}, \hat{A}^\dagger] = 1$ in the operator Eq. (29) leads to several other equivalent forms [2]. For example, if $f(\hat{A}, \hat{A}^\dagger) = \hat{A}\hat{A}^\dagger$, then use of the commutation relation $[\hat{A}, \hat{A}^\dagger] = 1$ implies $f(\hat{A}, \hat{A}^\dagger) = \hat{A}\hat{A}^\dagger = \hat{A}^\dagger\hat{A} + 1$. These two operator forms are different, but they are equivalent in view of that fact that they represent the same dynamical system in distinct forms. In what follows, we may write Eq. (29) as

$$\dot{\hat{A}}(t) = -\gamma_2(\hat{A}\hat{A}^\dagger\hat{A} - \hat{A}) + \hat{F}_2(t)\hat{A}^\dagger(t). \tag{34}$$
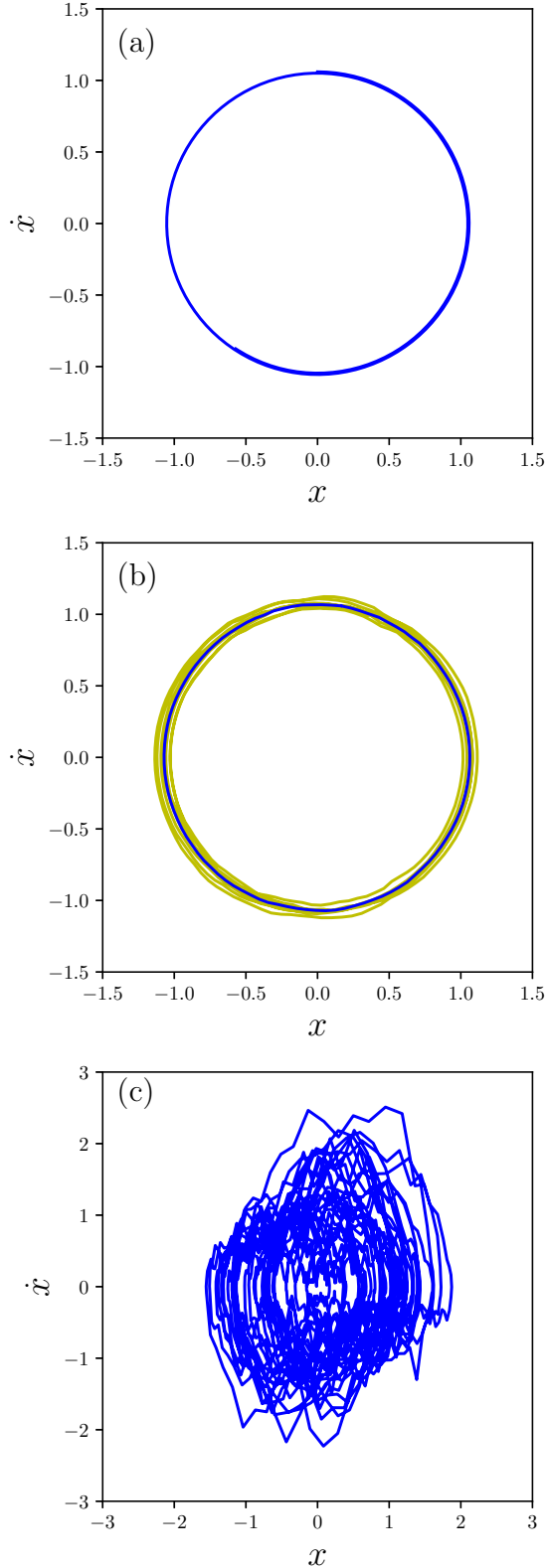
FIG. 1. The phase space trajectories of the (a) classical and the (b) quantum Liénard oscillators [Eq. (32)] with $f(x) = \epsilon x^2$ are plotted in the absence (a) and the presence (b) of intrinsic noise. The average trajectory is shown (blue) in (b) for internal noise. (c) The quantum Liénard oscillator in the presence of external noise: the lack of detailed balance destroys the limit cycle oscillation. We set $\epsilon = 0.01$ and $KT = 50$ (arbitrary units).

More generally we may write for Eq. (9),

$$\dot{\hat{A}}(t) = -\gamma_{n+1} \sum_{p,q\cdots r} \phi(p, q \cdots r)\hat{A}^p(\hat{A}^\dagger)^q \cdots \hat{A}^r$$
$$+ \hat{F}_{n+1}(t)(\hat{A}^\dagger)^n(t). \tag{35}$$

In the next step we derive the c-number equivalent of the above operator equation such that it corresponds to the specific classical Liénard form. This may be achieved by appropriate operator ordering followed by performing average with coherent states under the mean-field approximation. We are then led to a desired form of c-number equivalent of Eq. (35),

$$\dot{\alpha} = -\gamma_{n+1} \sum_{i,j} \psi(i, j)\alpha^i(\alpha^*)^j + f_{n+1}(t)(\alpha^*)^n. \tag{36}$$

A pertinent point is to be noted. Recognizing the right hand side of Eq. (36) as a polynomial in c numbers, the term with its highest power is *sufficient* for the existence of the limit cycle as shown in Eq. (30). The inclusion of terms of lower power is *necessary* to retain the shape of the limit cycle that conforms faithfully with its classical counterpart. The mean-field approximation, therefore, allows us to establish a quantum-classical correspondence. Furthermore, we emphasize that the stability of the limit cycles and their numbers remain invariant with respect to the inclusion of terms with lower power of the polynomial. Based on these considerations Eq. (34) takes the following form (for details see the Appendix):

$$\dot{\alpha} = -\frac{\gamma_2}{m_1}(m_1|\alpha|^2 + m_0)\alpha + f_2\alpha^*, \tag{37}$$

which may correspond to Liénard systems for specific values of m_1 and m_0. In what follows, we will show that van der Pol and Rayleigh oscillators appear from it as special cases.

A. van der Pol oscillator

The van der Pol oscillator is a prototypical self-sustained oscillator which has been used to model the dynamics of a variety of classical [29] and biological processes, such as heart [30], neurons [31], and circadian rhythms [32]. If we substitute $m_1 = 1$ and $m_0 = -1$ in Eq. (37), we find

$$\dot{\alpha}(t) = -\gamma_2(|\alpha|^2 - 1)\alpha + f_2(t)\alpha^*, \tag{38}$$

and its complex conjugate where the c-number noise expression follows from generalized Eq. (20) for $n = 1$ as

$$f_2(t) = 2 \sum_k g_k \mu_k(0) \exp[-i(\omega_k - 2\omega_0)t]. \tag{39}$$

If we further replace α and α^* in Eq. (38) by the usual form [Eq. (31)] and neglect the high frequency terms, one may obtain the following differential equation after a little bit of rearrangement,

$$\ddot{x} + \gamma_2(x^2 - 1)\dot{x} + \omega_0^2 x = \eta_2(x, \dot{x}), \tag{40}$$

where by definition $\gamma_2 > 0$.

The c-number Eq. (40) looks similar to the classical equation of the van der Pol oscillator which is simultaneously driven by the c-number quantum noise. This quantum van der Pol oscillator can, therefore, be regarded as a quantum

harmonic oscillator with two types of dissipation: negative $(-\dot{x})$ as well as nonlinear $(x^2\dot{x})$ dampings, and an internal noise—the combination of which leads to a noisy but self-sustained limit cycle oscillation. The appearance of the limit cycle indicates that for the system which follows the isolated asymptotic trajectory, the gain in energy due to self-excitation is equal to the loss of energy due to dissipation. On the other hand, the Langevin equation derived from an equilibrium many body quantum system without any external noise possesses time-reversal symmetry. The fluctuation-dissipation relation thereby entails the detailed balance condition which implies that the energy dissipated by the system on average is equal to the energy gained by it through fluctuation of the reservoir. The energy loss-gain in the former processes is purely dynamical whereas that as a result of detailed balance is statistical in nature.

Because of its simple form, such a quantum van der Pol oscillator has recently gained significant interest for the realization of synchronization phenomena on the quantum scale [33–36]. Interestingly, an experiment with trapped ions may serve as an ideal test bed for simulating collective dynamics for such oscillators [37–40]. The phase space diagrams for both the classical van der Pol oscillator and its quantum counterpart with Gaussian white noise $\eta_2(x, \dot{x})$ [Eq. (33)] are plotted in Fig. 2 for illustration.

B. Rayleigh oscillator

Similar to the van der Pol oscillator, the Rayleigh oscillator can also be obtained from the same interaction Hamiltonian Eq. (28) or in other words from the same generalized c-number Langevin dynamics [Eq. (37)]. This immediately suggests that both the noisy van der Pol and the Rayleigh oscillators are equivalent to the simplest possible form of the Liénard system [Eq. (30)] within a mean-field description. Thus, the characteristic features of both oscillators can be essentially captured by Eq. (32).

To recover the explicit form of the Rayleigh oscillator, we substitute $m_1 = 3$ and $m_0 = -1$ in Eq. (37) so that we get

$$\dot{\alpha}(t) = \frac{\gamma_2}{3}(1 - 3|\alpha|^2)\alpha + f_2\alpha^*, \tag{41}$$

and its complex conjugate. To arrive at the standard form of the Rayleigh oscillator equation from Eq. (41), instead of Eq. (31), we now use the following substitution for α and α^*:

$$\alpha = \frac{1}{2}(\omega_0 x - i\dot{x})\exp[-i\omega_0 t],$$
$$\alpha^* = \frac{1}{2}(\omega_0 x + i\dot{x})\exp[+i\omega_0 t]. \tag{42}$$

This difference in the substitution can be traced back to the dimensional relationship between the phase space dynamical variables of the two oscillators as understandable from Eqs. (26) and (27). With the help of Eqs. (42) and using the same treatment as before, we finally reduce the following equation of motion for the quantum Rayleigh oscillator,

$$\ddot{x} + \frac{1}{3}\gamma_2(\dot{x}^2 - 1)\dot{x} + \omega_0^2 x = \eta_2(x, \dot{x}). \tag{43}$$
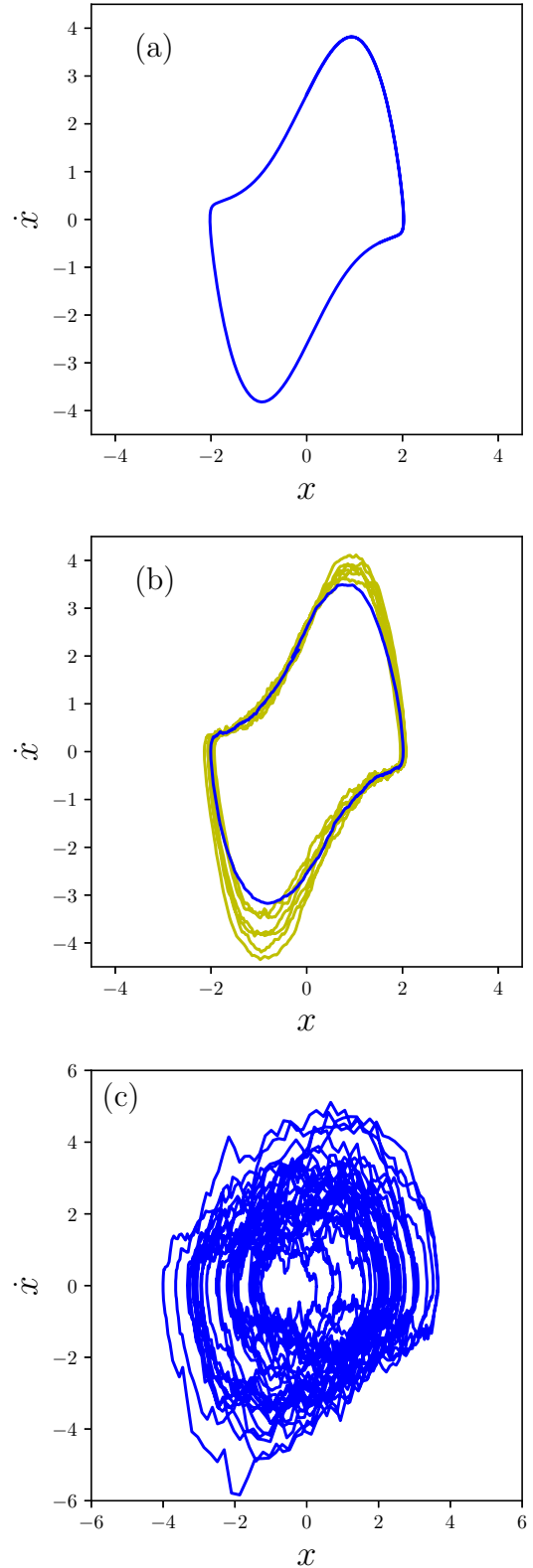

FIG. 2. The phase space trajectories of a (a) classical [Eq. (26)] and a (b) quantum van der Pol oscillator [Eq. (40)] are plotted for $\epsilon = 2$ and $KT = 0$. It is shown that quantum fluctuations can retain the general shape of the limit cycle oscillation even at absolute zero (arbitrary units). The average trajectory is shown (blue) in (b) closely resembles the classical one. (c) In the absence of detailed balance, an external noise completely destroys the limit cycle.

Similar to the van der Pol oscillator, the Rayleigh oscillator has a close kinship with the parametric oscillation in pipe organs as pioneered by Rayleigh [41] and in wave mixing in nonlinear optics [42]. The phase space plots for the classical Rayleigh oscillator and its quantum version with Gaussian white noise $\eta_2(x, \dot{x})$ [Eq. (33)] are depicted in Fig. 3. In general, we find that average quantum dynamics agrees with the phase boundary of the classical limit cycle even on the microscopic scale.

C. Construction of generalized quantum Liénard oscillators

Let us now extend our formalism to construct a broader class of Liénard oscillators. For this we adopt the following recipe: First we resort to operator Langevin Eq. (9) and discuss the first two cases [$n = 1$ and 2 of Eq. (9)] as illustrative examples. Next we generalize the concept (see the Appendix for the details) to construct the most generic form of the Liénard oscillators that can be derived from our model Hamiltonian Eq. (2).

(1) *Case 1.* $f(x) = \epsilon(a_1 x^2 + a_0)$. In the previous section we have shown that a general form of $f(x) = \epsilon(a_1 x^2 + a_0)$ can be obtained from the interaction Hamiltonian $\hat{H}_I = i\hbar \sum_k g_k[(\hat{a}^\dagger)^2 \hat{b}_k - (\hat{a})^2 \hat{b}_k^\dagger]$. The respective c-number Langevin dynamics is given by Eq. (37), and the corresponding equation of motion takes the form of $\ddot{x} + \omega_0^2 x = -(\gamma_2/m_1)(m_1 x^2 + m_0)\dot{x} + \eta_2(x, \dot{x})$ where we may choose m_1 and m_0 so that it is consistent with the values of a_1, a_0, and ϵ of $f(x)$. Essentially one sets $\epsilon = (\gamma_2/m_1)$, then m_1 and m_0 become identical to a_1 and a_0, respectively. This can be regarded as a representation of the Liénard system in a quantum scenario with the friction coefficient $f(x) = \epsilon(a_1 x^2 + a_0)$ such that our desired system exhibits a single quantum limit cycle.

(2) *Case 2.* $f(x) = \epsilon(a_2 x^4 + a_1 x^2 + a_0)$. Let us now consider the most general form of $f(x)$ with the next highest order power in accordance with Liénard's theorem. In a similar spirit one can show that the above form of $f(x)$ can be obtained from the Hamiltonian,

$$\hat{H}_I = i\hbar \sum_k g_k[(\hat{a}^\dagger)^3 \hat{b}_k - (\hat{a})^3 \hat{b}_k^\dagger], \qquad (44)$$

which corresponds to $n = 2$ of the generalized interaction Hamiltonian [Eq. (2)]. From the operator Langevin equations Eq. (9), one carries out the same procedure to derive the c-number Langevin equations for $n = 2$ (see the Appendix),

$$\dot{\alpha} = -\frac{\gamma_3}{m_2}[m_2|\alpha|^4 + m_1|\alpha|^2 + m_0]\alpha + f_3(\alpha^*)^2. \qquad (45)$$

Following the standard substitution as given in Eq. (31) we find

$$\ddot{x} + \omega_0^2 x = -\frac{\gamma_3}{m_2}\left[\frac{m_2}{2}x^4 + m_1 x^2 + m_0\right]\dot{x} + \eta_3(x, \dot{x}). \qquad (46)$$

This is the relevant quantum version of the Liénard system with $f(x) = \epsilon(a_2 x^4 + a_1 x^2 + a_0)$ where we have to choose the values of m_2, m_1, and m_0 judiciously such that it matches with the known form of $f(x)$. Some of the choices of $\{m_2, m_1, m_0\}$ are shown in Table I. Note that in each case the value of ϵ varies as γ_3/m_2. It was shown by Rychkov
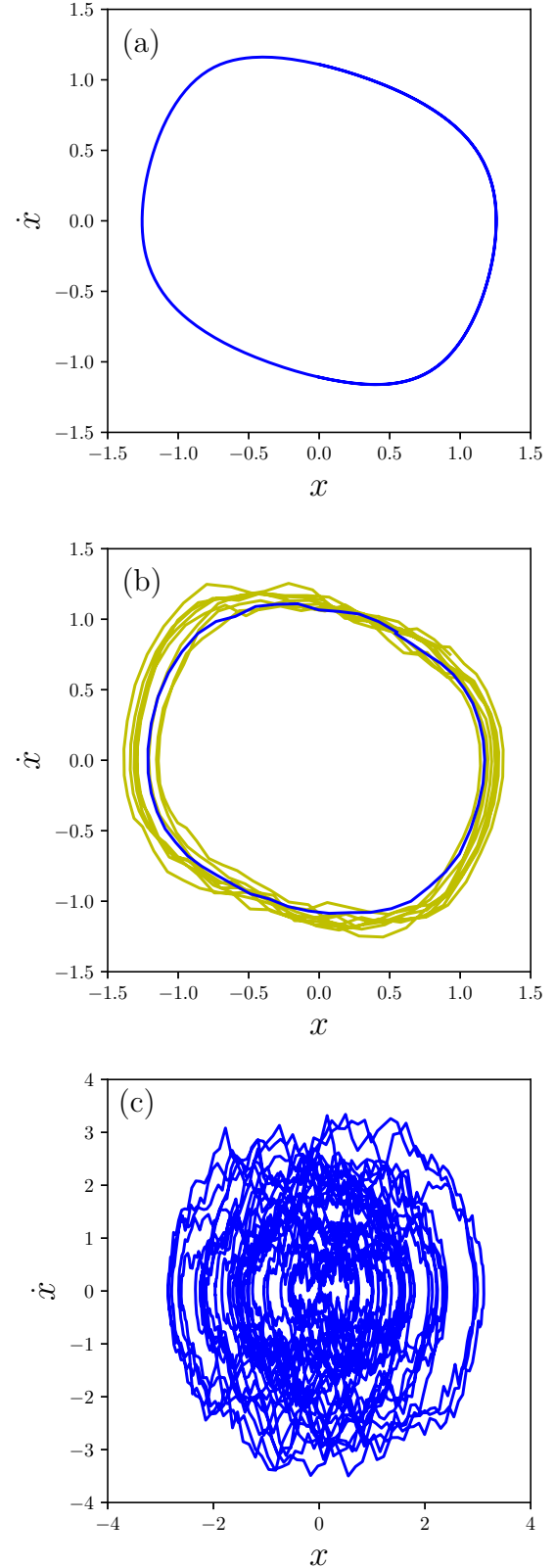

FIG. 3. (a) The phase space trajectories of a (a) classical [Eq. (27)] and a (b) quantum Rayleigh oscillator [Eq. (43)] are plotted for the parameters $\epsilon = 1$ and $KT = 10$ (arbitrary units). The statistical average over noisy trajectories in (b) is identical to its classical counterpart. (c) An external noise breaks the stable limit cycle.

TABLE I. Some of the choices of $\{m_2, m_1, m_0\}$ are shown.

Examples	m_2	m_1	m_0	$f(x)$
1	2	1	-1	$\epsilon(x^4 + x^2 - 1)$
2	2	-1	0	$\epsilon(x^4 - x^2)$
3	2	0	-1	$\epsilon(x^4 - 1)$

[43] that the number of limit cycles is at most 2 for $f(x) = \epsilon(a_2x^4 + a_1x^2 + a_0)$. Now extrapolating the form of the above two cases we may generalize the formula by the method of mathematical induction to an arbitrary polynomial $f(x)$ as given below:

(3) *Case 3.* $f(x) = a_nx^{2n} + a_{n-1}x^{2n-2} \cdots + a_1x^2 + a_0$. Let us now consider the most general form of an even function $f(x)$. From the operator equation of motion (9) one can evaluate the following equations for the c-number Langevin dynamics (see the Appendix),

$$\dot{\alpha} = -\frac{\gamma_{n+1}}{m_n}[m_n|\alpha|^{2n} + m_{n-1}|\alpha|^{2n-2} \cdots + m_1|\alpha|^2 + m_0]\alpha$$
$$+ f_{n+1}(\alpha^*)^n, \tag{47}$$

$$\dot{\alpha}^* = -\frac{\gamma_{n+1}}{m_n}[m_n|\alpha|^{2n} + m_{n-1}|\alpha|^{2n-2} \cdots + m_1|\alpha|^2 + m_0]\alpha^*$$
$$+ f_{n+1}^*\alpha^n, \tag{48}$$

where $m_0, m_1, \ldots m_n$ are independent of one another. Proceeding in a similar way, it is possible to construct the following dynamical equation of motion for the system,

$$\ddot{x} + \omega_0^2 x = -\frac{\gamma_{n+1}}{m_n}(a_nx^{2n} + a_{n-1}x^{2n-2} \cdots + a_1x^2 + a_0)\dot{x}$$
$$+ \eta_{n+1}(x, \dot{x}), \tag{49}$$

where

$$a_j = \frac{m_j}{2^j C_{j+1}}j, \quad \forall\; j = 1, 2 \cdots n \quad \text{and} \quad a_0 = m_0. \tag{50}$$

The above equation gives us the relation between different m_j's with a_j's. Note that for $n = 1$ (Case 1), $a_j = m_j$ irrespective of all j.

According to Refs. [16,44–46] the system of Liénard-Smith-Levinson form with $f(x)$, a polynomial of highest degree $2n$ where the coefficients $a_n, a_{n-1} \cdots a_1$ alternate in sign, can support at most n numbers of limit cycles. Equation (49) can be interpreted as a quantum generalization of the Liénard oscillator with the most general form of $f(x)$ which may have at most n numbers of limit cycles. Phase portraits of a system with multiple limit cycles in both classical and quantum scenarios are plotted in Fig. 4. Classically, any noisy limit cycle at $T = 0$ does not make any sense, but quantum mechanically, due to vacuum fluctuation, one can have stable limit cycle oscillation even at absolute zero. This is a pure quantum phenomenon which does not have any classical analogs. As an extension of the Liénard-Smith-Levinson form of oscillators, the generalized Rayleigh family of oscillators [13,47,48] can also be constructed from the above prescription (see the Appendix).
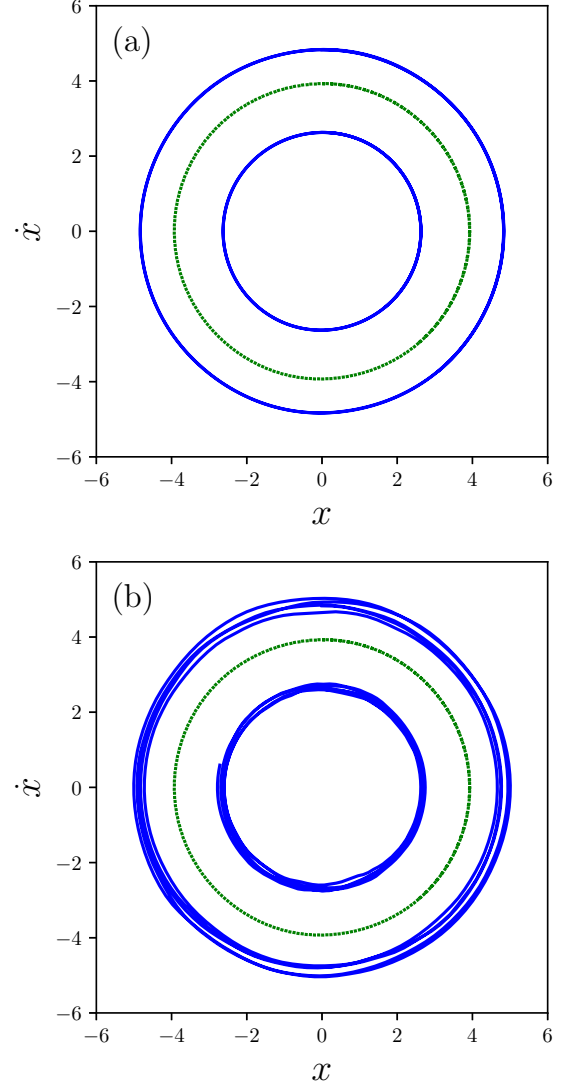

FIG. 4. The phase space trajectories of the (a) classical and the (b) quantum Liénard systems with $f(x) = 0.005 * x^6 - 0.144 * x^4 + x^2 - 1$ [16] are plotted for $\epsilon = 0.01$. In both cases we have two stable (blue line) and one unstable (green dotted line) limit cycles. Pure quantum fluctuations at $T = 0$ may lead to noisy but stable oscillations even for the two stable limit cycles (arbitrary units).

V. CONCLUSIONS

In this paper we have presented a generalized microscopic quantum description of the classical Liénard system. Our approach is based on two essential elements. First, the interaction Hamiltonian consisting of suitable nonlinear coupling terms serves as a universal paradigm for the description of several variants of the Liénard system with appropriate noise and nonlinear dissipation. Second, the harmonic oscillator coherent states facilitate the use of the c-number description for the reduced oscillator equation of motion. The major findings of our paper are as follows:

(i) A classical Liénard system without any noise is purely a nonlinear dynamical oscillator which depending on the specificity of nonlinear dissipation admits single or multiple limit cycles. As opposed to it, the quantum noise appears in the

dynamics as a consequence of the system-reservoir framework of our microscopic model and carries the signature of pure quantum effects.

(ii) The effect of quantum noise makes the stable limit cycle fuzzy; however, it does not lose its stability in the sense that the limit cycle trajectory neither collapses to a steady state nor diverges. The noninterference of the two energy balance processes is characteristic of the internal noise of a thermodynamically closed system. Thus, the detailed balance through the nonlinear fluctuation-dissipation relation keeps the dynamical stability of the limit cycle preserved. In contrast, the external noise tends to destroy the limit cycle or, in general, the dynamical stability of the Liénard system.

(iii) Within the framework of a mean-field description the stability and the number of limit cycles are invariant under the addition of terms of lower orders to the polynomial whereas their inclusion is crucial for retaining the shape of the limit cycle oscillation. As a result, despite the fact that van der Pol and Rayleigh represent two distinct oscillators with the single limit cycle at the classical level, they can be constructed from the same microscopic interaction Hamiltonian within the mean-field theory. In fact a large number of Liénard systems (both van der Pol and Rayleigh families of cycles) differing in the form of their nonlinear damping can be derived from the same microscopic Hamiltonian at the mean-field level. In all such cases the number of limit cycles is determined by the highest power of the polynomial governing the nonlinear dissipation as dictated by the Liénard theorem.

ACKNOWLEDGMENTS

APPENDIX

1. Derivation of Eq. (19)

Since $A|\alpha\rangle = \alpha|\alpha\rangle$ and $\langle\alpha|A^\dagger = \langle\alpha|\alpha^*$, Eq. (17) can be written as

$$\dot{\alpha} = -\gamma_{n+1}\alpha^{n+1}(\alpha^*)^n + f_{n+1}(t)(\alpha^*)^n$$
$$= -\gamma_{n+1}|\alpha|^{2n}\alpha + f_{n+1}(t)(\alpha^*)^n. \tag{A1}$$

One can similarly write for the operator Langevin equation $\hat{A}^\dagger(t)$,

$$\dot{\hat{A}}^\dagger(t) = -\gamma_{n+1}(\hat{A}^\dagger)^{n+1}(t)\hat{A}^n(t) + \hat{F}_{n+1}^\dagger(t)\hat{A}^n(t), \tag{A2}$$

as

$$\langle\dot{\hat{A}}^\dagger(t)\rangle = -\gamma_{n+1}\langle(\hat{A}^\dagger)^{n+1}(t)\hat{A}^n(t)\rangle + \langle\hat{F}_{n+1}^\dagger(t)\rangle\langle\hat{A}^n(t)\rangle. \tag{A3}$$

Following the same steps as in Eq. (A1), we obtain

$$\dot{\alpha}^*(t) = -\gamma_{n+1}(\alpha^*)^{n+1}\alpha^n + f_{n+1}^*(t)\alpha^n$$
$$= -\gamma_{n+1}|\alpha|^{2n}\alpha^* + f_{n+1}^*(t)\alpha^n.$$

2. Derivation of Eq. (37)

Consider the operator Langevin equation (29) which is also the special case of generalized Langevin Eq. (9) for $n = 1$. Taking the average on both sides of the equation with the initial product separable coherent states of the system and bath oscillators $|\alpha\rangle|\mu_1\rangle|\mu_2\rangle\cdots|\mu_N\rangle$ one finds

$$\dot{\alpha} = -\frac{\gamma_2}{m_1}(m_1+m_0)\langle\alpha|\hat{A}^\dagger\hat{A}^2|\alpha\rangle + \frac{\gamma_2}{m_1}m_0\langle\alpha|\hat{A}^\dagger\hat{A}^2|\alpha\rangle + f_2\alpha^*, \tag{A4}$$

where we have split $-\gamma_2\langle\alpha|\hat{A}^\dagger\hat{A}^2|\alpha\rangle$ into two terms by introducing two dimensionless numbers m_0 and m_1. m_0 and m_1 can assume positive or negative values depending on the specificity of the case and are independent of each other. This choice is determined by the characteristic polynomial of classical nonlinear damping. Rewriting the above equation (A4) we have

$$\dot{\alpha} = -\frac{\gamma_2}{m_1}(m_1 + m_0)|\alpha|^2\alpha + \frac{\gamma_2}{m_1}m_0\langle\alpha|\hat{A}(\hat{A}^\dagger\hat{A} - 1)|\alpha\rangle + f_2\alpha^*$$

$$\approx -\frac{\gamma_2}{m_1}(m_1 + m_0)|\alpha|^2\alpha + \frac{\gamma_2}{m_1}m_0\langle\alpha|\hat{A}|\alpha\rangle\langle\alpha|(\hat{A}^\dagger\hat{A} - 1)|\alpha\rangle$$
$$+ f_2\alpha^*$$

$$= -\frac{\gamma_2}{m_1}(m_1 + m_0)|\alpha|^2\alpha + \frac{\gamma_2}{m_1}m_0(|\alpha|^2 - 1)\alpha + f_2\alpha^*$$

$$= -\frac{\gamma_2}{m_1}(m_1|\alpha|^2 + m_0)\alpha + f_2\alpha^*. \tag{A5}$$

In deriving Eq. (A5) it has been assumed $\langle\hat{A}(\hat{A}^\dagger\hat{A} - 1)\rangle \approx \langle\hat{A}\rangle\langle(\hat{A}^\dagger\hat{A} - 1)\rangle$, which is a valid approximation within the mean-field theory. This corresponds to neglect the higher order quantum correlation in the dynamics of nonlinear dissipation; quantum noise due to the heat bath, however, completely unaffected.

3. Derivation of Eq. (45)

Setting $n = 2$ in Eq. (9) and averaging with $|\alpha\rangle|\mu_1\rangle|\mu_2\rangle\cdots|\mu_k\rangle\cdots|\mu_N\rangle$, we write down the c-number Langevin equation within the mean-field approximation as

$$\dot{\alpha} = -\frac{\gamma_3}{m_2}(m_2 + m_1)\langle\alpha|(\hat{A}^\dagger)^2\hat{A}^3|\alpha\rangle + \frac{\gamma_3}{m_2}m_1\langle\alpha|(\hat{A}^\dagger)^2\hat{A}^3|\alpha\rangle$$
$$+ f_3(\alpha^*)^2$$

$$= -\frac{\gamma_3}{m_2}(m_2 + m_1)|\alpha|^4\alpha + \frac{\gamma_3}{m_2}m_1\langle\alpha|\hat{A}^\dagger(\hat{A}\hat{A}^\dagger - 1)\hat{A}^2|\alpha\rangle$$
$$+ f_3(\alpha^*)^2$$

$$\approx -\frac{\gamma_3}{m_2}(m_2 + m_1)|\alpha|^4\alpha + \frac{\gamma_3}{m_2}m_1\langle\alpha|\hat{A}^\dagger\hat{A}|\alpha\rangle\langle\alpha|\hat{A}^\dagger\hat{A}^2|\alpha\rangle$$
$$- \frac{\gamma_3}{m_2}m_1\langle\alpha|\hat{A}^\dagger\hat{A}^2|\alpha\rangle + f_3(\alpha^*)^2$$

$$= -\frac{\gamma_3}{m_2}(m_2 + m_1)|\alpha|^4\alpha + \frac{\gamma_3}{m_2}m_1|\alpha|^4\alpha$$
$$- \frac{\gamma_3}{m_2}(m_1 + m_0)\langle\alpha|\hat{A}^\dagger\hat{A}^2|\alpha\rangle$$
$$+ \frac{\gamma_3}{m_2}m_0\langle\alpha|(\hat{A}\hat{A}^\dagger\hat{A} - \hat{A}) + f_3(\alpha^*)^2$$

$$\approx -\frac{\gamma_3}{m_2} m_2 |\alpha|^4 \alpha - \frac{\gamma_3}{m_2} (m_1 + m_0) |\alpha|^2 \alpha$$

$$+ \frac{\gamma_3}{m_2} m_0 \langle \alpha | \hat{A} | \alpha \rangle \langle \alpha | \hat{A}^\dagger \hat{A} - 1 | \alpha \rangle + f_3 (\alpha^*)^2$$

$$= -\frac{\gamma_3}{m_2} m_2 |\alpha|^4 \alpha - \frac{\gamma_3}{m_2} (m_1 + m_0) |\alpha|^2 \alpha$$

$$+ \frac{\gamma_3}{m_2} m_0 (|\alpha|^2 - 1) \alpha + f_3 (\alpha^*)^2$$

$$= -\frac{\gamma_3}{m_2} (m_2 |\alpha|^4 + m_1 |\alpha|^2 + m_0) \alpha + f_3 (\alpha^*)^2. \tag{A6}$$

Here the mean-field approximation has been applied twice to obtain Eqs. (A6) or (45). m_2, m_1, and m_0 are the integers to be chosen as per requirement of the form of the polynomial describing nonlinear dissipation.

4. Derivation of Eq. (47)

Following the same procedure as above, it may be anticipated that we can put $n = k$ in Eq. (9) and generate terms, such as $|\alpha|^{2k}\alpha$, $|\alpha|^{2k-2}\alpha \cdots$ up to $|\alpha|^2\alpha$ using c-number formalism. The basis of this assumption lies in the previous two cases. Therefore, we can presume the structure for $n = k$ within the purview of the mean-field approximation as follows:

$$\dot{\alpha} = -\gamma_{k+1} \langle \alpha | (\hat{A}^\dagger)^k \hat{A}^{k+1} | \alpha \rangle + f_{k+1} (\alpha^*)^k$$

$$= -\frac{\gamma_{k+1}}{l_k} (l_k ||\alpha|^{2k} + l_{k-1} |\alpha|^{2k-2} \cdots + l_1 |\alpha|^2 + l_0) \alpha$$

$$+ f_{k+1} (\alpha^*)^k, \tag{A7}$$

where the coefficients l_k, $l_{k-1} \ldots, l_1$ and l_0 may or may not be interrelated.

Now having obtained the above equation for $n = k$, if we show that the same structure holds for $n = k + 1$, we may claim that we prove our desired result Eq. (47). For that we go back to Eq. (9), take $n = k + 1$, and carry out the averaging

with respect to coherent states as before

$$\dot{\alpha} = -\gamma_{k+2} \langle \alpha | (\hat{A}^\dagger)^{k+1} \hat{A}^{k+2} | \alpha \rangle + f_{k+2} (\alpha^*)^{k+1}$$

$$= -\frac{\gamma_{k+2}}{m_{k+1}} (m_{k+1} - m) \langle \alpha | (\hat{A}^\dagger)^{k+1} \hat{A}^{k+2} | \alpha \rangle$$

$$- \frac{\gamma_{k+2}}{m_{k+1}} m \langle \alpha | (\hat{A}^\dagger)^k (\hat{A}\hat{A}^\dagger - 1) \hat{A}^{k+1} | \alpha \rangle + f_{k+2} (\alpha^*)^{k+1}$$

$$\approx -\frac{\gamma_{k+2}}{m_{k+1}} (m_{k+1} - m) |\alpha|^{2k+2} \alpha$$

$$- \frac{\gamma_{k+2}}{m_{k+1}} m \langle \alpha | (\hat{A}^\dagger)^k \hat{A} | \alpha \rangle \langle \alpha | \hat{A}^\dagger \hat{A}^{k+1} | \alpha \rangle$$

$$+ \frac{\gamma_{k+2}}{m_{k+1}} m \langle \alpha | (\hat{A}^\dagger)^k \hat{A}^{k+1} | \alpha \rangle + f_{k+2} (\alpha^*)^{k+1}$$

$$= -\frac{\gamma_{k+2}}{m_{k+1}} (m_{k+1} - m) |\alpha|^{2k+2} \alpha - \frac{\gamma_{k+2}}{m_{k+1}} m |\alpha|^{2k+2} \alpha$$

$$+ \frac{\gamma_{k+2}}{m_{k+1}} \frac{m}{l_k} \{l_k |\alpha|^{2k} + l_{k-1} |\alpha|^{2k-2} \cdots + l_1 |\alpha|^2 + l_0\} \alpha$$

$$+ f_{k+2} (\alpha^*)^{k+1} \tag{A8}$$

$$= -\frac{\gamma_{k+2}}{m_{k+1}} (m_{k+1} - m) |\alpha|^{2k+2} \alpha - \frac{\gamma_{k+2}}{m_{k+1}} m |\alpha|^{2k+2} \alpha$$

$$- \frac{\gamma_{k+2}}{m_{k+1}} (m_k |\alpha|^{2k} + m_{k-1} |\alpha|^{2k-2} \cdots + m_1 |\alpha|^2 + m_0) \alpha$$

$$+ f_{k+2} (\alpha^*)^{k+1} \tag{A9}$$

$$= -\frac{\gamma_{k+2}}{m_{k+1}} (m_{k+1} |\alpha|^{2k+2} + m_k |\alpha|^{2k} \cdots + m_1 |\alpha| + m_0) \alpha$$

$$+ f_{k+2} (\alpha^*)^{k+1}. \tag{A10}$$

Here we use Eq. (A7) in Eq. (A8). Further defining

$$m_j = -\frac{l_j}{l_k} m, \quad \forall \ j = 0, 1, 2 \cdots k. \tag{A11}$$

In Eq. (A9) we obtain Eq. (A10). Thus, we find the same structure for $n = k + 1$ as for $n = k$. Therefore, by the method of mathematical induction we have proved that the structure
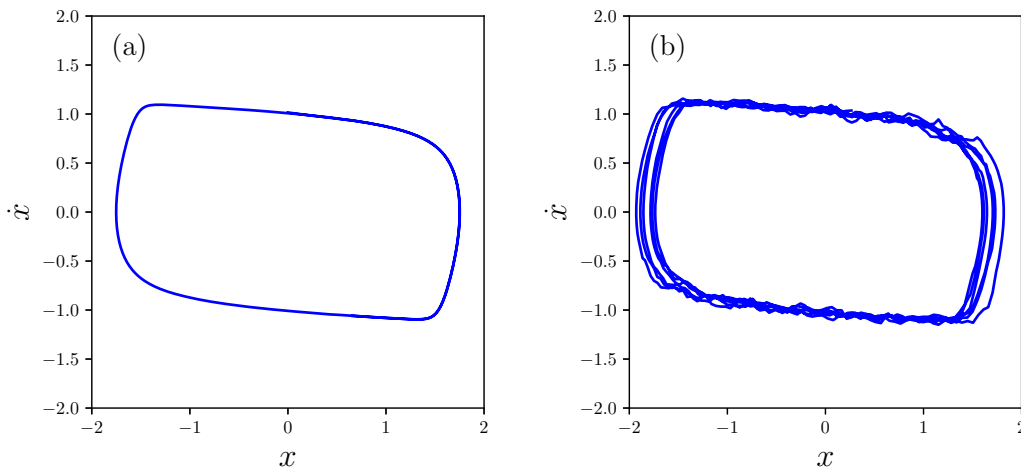

FIG. 5. The phase space trajectories of the (a) classical and the (b) quantum versions of a generalized Rayleigh oscillator for polynomial $\dot{x}^6 + \dot{x}^4 - \dot{x}^2 - 1$ are plotted for $\epsilon = 1.5$. We have put $KT = 20$ for plotting the quantum case (arbitrary units).

assumed in Eq. (47) is true for all $n \in \mathbb{Z}^+$. A similar technique can be followed to derive its complex conjugate equation.

5. Rayleigh family of quantum oscillators

From c-number Langevin Eqs. (47) and (48) we construct the following form of the quantum equation with the help of substitution (42):

$$\dddot{x} + \frac{\gamma_{n+1}}{m_n} \{\beta_n \dot{x}^{2n} + \beta_{n-1} \dot{x}^{2n-2} \cdots + \beta_1 \dot{x}^2 + \beta_0\} \dot{x} + \omega_0^2 x$$
$$= \omega_0^{n-1} \eta_{n+1}. \tag{A12}$$

Equation (A12) can be considered as a quantum analog of the generalized Rayleigh family of oscillators [13,47], where

$$\beta_j = \frac{m_j j}{2^j C_{j+1}(2j+1)}, \quad \forall \, j = 1, 2, \ldots n \quad \text{and} \quad \beta_0 = m_0. \tag{A13}$$

The family of oscillators could also support at most n numbers of limit cycles, where $2n$ is the highest degree of the polynomial [48]. The phase space trajectories of a representative example of such a generalized Rayleigh oscillator is shown in both the (a) classical and the (b) quantum pictures in Fig. 5.

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