First-passage probabilities and mean number of sites visited by a persistent random walker in one- and two-dimensional lattices

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We study the first passage probability and mean number of sites visited by a discrete persistent random walker on a lattice in one and two dimensions. This is performed by using the multistate formulation of the process. We obtain explicit expressions for the generating functions of these quantities. To evaluate these expressions, we study the system in the strongly persistent limit. In the one-dimensional case, we recover the behavior of the continuous one-dimensional persistent random walk (telegrapher process). In two dimensions we obtain an explicit expression for the probability distribution in the strongly persistent limit, however, the Laplace transforms required to evaluate the first-passage processes could only be evaluated in the asymptotic limit corresponding to long times in which regime we recover the behavior of normal two-dimensional diffusion.

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I. INTRODUCTION

Persistent random walkers—frequently, also referred to as correlated random walks—are, perhaps, the simplest models of stochastic transport with memory [1–5]. These processes, that mimic the effects of inertia for Brownian particles, are extensively used in biology [6,7], where the tail to head axis induces a preferred direction of motion in many organisms; they can also be useful as models for self propelled particles [8,9] and as models for the propagation of light [10], among other things.

In this paper we consider a discrete time persistent random walk on a lattice. We calculate the first passage probability to a site [1,11], the probability of first return to the origin, as well as the mean number of distinct sites visited as a function of time [1]. First passage probabilities for persistent random walkers in one dimension have been studied previously in the continuum limit, mostly by implementing absorbing boundary conditions for the telegrapher's equation—that is, treating a site as a trap, so the probability that the walker is trapped at that site at a given time is the probability of first passage [12–15]—or by using variations of Siegert's formula [1,4,16–18]. In this paper we apply arguments similar to those presented in Ref. [1] (which are akin to Siegert's formula) to persistent random walks in one- and two-dimensional square lattices. The generating functions of the first passage probabilities, the first return probabilities, and the mean number of distinct sites visited are obtained explicitly. To evaluate these expressions, we consider the strongly persistent limit. In this limit, where the probability that the walker changes direction is very low and the process performs long sojourns between turns, we recover the behavior of the continuous persistent random walk in one dimension [18]. In two dimensions we consider a simplified persistent random walker that cannot reverse direction. We obtain the explicit asymptotic probability distribution function describing the process in the strongly persistent limit. Unfortunately, the expressions required to evaluate the first passage probabilities and the number of distinct sites in the time domain could only be evaluated in the asymptotic limit corresponding to long times.

II. ONE DIMENSION

To illustrate the procedure, we begin with the simple one-dimensional case.

We denote by $+$ $(-)$ the state of the walker when it reaches a site having stepped to the right (left, respectively). A complete detailed description of the process requires the specification of the initial conditions, which consist of the initial position of the walker as well as its initial state. Thus, $P_+(x, n|y, +)$ represents the probability of finding the walker at site x at step n, the last step being to the right, given that the walker was initially at site y in the $+$ state. In what follows, as notation, I omit the initial position when the walker begins at the origin, i.e., $P_+(x, n|+) \equiv P_+(x, n|0, +)$. Similarly, $P_-(x, n|+)$ represents the probability of finding the walker at site x at step n, the last step being to the left, given that the walker was initially at the origin in the $+$ state. Corresponding definitions hold for $P_+(x, n|-)$ and $P_-(x, n|-)$.

If we let α be the probability of changing from one state to the other, then $P_-(x, n|+)$ and $P_+(x, n|+)$ satisfy

$$P_+(x, n+1|+) = (1-|, \alpha)P_+(x-1, n|+) + \alpha P_-(x-1, n|+),$$
$$P_-(x, n+1|+) = \alpha P_+(x+1, n|+) + (1-\alpha)P_-(x+1, n|+) \quad (1)$$

subject to the initial conditions $P_+(x, 0|+) = \delta_{x,0}$ and $P_-(x, 0|+) = 0$. Similar equations are satisfied by $P_-(x, n|-)$ and $P_+(x, n|-)$ with appropriate initial conditions.

On the other hand, if we define $F_+(x, n|+)$ and $F_-(x, n|+)$ as the probability that the walker reaches site x in state $+$ (respectively, $-$) for the first time at step n, given that it started

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at the origin in state +, then we can write [1]

$$P_+(x,n|+) = \delta_{x,0}\delta_{n,0} + \sum_{m=0}^{n}[F_+(x,m|+)P_+(x,n-m|x,+) + F_-(x,m|+)P_+(x,n-m|x,-)],$$
$$P_-(x,n|+) = \sum_{m=0}^{n}[F_+(x,m|+)P_-(x,n-m|x,+) + F_-(x,m|+)P_-(x,n-m|x,-)]. \quad (2)$$

These equations state that the probability for the walker to be at site x in state +, say, starting at the origin in state + is $\delta_{x,0}$ at step zero, or it is given by a first passage from the origin in state + to site x in state + (respectively, −) in m steps, and then a return from site x to site x again in state +, in the remaining $n-m$ steps, given that the initial state at x was + (respectively, −), added over all possible values of m. These equations simplify slightly by noting that since the system is isotropic, $P_\pm(x,n|x,+) = P_\pm(0,n|0,+) \equiv P_\pm(0,n|+)$ and $P_\pm(x,n|x,-) = P_\pm(0,n|0,-) \equiv P_\pm(0,n|-)$.

Thus, the idea is to solve the recurrence relations of the type given in Eq. (1) and use the solutions in Eq. (2) to find the first passage probabilities. Although the one-dimensional case is simple enough, the following observation further simplifies the calculations. We define $P(x,n|+) = P_+(x,n|+) + P_-(x,n|+)$ the probability to find the walker at site x at step n irrespective of the state, given that it was initially at the origin in state +; and $F(x,m|+) = F_+(x,m|+) + F_-(x,m|+)$ the first passage probability to site x irrespective of the state, again given that it was initially at the origin in state +. Then, adding both equations in (2), we have

$$P(x,n|+) = \delta_{x,0}\delta_{n,0} + \sum_{m=0}^{n}[F_+(x,m|+)P(0,n-m|+) + F_-(x,m|+)P(0,n-m|-)].$$

Now, since the system is symmetric, we have $P(0,n-m|+) = P(0,n-m|-)$, so

$$P(x,n|+) = \delta_{x,0}\delta_{n,0} + \sum_{m=0}^{n} F(x,m|+)P(0,n-m|+). \quad (3)$$

In what follows, as notation, we will denote in lowercase the generating function $g(z)$ of a given function $G(n)$,

$$g(z) \equiv \sum_{n=0}^{\infty} z^n G(n), \quad (4)$$

and by a tilded symbol, say $\tilde{H}(\theta)$, the Fourier transform of a function $H(x)$,

$$\tilde{H}(\theta) \equiv \sum_{n=-\infty}^{\infty} e^{ix\theta} H(x). \quad (5)$$

Thus, calculating the generating function of the Fourier transform of equations in (1) we obtain

$$\tilde{p}_+(\theta,z|+) = 1 + ze^{i\theta}[(1-\alpha)\tilde{p}_+(\theta,z|+) + \alpha\tilde{p}_-(\theta,z|+)],$$
$$\tilde{p}_-(\theta,z|+) = ze^{-i\theta}[(1-\alpha)\tilde{p}_-(\theta,z|+) + \alpha\tilde{p}_+(\theta,z||+)] \quad (6)$$

from which we find

$$\tilde{p}(\theta,z|+) = \tilde{p}_+(\theta,z|+) + \tilde{p}_-(\theta,z|+) = \frac{1 - ze^{-i\theta}(1-2\alpha)}{1 - 2z(1-\alpha)\cos\theta + z^2(1-2\alpha)}. \quad (7)$$

The inverse Fourier transform can be easily calculated [19]

$$p(x,z|+) = \frac{1}{(1-A^2)^{1/2}[1+z^2(1-2\alpha)]}\left[\left(\frac{A}{1+(1-A^2)^{1/2}}\right)^{|x|} - z(1-2\alpha)\left(\frac{A}{1+(1-A^2)^{1/2}}\right)^{|x+1|}\right], \quad (8)$$

where

$$A = \frac{2z(1-\alpha)}{1+z^2(1-2\alpha)}. \quad (9)$$

With this expression we can now calculate the generating function for the first passage probability. Multiplying Eq. (3) by z^n and summing over n, we can write

$$f(x,z|+) = \frac{p(x,z|+)}{p(0,z|+)} = \frac{1+(1-A^2)^{1/2}}{1+(1-A^2)^{1/2} - z(1-2\alpha)A}\left[\left(\frac{A}{1+(1-A^2)^{1/2}}\right)^{|x|} - z(1-2\alpha)\left(\frac{A}{1+(1-A^2)^{1/2}}\right)^{|x+1|}\right] \quad (10)$$

for $x \neq 0$, and the generating function of the probability of first return to the origin is

$$f(0,z|+) = 1 - \frac{1}{p(0,z|+)} = 1 - \frac{(1-A^2)^{1/2}[1+z^2(1-2\alpha)][1+(1-A^2)]}{1+(1-A^2)^{1/2} - z(1-2\alpha)A}. \quad (11)$$

First, we note that since $A \to 1$ when $z \to 1$, then, for $\alpha > 0$ we have

$$f(x,1|+) = 1 \quad \forall\, x, \quad (12)$$

which implies that the walker will visit every site as well as return to the origin with probability 1. This is not unexpected since in the long time regime, the behavior of the persistent random walker is similar to normal diffusion, which is known to be recurrent in one and two dimensions [1]. Note, however, that if $\alpha = 0$, then $A = 2z/(1+z^2)$ and $f(x,z|+) = 0\ \forall\, x < 0$ as should be expected since in this case, the walker never changes state, and having started in the + state, will only move in the positive x direction.

Furthermore, we can calculate $\gamma(z)$ the generating function of the mean number of new sites visited at step n, $\Gamma(n)$ as [1]

$$\gamma(z) \equiv \sum_{n=0}^{\infty}\Gamma(n)z^n = \sum_{x=-\infty}^{\infty} f(x,z|+) = \frac{z}{(1-z)p(0,z|+)}. \quad (13)$$

The average number of distinct sites visited by the random walker at step n will be given by $\langle \mathbf{S}(n)\rangle = \sum_{m=0}^{n}\Gamma(m)$. Thus, the generating function of this quantity is given by

$$\langle \mathbf{s}(z)\rangle = \frac{z}{(1-z)^2 p(0,z|+)} = \frac{z(1-A^2)^{1/2}[1+z^2(1-2\alpha)][1+(1-A^2)^{1/2}]}{(1-z)^2[1+(1-A^2)^{1/2}-z(1-2\alpha)A]}. \tag{14}$$

Inverting the generating function to obtain the explicit time behavior of the results obtained thus far is not feasible. What we can do, however, is evaluate the process' behavior in the strongly persistent limit $\alpha \ll 1$. In this approximation, times and distances are much longer than the single steps so that we can consider the position and the number of steps as continuous variables, and we can interpret the generating function as a usual Laplace transform. To this end, we write $z = 1 - s$, anticipating that $s \ll 1$. To appropriately describe this regime, we introduce the rescaled variables $\sigma = s/\alpha$, $\vartheta = \theta/\alpha$, and $\xi = \alpha x$, which allow us to obtain explicit expressions for the various quantities discussed, thus far, in the limit $\alpha \to 0$. First of all, in this limit, the transform of the probability density in Eq. (7), expressed in the rescaled variables, correct to leading order in α, becomes

$$\alpha\tilde{p}(\theta,z|+) \to \tilde{p}(\vartheta,\sigma|+) \approx \frac{\sigma+i\vartheta+2}{\sigma^2+2\sigma+\vartheta^2}. \tag{15}$$

which is the Laplace-Fourier transform of the solution to the corresponding telegrapher's equation with the appropriate initial condition [3].

Next, to order α^2 we have $A \approx 1-\alpha^2(\sigma+\sigma^2/2)$, and from Eq. (10), the Laplace transform of the asymptotic distribution for the first passage process in terms of the rescaled variables, $f(\xi,\sigma|+)$, becomes

$$f(\xi,\sigma|+) \sim \begin{cases} e^{-\xi(\sigma^2+2\sigma)^{1/2}}, & \xi > 0, \\ \frac{(\sigma+2)^{1/2}-\sigma^{1/2}}{(\sigma+2)^{1/2}+\sigma^{1/2}} e^{\xi(\sigma^2+2\sigma)^{1/2}}, & \xi < 0, \end{cases} \tag{16}$$

where we see the effect of the asymmetry induced by the initial state of the walker. This asymmetry is eventually lost as $\sigma \to 0$. Inverse Laplace transforming this expression yields [20], in the original variables,

$$F(x,n|+) \sim \begin{cases} -\frac{d}{dx}\{e^{-\alpha n} I_0(\alpha\sqrt{n^2-x^2})u(n-x)\}, & x > 0, \\ \frac{d}{dx}\left\{e^{-\alpha n}\left(\frac{n+x}{n-x}\right)^{1/2} I_1(\alpha\sqrt{n^2-x^2})u(n+x)\right\}, & x < 0, \end{cases} \tag{17}$$

where $I_\nu(y)$ is the modified Bessel function of order ν and $u(y)$ is the step function. These results agree with those reported in Refs. [15,18] for continuous time versions of the process.

From Eq. (11), the Laplace transform of the asymptotic probability distribution function for first return to the origin, expressed in the rescaled variable σ will be given by

$$f(0,\sigma|+) \approx \frac{(\sigma+2)^{1/2}-\sigma^{1/2}}{(\sigma+2)^{1/2}+\sigma^{1/2}}, \tag{18}$$

as $\alpha \to 0$. This expression corresponds to the limit $\xi \to 0^-$ of $f(\xi,\sigma|+)$. The inverse Laplace transform of this quantity leads to [20]

$$F(0,n|+) \approx \frac{1}{n} e^{-\alpha n} I_1(\alpha n) \sim \frac{1}{\sqrt{2\pi\alpha} n^{3/2}}, \quad \alpha n \gg 1. \tag{19}$$

In this approximation, to leading order, the generating function of $\langle \mathbf{s}(z)\rangle$, the mean number of distinct sites visited by the random walker becomes

$$\alpha\langle \mathbf{s}(s)\rangle \to \langle \mathbf{s}(\sigma)\rangle \approx \frac{(\sigma+2)^{1/2}-\sigma^{1/2}}{\sigma^{3/2}}, \tag{20}$$

which after inverse Laplace transforming [20], can be written in the original variables as

$$\langle \mathbf{S}(n)\rangle \sim \frac{1}{\alpha}\{I_0(\alpha n)e^{-\alpha n} - 1 + 2\alpha n e^{-\alpha n}[I_0(\alpha n)+I_1(\alpha n)]\} \tag{21}$$

for $n \gg 1$ from which we can infer the limiting behaviors [20],

$$\langle \mathbf{S}(n)\rangle \sim \begin{cases} n, & \alpha n \ll 1, \\ 2\left(\frac{2n}{\alpha\pi}\right)^{1/2}, & \alpha n \gg 1. \end{cases} \tag{22}$$

This result reflects the expected diffusive behavior of the process at long times and the ballistic behavior at times shorter than the persistence time $1/\alpha$.

III. TWO DIMENSIONS

We now turn to the two-dimensional persistent random walker on a square lattice. This process can be described as a four-state random walker. Specifically, a walker found at site (x, y) will be in state N—for **N**orthbound—, if it reached that site from site $(x, y-1)$, similarly, it will be in state S—for **S**outhbound—, if it arrived from site $(x, y+1)$, and so on for **E**astbound and **W**estbound.

To avoid unnecessary lengthy and messy algebra, in this paper we consider a particularly simple symmetric persistent random walker in which the probability of reversing direction is zero (the more general case will be treated in Ref. [21]). Thus, in this paper, a walker can remain in the same state with probability $(1-\beta)$ or change into a transverse direction with probability $\beta/2$. We denote $P_K(x, y, n|E)$, the probability to be at site (x, y) at step n in state $K = N, E, W, S$, given that the walker started at the origin in state E. These probabilities satisfy

$$P_E(x,y,n+1|E) = (1-\beta)P_E(x-1,y,n|E) + \frac{\beta}{2}[P_S(x-1,y,n|E)+P_N(x-1,y,n|E)],$$

$$P_W(x,y,n+1|E) = (1-\beta)P_W(x+1,y,n|E) + \frac{\beta}{2}[P_S(x+1,y,n|E)+P_N(x+1,y,n|E)],$$

$$P_N(x,y,n+1|E) = (1-\beta)P_N(x,y-1,n||E) + \frac{\beta}{2}[P_E(x,y-1,n|E)+P_W(x,y-1,n|E)],$$

$$P_S(x,y,n+1|E) = (1-\beta)P_S(x,y+1,n|E) + \frac{\beta}{2}[P_E(x,y+1,n|E)+P_W(x,y+1,n|E)],$$

with initial conditions given by

$$P_E(x,y,0|E) = \delta_{x,0}\delta_{y,0},$$
$$P_N(x,y,0|E) = P_W(x,y,0|E)$$
$$= P_S(x,y,0|E) = 0.$$

By considering the generating functions of the Fourier transform of the probabilities, with a little patience, the four coupled equations can be solved. If we denote $P(x,y,n|E) = P_E(x,y,n|E) + P_N(x,y,n|E) + P_W(x,y,n|E) + P_S(x,y,n|E)$, the probability of finding the walker at site (x,y) irrespective of what state it is in, then the generating function of the Fourier transform of this quantity can be written as

$$\tilde{p}(\theta_x,\theta_y,z|E) = \frac{e^{-i\theta_x}\{|B(\theta_y)|^2 + z\beta\ \mathrm{Re}[B(\theta_y)]\}B(\theta_x)}{|B(\theta_x)|^2|B(\theta_y)|^2 - (\beta z)^2\mathrm{Re}[B(\theta_x)]\mathrm{Re}[B(\theta_y)]}, \quad (23)$$

where $B(\theta) \equiv e^{i\theta} - (1-\beta)z$.

Now, although the equations for the first passage probabilities, the two-dimensional equivalent to Eqs. (2) and (3), and the mean number of distinct sites visited, Eq. (14), still hold, the explicit inverse Fourier transform can only be evaluated approximately. To advance, we proceed as we did in the one-dimensional case: we write $z = 1 - s$, we consider the rescaled variables $\vartheta_{x,y} = \theta_{x,y}/\beta$, $\xi = \beta x$, $\eta = \beta y$, $\tau = \beta n$, $\sigma = s/\beta$, and we take the strongly persistent limit $\beta \to 0$. In terms of these variables, to leading order in β we can write $|B(\vartheta)|^2 \approx \beta^2[(\sigma+1)^2 + \vartheta^2]$, and $\mathrm{Re}\,B(\theta) \approx \beta(\sigma+1)$. Then, to leading order in β, the transform of the probability $\tilde{p}(\theta_x,\theta_y,z|E)$ in Eq. (23) becomes

$$\beta\tilde{p}(\theta_x,\theta_y,z|E) \to \tilde{p}(\vartheta_x,\vartheta_y,\sigma|E) \approx \frac{[(\sigma+1)^2 + \vartheta_y^2 + (\sigma+1)][(\sigma+1) + i\vartheta_x]}{(\sigma+1)^4 + (\sigma+1)^2(\vartheta_x^2+\vartheta_y^2) + \vartheta_x^2\vartheta_y^2 - (\sigma+1)^2}. \quad (24)$$

We focus first on the inverse transform of

$$\tilde{m}(\vartheta_x,\vartheta_y;\sigma) \equiv \frac{\sigma}{\sigma^4 + \sigma^2(\vartheta_x^2+\vartheta_y^2) + \vartheta_x^2\vartheta_y^2 - \sigma^2}. \quad (25)$$

First, we note that $\tilde{m}(\vartheta_x,\vartheta_y;\sigma)$ can be rewritten as

$$\tilde{m}(\vartheta_x,\vartheta_y;\sigma) = \frac{1}{\Delta_+\Delta_-}\left(\frac{\sigma}{\sigma^2 - \frac{1}{4}[\Delta_+ + \Delta_-]^2} - \frac{\sigma}{\sigma^2 - \frac{1}{4}[\Delta_+ - \Delta_-]^2}\right), \quad (26)$$

where $\Delta_\pm = \sqrt{1 - [\vartheta_x \pm \vartheta_y]^2}$. Inverse Laplace transforming yields

$$\tilde{M}(\vartheta_x,\vartheta_y;\tau) = \frac{1}{\Delta_+\Delta_-}\left(\cosh\frac{[\Delta_+ + \Delta_-]\tau}{2} - \cosh\frac{[\Delta_+ - \Delta_-]\tau}{2}\right) = \frac{2}{\Delta_+\Delta_-}\left(\sinh\frac{\Delta_+\tau}{2}\sinh\frac{\Delta_-\tau}{2}\right). \quad (27)$$

The inverse Fourier transform of this expression is

$$M(\xi,\eta,\tau) = \frac{2}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{e^{-i\vartheta_x\xi - i\vartheta_y\eta}}{\Delta_+\Delta_-} \times\left(\sinh\frac{\Delta_+\tau}{2}\sinh\frac{\Delta_-\tau}{2}\right)d\vartheta_x d\vartheta_y.$$

Making the change in variables $Q_+ = (\vartheta_x + \vartheta_y)/2$ and $Q_- = (\vartheta_x - \vartheta_y)/2$, the above expression separates

$$M(\xi,\eta,\tau) = H(\xi+\eta,\tau)H(\xi-\eta,\tau), \quad (28)$$

where the function $H(X,\tau)$ is defined by

$$H(X,\tau) \equiv \frac{1}{\pi}\int_{\infty}^{\infty}\frac{e^{-iQX}}{\sqrt{1-4Q^2}}\sinh\frac{\tau\sqrt{1-4Q^2}}{2}dQ. \quad (29)$$

The Fourier-Laplace transform of this expression is

$$\tilde{h}(Q,\sigma) = \frac{1}{\sigma^2 - \frac{1}{4} + Q^2}, \quad (30)$$

which is closely related with the propagator of a telegrapher's equation. Inverse Fourier transforming yields

$$h(X,\sigma) = \frac{e^{-|X|\sqrt{\sigma^2 - 1/4}}}{2\sqrt{\sigma^2 - 1/4}}, \quad (31)$$

and finally, inverse Laplace transforming we find

$$H(X,\tau) = \frac{1}{2}I_0\left(\frac{1}{2}\sqrt{\tau^2 - X^2}\right)u(\tau - |X|). \quad (32)$$

A very similar calculation can be carried out for

$$\tilde{\mu}(\vartheta_x,\vartheta_y;\sigma) \equiv \frac{\vartheta_x\vartheta_y}{\sigma^4 + \sigma^2(\vartheta_x^2+\vartheta_y^2) + \vartheta_x^2\vartheta_y^2 - \sigma^2}, \quad (33)$$

the inverse Laplace-Fourier transform of which is

$$\mathcal{M}(\xi,\eta,\tau) = H(\xi+\eta,\tau)\frac{\partial H(\xi-\eta,\tau)}{\partial\tau} - H(\xi-\eta,\tau)\frac{\partial H(\xi+\eta,\tau)}{\partial\tau}. \quad (34)$$

In terms of these functions, using the fact that $\tilde{M}(\vartheta_x,\vartheta_y,\tau=0) = 0$ and $\frac{\partial}{\partial\tau}\tilde{M}(\vartheta_x,\vartheta_y,\tau=0) = 0$ [see Eq. (27)], we can finally write

$$P(\xi,\eta,\tau|E) \sim e^{-\tau}\left\{\left[\frac{\partial^2}{\partial\tau^2} + \frac{\partial}{\partial\tau} - \frac{\partial}{\partial\xi}\left(1 + \frac{\partial}{\partial\tau}\right) - \frac{\partial^2}{\partial\eta^2}\right] \times M(\xi,\eta,\tau) - \frac{\partial}{\partial\eta}\mathcal{M}(\xi,\eta,\tau)\right\}. \quad (35)$$

This is the *exact* analytic expression for the asymptotic probability distribution function for the two-dimensional persistent random walk in the strongly persistent limit $\beta \to 0$, expressed in the rescaled variables. It is worth stressing that this function has support in the tilted square contained within the diagonals $\tau = |\xi + \eta|$ and $\tau = |\xi - \eta|$. Unfortunately, to evaluate the first passage probability [cf. first line of Eq. (10)], we require the Laplace transform $p(\xi,\eta,\sigma|E)$ of $P(\xi,\eta,\tau|E)$, and this can only be evaluated in the limit of large τ, ξ, and η. The easiest way to perform this is taking the limits $\vartheta_x \to 0$, $\vartheta_y \to 0$, $\sigma \to 0$ directly in Eq. (24). Then, to leading order we have

$$\tilde{p}(\vartheta_x, \vartheta_y, \sigma|E) \approx \frac{2}{2\sigma + \vartheta_x^2 + \vartheta_y^2}, \tag{36}$$

which can be inverse Fourier transformed to yield

$$p(\xi, \eta, \sigma|E) \sim \frac{1}{\pi} K_0[\sqrt{2\sigma(\xi^2 + \eta^2)}], \tag{37}$$

where $K_0(x)$ is the modified Bessel function of the second kind of order zero [20]. From this expression we conclude that in this regime, the process tends to a diffusive process described by an asymptotic probability distribution function given, in the original variables, by

$$P(x, y; n|E) \sim \frac{\beta}{2\pi n} \exp\left[-\frac{\beta(x^2 + y^2)}{2n}\right]. \tag{38}$$

It should be stressed that the expression in Eq. (37) is only valid in the long distance regime, the apparent divergence at $\xi^2 + \eta^2 = 0$ is spurious. It also comes as no surprise that in this limit, the asymmetry due to the initial condition is lost. Now, to evaluate the behavior of the first passage probability, the probability of first return to the origin and the mean number of distinct sites visited, we require $p(0, 0, \sigma|E)$, the Laplace transform of $P(0, 0, \tau|E)$. This quantity can be evaluated in the small σ limit, corresponding to the behavior at long times. This is achieved by, first, taking the Laplace transform as

$$p(0, 0, \sigma|E) \sim \int_{0^+}^{\infty} e^{-\sigma\tau} P(0, 0, \tau|E) d\tau. \tag{39}$$

By doing so we avoid dealing with the singularities that occur at $\tau = 0$, which contribute, at most, terms of order one as $\sigma \to 0$ [21]. Next, we note that both $\frac{\partial}{\partial\xi} M(\xi, \eta, \tau)$ and $\mathcal{M}(\xi, \eta, \tau)$ are odd functions of ξ, thus, they vanish at the origin.

Then $p(0, 0, \sigma|E)$ can be evaluated using the following identities:

$$\int_0^{\infty} e^{-az} I_0^2(bz) dz = \frac{2}{\pi a} \mathbf{K}(2b/a), \tag{40}$$

and

$$\int_0^{\infty} \frac{e^{-az}}{z} \frac{d}{dz} I_0^2(bz) dz = a\left[1 - \frac{2}{\pi}\mathbf{E}(2b/a)\right] \tag{41}$$

for $a > 2b$, where $\mathbf{K}(k)$ and $\mathbf{E}(k)$ are the complete elliptic integrals of the first and second kinds of modulus k, respectively [19,22]. The validity of these identities can be established by expanding $I_0^2(bz)$ in a Maclaurin series [20,23] and integrating term by term. Thus, we find

$$p(0, 0, \sigma|E) \sim \frac{(\sigma + 2)}{4}\left[\frac{2}{\pi}\mathbf{K}\left(\frac{1}{\sigma + 1}\right) - 1\right] + \frac{(\sigma + 1)}{4}\left[1 - \frac{2}{\pi}\mathbf{E}\left(\frac{1}{\sigma + 1}\right)\right]. \tag{42}$$

With this expression, we can obtain the asymptotic behavior for the Laplace transforms of the probability of first passage probability,

$$f(\xi, \eta, \sigma|E) = \frac{p(\xi, \eta, \sigma|E)}{p(0, 0, \sigma|E)},$$

and the mean number of distinct sites visited $\langle \mathbf{s}(s) \rangle$ [cf. the first line of Eqs. (10) and (14), respectively]. To proceed, we require the asymptotic behavior of the complete elliptic integrals as $k \to 1$ [22],

$$\mathbf{K}(k) \sim \left(\ln \frac{1}{k'} + \ln 4\right) + \frac{1}{4} k'^2 \left(\ln \frac{1}{k'} + \ln 4 - \frac{2}{3 \times 4}\right) + \cdots, \tag{43}$$

and

$$\mathbf{E}(k) \sim 1 + \frac{1}{2} k'^2 \left(\ln \frac{1}{k'} + \ln 4 - \frac{1}{1 \times 2}\right) + \cdots, \tag{44}$$

where $k' = \sqrt{1 - k^2}$ is the complementary modulus. Keeping the leading terms, we find

$$p(0, 0; \sigma|E) \sim \frac{1}{2\pi} \ln \frac{1}{\sigma} + O(1), \quad \text{as } \sigma \to 0. \tag{45}$$

The first thing we note, using Eq. (37) and the asymptotic behavior of $K_0(x)$ as $x \to 0$ [20], is that the first passage probability satisfies

$$\lim_{\sigma \to 0} f(\xi, \eta; \sigma|E) = 1 \quad \forall\, (\xi, \eta), \tag{46}$$

which implies that every site will be visited with probability 1 and that the process is recurrent as in the one-dimensional case.

The long time behavior of the mean number of distinct sites visited can be evaluated using a Tauberian theorem as discussed in Ref. [1]. Namely, if the $f(s)$ is the Laplace transform of a monotonous function $F(\tau)$ and $f(s) \sim \frac{1}{s^{\alpha}} L(1/s)$ as $s \to 0$, where $L(x)$ is a slowly varying function in the sense that $\lim_{x\to\infty} L(cx)/L(x) = 1$ for any $c > 0$, then

$$F(\tau) \sim \frac{\alpha\tau^{\alpha-1} L(\tau) + \tau^{\alpha} L'(\tau)}{\Gamma(\alpha + 1)} \quad \text{as } \tau \to \infty. \tag{47}$$

Thus, in the original variables, we obtain

$$\langle S(n) \rangle \sim \frac{2\pi n}{\ln(\beta n)} \quad \text{as } \beta n \to \infty, \tag{48}$$

which has the same functional form as the corresponding quantity for a normal random walker in two dimensions [1].

IV. CONCLUSIONS

By using the multistate formulation, we present a formalism for calculating the generating functions for the first-passage probabilities, first return probabilities, and mean number of distinct sites visited by a persistent random walks on a lattice. We evaluate these quantities, in the strongly persistent limit, i.e., when the probability that the walker changes direction is very small. In this limit, all the Laplace transforms needed to evaluate the first-passage probabilities, and the mean number of distinct sites visited can be calculated explicitly in the one-dimensional case, and the results reproduce those obtained for the continuous one-dimensional persistent random walk (or telegrapher process). In two dimensions, however, even though the explicit probability distribution function can actually be calculated explicitly in the strongly persistent limit, the required Laplace transforms can only be calculated asymptotically in the long distance-large number of steps limit. In this regime, the results reproduce the behavior of the normal two-dimensional diffusive process to which the persistent random walk tends asymptotically. Thus, for the two-dimensional case, more clever approximations will be needed to describe the behavior of the first-passage processes at shorter times than those corresponding to the effective long time diffusive limit. Nevertheless, the approach used in this paper may also be useful to study other interesting statistics for these processes, such as the occupancy of a set of sites, the number of visits to a given site [1], and the coverage of a finite or periodic lattice [24].

[1] G. H. Weiss, *Aspects and Applications of the Random Walk* (North-Holland, Amsterdam, 1994).

[2] G. H. Weiss, Some applications of persistent random walks and the telegrapher's equation, Physica A **311**, 381 (2002).

[3] J. Masoliver, K. Lindenberg, and G. H. Weiss, A continuous-time generalization of the persistent random walk, Physica A **157**, 891 (1989).

[4] V. Rossetto, The one-dimensional asymmetric persistent random walk, J. Stat. Mech. (2018) 043204.

[5] M. Boguna, J. M. Porra, J. Masoliver, Generalization of the persistent random walk to dimensions greater than 1, Phys. Rev. E **58**, 6992 (1998).

[6] E. A. Codling, M. J. Plank, and S. Benhamou, Random walk models in biology, J. R. Soc., Interface **5**, 813 (2008).

[7] H.-i. Wu, B.-L. Li, T. A. Springer, and W. H. Neill, Modelling animal movement as a persistent random walk in two dimensions: Expected magnitude of net displacement, Ecol. Modell. **132**, 115 (2000).

[8] F. J. Sevilla and L. A. G. Nava, Theory of diffusion of active particles that move at constant speed in two dimensions, Phys. Rev. E **90**, 022130 (2014).

[9] F. Peruani and L. G. Morelli, Self-Propelled Particles with Fluctuating Speed and Direction of Motion in Two Dimensions, Phys. Rev. Lett. **99**, 010602 (2007).

[10] M. F. Miri and H. Stark, The role of liquid films for light transport in dry foams, Europhys. Lett. **65**, 567 (2004).

[11] S. Redner, *A Guide to First-Passage Processes* (Cambridge University Press, Cambridge, UK, 2001).

[12] G. H. Weiss, First passage time problems for one-dimensional random walks, J. Stat. Phys. **24**, 587 (1981).

[13] J. Masoliver and G. H. Weiss, First passage times for a generalized telegrapher's equation, Physica A **183**, 537 (1992).

[14] V. Balakrishnan and S. Chaturvedi, Persistent diffusion on a line, Physica A **148**, 581 (1988).

[15] D. J. Bicout and A. Szabo, On the Wang-Uhlenbeck problem in discrete velocity space, J. Stat. Phys. **91**, 1047 (1998).

[16] A. J. Siegert, On the first passage time probability problem, Phys. Rev. **81**, 617 (1951).

[17] S. K. Foong, First-passage time, maximum displacement and Kac's solution of the telegrapher equation, Phys. Rev. A **46**, 707(R) (1992).

[18] S. K. Foong and S. Kanno, Properties of the telegrapher's random process with or without a trap, Stochastic Processes Appl. **53**, 147 (1994).

[19] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products* (Academic Press, Burlington, MA, 2014).

[20] M. Abramowitz and I. A. Stegun (eds.), *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, Vol. 55 (U.S. Government Printing Office, Washington, DC, 1948).

[21] H. Larralde and F. Leyvraz (unpublished).

[22] NIST Digital Library of Mathematical Functions, http://dlmf.nist.gov/, Release 1.0.28 of 2020-09-15, edited by F. W. J. Olver, A. B. Olde Daalhuis, D. W. Lozier, B. I. Schneider, R. F. Boisvert, C. W. Clark, B. R. Miller, B. V. Saunders, H. S. Cohl, and M. A. McClain.

[23] C. M. Bender, D. C. Brodya, and B. K. Meister, On powers of Bessel functions, J. Math. Phys. **44**, 309 (2003).

[24] M. Chupeau, O. Bénichou, and R. Voituriez, Cover times of random searches, Nat. Phys. **11**, 844 (2015).