Exact variance of von Neumann entanglement entropy over the Bures-Hall measure

Lu Wei*
Department of Electrical and Computer Engineering, University of Michigan, Dearborn, Michigan 48128, USA

(Received 25 June 2020; accepted 23 November 2020; published 14 December 2020)

The Bures-Hall distance metric between quantum states is a unique measure that satisfies various useful properties for quantum information processing. In this work, we study the statistical behavior of quantum entanglement over the Bures-Hall ensemble as measured by von Neumann entropy. The average von Neumann entropy over such an ensemble has been recently obtained, whereas the main result of this work is an explicit expression of the corresponding variance that specifies the fluctuation around its average. The starting point of the calculations is the connection between correlation functions of the Bures-Hall ensemble and those of the Cauchy-Laguerre ensemble. The derived variance formula, together with the known mean formula, leads to a simple but accurate Gaussian approximation of the distribution of von Neumann entropy of finite-size systems. This Gaussian approximation is also conjectured to be the limiting distribution for large dimensional systems.

DOI: 10.1103/PhysRevE.102.062128

I. INTRODUCTION AND THE MAIN RESULT

Quantum information theory aims at explaining the theoretical underpinnings of quantum technologies such as quantum computing and quantum communications. Crucial to successful exploitation of the quantum revolutionary advances is the understanding of the nonclassical phenomenon of quantum entanglement. Entanglement is the most fundamental characteristic trait of quantum mechanics, which is also the resource and medium that enable quantum technologies.

In this work, we aim to understand the statistical behavior of the entanglement of quantum bipartite systems over the Bures-Hall measure [1–3]. In particular, we study the degree of entanglement as measured by the von Neumann entropy over such a measure. The mean value of von Neumann entropy over the Bures-Hall measure has been recently obtained [4,5]. As an important step towards understanding its statistical distribution, we derive the corresponding variance in this paper. The variance describes the fluctuation of the entropy around its mean value, which also provides crucial information such as whether the average entropy is typical.

The density matrix formulism [6], introduced by von Neumann, that has led to the Bures-Hall ensemble is described as follows. Consider a composite (bipartite) system that consists of two subsystems A and B of Hilbert space (complex vector space) with dimensions m and n, respectively. The Hilbert space $\mathcal{H}_{A+B}$ of the composite system is given by the tensor product of the subsystems, $\mathcal{H}_{A+B}=\mathcal{H}_A\otimes\mathcal{H}_B$. A random pure state of the composite system $\mathcal{H}_{A+B}$ is defined as a linear combination of the random coefficients $z_{i,j}$ and the complete basis $\{|i^A\rangle\}$ and $\{|j^B\rangle\}$ of $\mathcal{H}_A$ and $\mathcal{H}_B$ [7],

$$|\psi\rangle=\sum_{i=1}^{m}\sum_{j=1}^{n}z_{i,j}|i^A\rangle\otimes|j^B\rangle, \tag{1}$$

where each $z_{i,j}$ follows the standard complex Gaussian distribution. We now consider a superposition of the state (1),

$$|\varphi\rangle=|\psi\rangle+(\mathbf{U}\otimes\mathbf{I}_m)|\psi\rangle, \tag{2}$$

where $\mathbf{U}$ is an $m\times m$ unitary random matrix with the measure proportional to $\det(\mathbf{I}_m+\mathbf{U})^{2\alpha+1}$ [4]. The corresponding density matrix of the pure state (2) is

$$\rho=|\varphi\rangle\langle\varphi|, \tag{3}$$

which has the natural probability constraint

$$\mathrm{tr}(\rho)=1. \tag{4}$$

We assume, without loss of generality, that $m\leqslant n$. The reduced density matrix ρ_A of the smaller subsystem A is computed by partial tracing (purification) of the full density matrix (3) over the other subsystem B (environment) as

$$\rho_A=\mathrm{tr}_B\rho. \tag{5}$$

The resulting density of the eigenvalues of ρ_A ($\lambda_i\in[0,1]$, $i=1,\dots,m$) is the (generalized) complex Bures-Hall measure [1–4],

$$f(\boldsymbol{\lambda})=\frac{1}{C}\,\delta\left(1-\sum_{i=1}^{m}\lambda_i\right)\prod_{1\leqslant i<j\leqslant m}\frac{(\lambda_i-\lambda_j)^2}{\lambda_i+\lambda_j}\prod_{i=1}^{m}\lambda_i^{\alpha}, \tag{6}$$

where the parameter α takes half-integer values,

$$\alpha=n-m-\tfrac{1}{2}, \tag{7}$$

and the constant C is

$$C=\frac{2^{-m(m+2\alpha)}\pi^{m/2}}{\Gamma[m(m+2\alpha+1)/2]}\prod_{i=1}^{m}\frac{\Gamma(i+1)\Gamma(i+2\alpha+1)}{\Gamma(i+\alpha+1/2)}. \tag{8}$$

A relatively detailed derivation of the Bures-Hall density (6) can be found in Sec. 3 of Ref. [8]. In Eq. (6), the presence of the Dirac delta function $\delta(\cdot)$ reflects the constraint (4). Note that another approach to define the Bures-Hall measure (6)

*luwe@umich.edu

is to introduce a distance metric (Bures-Hall metric) over the reduced density matrices [9].

Before discussing the state of the art on the study of entanglement entropies over the Bures-Hall measure, below we outline the physical relevance of the considered model as well as the physical implications of the results of this work:

(i) The quantum bipartite model is useful in describing the entanglement between the two subsystems of different bipartite systems, in which one subsystem represents a physical object (such as spins) and the other subsystem is the environment (such as a heat bath).

(ii) The Bures-Hall measure enjoys the property that without any prior knowledge of a density matrix, the optimal way to estimate the density matrix is to generate a state at random with respect to this measure [2,8,10]. The corresponding Bures-Hall metric turns out to be the only monotone metric that is simultaneously Fisher adjusted and Fubini-Study adjusted [2,10].

(iii) Our proposed framework can be directly applied to study other entanglement entropies such as quantum purity [4,5,8,10,11] and Tsallis entropy [12] over the Bures-Hall ensemble. The results would lead towards a complete picture of how different entropies and random environments affect the degree of entanglement in bipartite systems.

(iv) The framework also makes it possible to investigate the phase transition phenomenon of entanglement thresholds between separable states and entangled states under the Bures-Hall measure as the needed random matrix results are available [13,14]. This phenomenon has been observed and studied under the Hilbert-Schmidt measure in Ref. [15].

(v) The Bures-Hall measure is a function of fidelity [9,16], which is a key performance indicator in quantum information processing. For example, in quantum computing, it is often necessary to quantify the distance between two density matrices that can be considered as the initial and desired states. Fidelity also gauges the algorithm performance, i.e., how a density matrix may be approximated by another one. Based on the results of this work, one may extend the results in Ref. [16] to arbitrary subsystem dimensions and study the higher order fluctuations of fidelity around the average value.

(vi) Since the determination of whether a state is entangled or separable is, in general, a difficult problem, the conditions for separability become very important. These conditions are often formulated in terms of volumes [9,17]. In addition to the asymptotic lower and upper bounds on the volumes [17], one could also compute the corresponding finite-size Bures-Hall volume with the help of the results in this work and the recent progress in understanding various aspects of the Bures-Hall ensemble [4,5,13,14].

The degree of entanglement of the bipartite subsystems can be measured by entanglement entropies, which are functions of eigenvalues (entanglement spectrum) of the reduced density matrix. We consider the standard measure of von Neumann entropy of the subsystem [18]

$$S = -\mathrm{tr}(\rho_A \ln \rho_A) = -\sum_{i=1}^{m} \lambda_i \ln \lambda_i, \tag{9}$$

supported in $S \in [0, \ln m]$, which achieves the separable state ($S = 0$) when $\lambda_1 = 1, \lambda_2 = \cdots = \lambda_m = 0$, and the maximally entangled state ($S = \ln m$) when $\lambda_1 = \cdots = \lambda_m = 1/m$. The statistical information of the entropies is encoded through the moments. In particular, the first moment (average value) implies the typical behavior of entanglement and the second moment (variance) specifies the fluctuation around the typical value. The average von Neumann entropy, valid for any subsystem dimensions $m \leqslant n$, has been recently obtained as [4,5]

$$\mathbb{E}_f[S] = \psi_0\left(mn - \frac{m^2}{2} + 1\right) - \psi_0\left(n + \frac{1}{2}\right), \tag{10}$$

where the expectation $\mathbb{E}_f[\cdot]$ is taken over the Bures-Hall ensemble (6). Here, $\psi_0(x) = d \ln \Gamma(x)/dx$ is the digamma function [19] and, for a positive integer l,

$$\psi_0(l) = -\gamma + \sum_{k=1}^{l-1} \frac{1}{k}, \tag{11a}$$

$$\psi_0\left(l + \frac{1}{2}\right) = -\gamma - 2\ln 2 + 2\sum_{k=0}^{l-1} \frac{1}{2k+1}, \tag{11b}$$

where $\gamma \approx 0.5772$ is the Euler's constant. The main result of this work on the corresponding variance is summarized in the following proposition.

Proposition 1. The exact variance of von Neumann entropy (9) under the Bures-Hall ensemble (6) is given by

$$\mathbb{V}_f[S] = -\psi_1\left(mn - \frac{m^2}{2} + 1\right) + \frac{2n(2n+m) - m^2 + 1}{2n(2mn - m^2 + 2)}\psi_1\left(n + \frac{1}{2}\right), \tag{12}$$

where $\psi_1(x) = d^2 \ln \Gamma(x)/dx^2$ denotes the trigamma function [19].

The proof of Proposition 1 is given in Sec. II. Note that for a positive integer l, the trigamma function can be written as finite sums as

$$\psi_1(l) = \frac{\pi^2}{6} - \sum_{k=1}^{l-1} \frac{1}{k^2}, \tag{13a}$$

$$\psi_1\left(l + \frac{1}{2}\right) = \frac{\pi^2}{2} - 3\sum_{k=1}^{l-1} \frac{1}{k^2} - 4\sum_{k=l}^{2l-1} \frac{1}{k^2}. \tag{13b}$$

Under the Bures-Hall measure, other entropies such as the quantum purity have also been studied in the literature. In particular, the first few exact moments of quantum purity [4,5,8,10] as well as its asymptotic distribution [11] are known [20]. Besides the Bures-Hall measure, the exact moments of von Neumann entropy [21–26] and quantum purity [27,28] have been well investigated over the less complicated [29] Hilbert-Schmidt measure [9]. Finally, we note that results of the real Bures-Hall random matrix ensemble may be parallelly obtained if the corresponding correlation functions can be found.

With the expressions of the mean (10) and variance (12), simple approximations can be constructed to understand the distribution of the von Neumann entropy. For convenience,

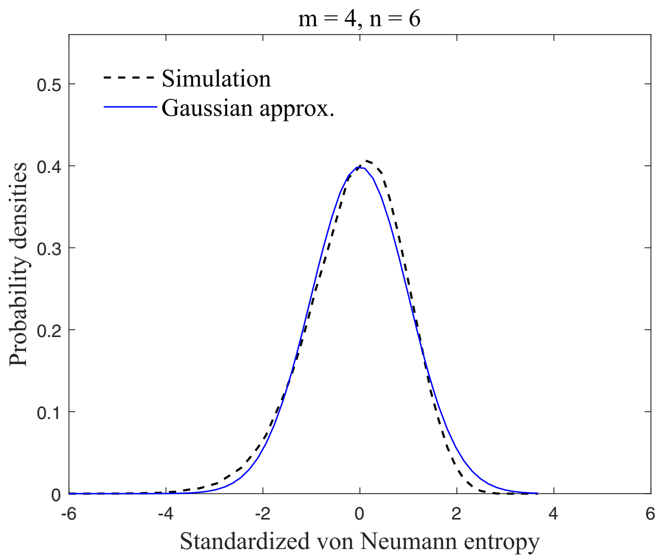

FIG. 1. Probability densities of the standardized von Neumann entropy (14) of subsystem dimensions $m = 4$ and $n = 6$: A comparison of the simulated true distribution (dashed line in black) and the Gaussian approximation (15) (solid line in blue).

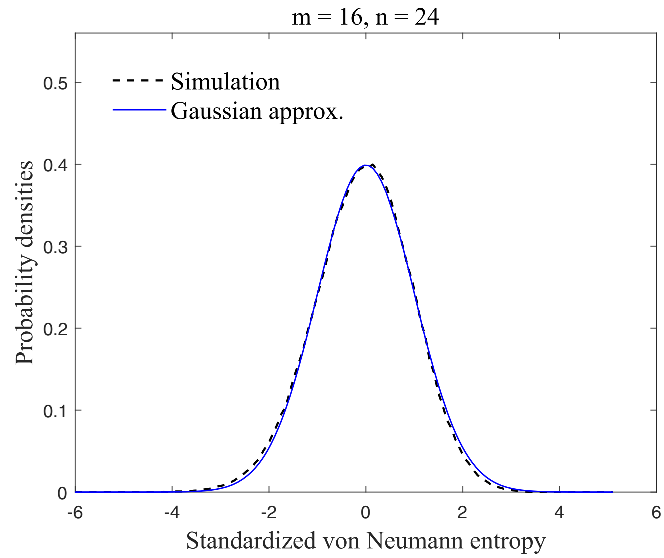

FIG. 2. Probability densities of the standardized von Neumann entropy (14) of subsystem dimensions $m = 16$ and $n = 24$: A numerical support to the conjectured Gaussian limit. The dashed line in black and the solid line in blue represent the simulated true distribution and the standard Gaussian distribution (15), respectively.

we standardize the von Neumann entropy as

$$X = \frac{S - \mathbb{E}_f[S]}{\sqrt{\mathbb{V}_f[S]}}, \tag{14}$$

so that the random variable X, supported in $X \in (-\infty, \infty)$, has zero mean and unit variance. Thus, a natural approximation to the distribution of X would be a standard Gaussian distribution,

$$\varphi_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}, \tag{15}$$

i.e., the distribution of S is approximated by a Gaussian distribution with mean $\mathbb{E}_f[S]$ and variance $\mathbb{V}_f[S]$. In Fig. 1, we compare the simulated true distribution of the standardized von Neumann entropy (14) to the Gaussian approximation (15), where the dimensions of the subsystems are $m = 4$ and $n = 6$. As opposed to the Gaussian distribution, we see from Fig. 1 that the true distribution of von Neumann entropy is nonsymmetric, which appears to be left skewed (a negative skewness). With the knowledge of higher order moments, the Gaussian approximation (15) can be systematically improved to provide more accurate approximations to finite-size systems. On the other hand, motivated by the case of the Hilbert-Schmidt measure [26], here we also conjecture that the first two moments, given by Eqs. (10) and (12), are sufficient to fully describe the distribution of von Neumann entropy as the dimensions of the subsystems become large. Formally, in the limit

$$m \to \infty, \quad n \to \infty, \quad \frac{m}{n} = c \in (0, 1], \tag{16}$$

we conjecture that the standardized von Neumann entropy (14) converges in distribution to a Gaussian random variable with zero mean and unit variance. Note that the high-dimensional asymptotic regime (16) is different from the classical asymptotic regime [21], where the dimension m is fixed as n goes to infinity. One way to prove the above conjecture is to show that all the higher order (beyond the first two) moments of the random variable (14) vanish in the limit (16). Another potential approach to show the Gaussian limiting behavior is via tools from asymptotic geometric analysis and concentration of measure techniques [30]. These tools can be used to establish, for example, the tail bounds of a Gaussian type to the distribution of entanglement entropies as have been done under the Hilbert-Schmidt measure [15]. A numerical evidence to support the conjecture is provided in Fig. 2, where we simultaneously increase the subsystem dimensions to $m = 16$ and $n = 24$ with their ratio $c = m/n = 2/3$ kept the same as in Fig. 1. Comparing Fig. 1 with Fig. 2, it is seen that the distribution of von Neumann entropy approaches rather rapidly to the conjectured limiting Gaussian distribution.

The rest of the paper is organized as follows. In Sec. II, we derive the main result (12) of the exact variance of von Neumann entropy over the Bures-Hall measure. Specifically, in Sec. II A, we relate the computation of the variance to that over a more convenient ensemble with no δ function constraint. Calculating the corresponding variance boils down, in Sec. II B, to computing four integrals over the correlation functions of the unconstraint ensemble. In Sec. II C, the four integrals are evaluated into terms involving polygamma functions by utilizing recent results on the unconstrained ensemble as well as some summation formulas of polygamma functions. We outline potential future works in Sec. III after summarizing the main findings of the paper. The polygamma summation formulas utilized in Sec. II C are listed and discussed in Appendix A. The coefficient lists of the four integrals are provided in the tables in Appendix B.

II. VARIANCE CALCULATION

A. Variance relation

Finding moment relations is a rather standard calculation (see, e.g., Refs. [4,5,8,21,23,25,26]) that relates moment computation to that over an ensemble without the constraint $\delta(1-\sum_{i=1}^{m}\lambda_i)$. As will be seen, the unconstrained ensemble of the Bures-Hall measure (6) is [4]

$$h(\boldsymbol{x}) = \frac{1}{C'}\prod_{1\leqslant i<j\leqslant m}\frac{(x_i-x_j)^2}{x_i+x_j}\prod_{i=1}^{m}x_i^{\alpha}e^{-x_i}, \tag{17}$$

where $x_i\in[0,\infty)$, $i=1,\ldots,m$, and the constant C' depends on the constant (8) as

$$C' = C\,\Gamma(d), \tag{18}$$

with d denoting

$$d = \tfrac{1}{2}m(m+2\alpha+1). \tag{19}$$

Despite being only interested in the physically relevant α values in Eq. (7), the results hereafter, in particular the expression (74), are valid for any $\alpha>-1$ in which the density (17) is defined.

We first derive the density $g(\theta)$ of trace

$$\theta = \sum_{i=1}^{m}x_i, \quad \theta\in[0,\infty), \tag{20}$$

of the unconstraint ensemble (17) as

$$g(\theta) = \int_{\boldsymbol{x}} h(\boldsymbol{x})\delta\left(\theta-\sum_{i=1}^{m}x_i\right)\prod_{i=1}^{m}dx_i \tag{21}$$

$$= \frac{C}{C'}e^{-\theta}\theta^{d-1}\int_{\boldsymbol{\lambda}}f(\boldsymbol{\lambda})\prod_{i=1}^{m}d\lambda_i \tag{22}$$

$$= \frac{1}{\Gamma(d)}e^{-\theta}\theta^{d-1}, \tag{23}$$

where we have employed the change of variables

$$x_i = \theta\lambda_i, \quad i=1,\ldots,m. \tag{24}$$

The above calculation implies that the density $h(\boldsymbol{x})$ is factored as [31]

$$h(\boldsymbol{x})\prod_{i=1}^{m}dx_i = f(\boldsymbol{\lambda})g(\theta)\,d\theta\prod_{i=1}^{m}d\lambda_i, \tag{25}$$

which leads to the fact that θ is independent of each λ_i (hence independent of S).

To exploit this independence in calculating the variance, we first write by the change of variables (24) that

$$S^2 = \theta^{-2}T^2 + 2S\ln\theta - \ln^2\theta, \tag{26}$$

where

$$T = \sum_{i=1}^{m}x_i\ln x_i \tag{27}$$

defines the induced von Neumann entropy over the unconstrained ensemble (17). The second moment relation can now be found, by multiplying an appropriate constant [cf. Eq. (23)],

$$1 = \int_0^\infty \frac{1}{\Gamma(d+2)}e^{-\theta}\theta^{d+1}\,d\theta, \tag{28}$$

as

$$\mathbb{E}_f[S^2] = \int_0^\infty\int_{\boldsymbol{\lambda}}\frac{e^{-\theta}\theta^{d+1}}{\Gamma(d+2)}S^2 f(\boldsymbol{\lambda})\,d\theta\prod_{i=1}^{m}d\lambda_i \tag{29}$$

$$= \frac{\Gamma(d)}{\Gamma(d+2)}\mathbb{E}_h[T^2] + 2\mathbb{E}_f[S]\mathbb{E}_g[\ln\theta] - \mathbb{E}_g[\ln^2\theta],$$

$$= \frac{1}{d(d+1)}\mathbb{E}_h[T^2] + 2\psi_0(d+2)\mathbb{E}_f[S] - \psi_0^2(d+2) - \psi_1(d+2), \tag{30}$$

where we have used the results given by Eqs. (25) and (26) and the identities [valid for $\mathrm{Re}(a)>0$]

$$\int_0^\infty e^{-\theta}\theta^{a-1}\ln\theta\,d\theta = \Gamma(a)\psi_0(a), \tag{31a}$$

$$\int_0^\infty e^{-\theta}\theta^{a-1}\ln^2\theta\,d\theta = \Gamma(a)\left[\psi_0^2(a)+\psi_1(a)\right]. \tag{31b}$$

By the known mean formulas (10) and [5],

$$\mathbb{E}_h[T] = \frac{m(m+2\alpha+1)}{2}\psi_0(m+\alpha+1), \tag{32}$$

the derived moment relation (30) translates showing the claimed variance formula (12) to prove an induced variance formula,

$$\mathbb{V}_h[T] = m(2n-m)\left[\psi_0\left(n+\frac{1}{2}\right)+\frac{1}{2}\psi_0^2\left(n+\frac{1}{2}\right) + \frac{4n^2+2mn-m^2+1}{8n}\psi_1\left(n+\frac{1}{2}\right)\right], \tag{33}$$

where we have also used the fact $\mathbb{V}[X]=\mathbb{E}[X^2]-\mathbb{E}^2[X]$ and the identities

$$\psi_0(l+n) = \psi_0(l) + \sum_{k=0}^{n-1}\frac{1}{l+k}, \tag{34a}$$

$$\psi_1(l+n) = \psi_1(l) - \sum_{k=0}^{n-1}\frac{1}{(l+k)^2}. \tag{34b}$$

B. Variance of unconstraint ensemble

Calculating $\mathbb{E}_h[T^2]$ requires one and two arbitrary eigenvalue densities, denoted, respectively, by $h_1(x)$ and $h_2(x,y)$, of the unconstrained Bures-Hall ensemble (17) as

$$\mathbb{E}_h[T^2] = m\int_0^\infty x^2\ln^2 x\,h_1(x)\,dx + m(m-1)\times\int_0^\infty\int_0^\infty xy\ln x\ln y\,h_2(x,y)\,dx\,dy. \tag{35}$$

In general, the density of k arbitrary eigenvalues (k-point correlation function) of the ensemble (17) is described by a Pfaffian point process of a $2k\times 2k$ antisymmetric matrix

[13]. The corresponding correlation kernels are written in terms of those of the Cauchy-Laguerre biorthogonal ensemble [14], which is a determinantal point process. In particular, the needed eigenvalue densities in Eq. (35) are written as [13,14]

$$h_1(x) = \frac{1}{2m}[K_{01}(x,x) + K_{10}(x,x)], \tag{36}$$

$$\begin{aligned} h_2(x,y) = &\frac{1}{4m(m-1)}\{[K_{01}(x,x) + K_{10}(x,x)] \\ &\times[K_{01}(y,y) + K_{10}(y,y)] - 2K_{01}(x,y) \\ &\times K_{01}(y,x) - 2K_{10}(x,y)K_{10}(y,x) \\ &-2K_{00}(x,y)K_{11}(x,y) - 2K_{00}(y,x)K_{11}(y,x)\}, \end{aligned} \tag{37}$$

where the correlation kernels are

$$K_{00}(x,y) = \sum_{k=0}^{m-1} p_k(x)q_k(y), \tag{38a}$$

$$K_{01}(x,y) = -x^\alpha e^{-x}\sum_{k=0}^{m-1} p_k(x)Q_k(-y), \tag{38b}$$

$$K_{10}(x,y) = -y^{\alpha+1}e^{-y}\sum_{k=0}^{m-1} P_k(-x)q_k(y), \tag{38c}$$

$$K_{11}(x,y) = x^\alpha y^{\alpha+1}e^{-x-y}\sum_{k=0}^{m-1} P_k(-y)Q_k(-x) - w(x,y), \tag{38d}$$

with the weight function $w(x,y)$ of the biorthogonal polynomials $p_k(x)$, $q_l(y)$,

$$\int_0^\infty\int_0^\infty p_k(x)q_l(y)w(x,y)\,dx\,dy = \delta_{kl}, \tag{39}$$

given by

$$w(x,y) = \frac{x^\alpha y^{\alpha+1}e^{-x-y}}{x+y}. \tag{40}$$

The functions in Eq. (38) are further related by [13,14]

$$P_k(x) = \int_0^\infty \frac{v^\alpha e^{-v}}{x-v}p_k(v)\,dv, \tag{41a}$$

$$Q_k(y) = \int_0^\infty \frac{w^{\alpha+1}e^{-w}}{y-w}q_k(w)\,dw, \tag{41b}$$

which, together with the orthogonality condition (39), can be employed to verify that the functions (36) and (37) are indeed probability density functions. Moreover, these functions are expressed explicitly via Meijer G-functions as [13,14]

$$p_j(x) = \sqrt{2}(-1)^j G^{1,1}_{2,3}\left(\begin{matrix}-2\alpha-1-j;\, j+1 \\ 0;\, -\alpha,\, -2\alpha-1\end{matrix}\middle| x\right),$$

$$q_j(x) = \sqrt{2}(-1)^j (j+\alpha+1) G^{1,1}_{2,3}\left(\begin{matrix}-2\alpha-1-j;\, j+1 \\ 0;\, -\alpha-1,\, -2\alpha-1\end{matrix}\middle| x\right),$$

$$P_j(x) = \sqrt{2}(-1)^{j+1}e^{-x} G^{2,1}_{2,3}\left(\begin{matrix}-\alpha-j-1;\, \alpha+j+1 \\ 0,\, \alpha;\, -\alpha-1\end{matrix}\middle| -x\right),$$

$$\begin{aligned} Q_j(x) = &\sqrt{2}(-1)^{j+1}(j+\alpha+1)e^{-x} \\ &\times G^{2,1}_{2,3}\left(\begin{matrix}-\alpha-j;\, \alpha+j+2 \\ 0,\, \alpha+1;\, -\alpha\end{matrix}\middle| -x\right), \end{aligned} \tag{42}$$

where the Meijer G-function is defined by the contour integral [19]

$$\begin{aligned} &G^{m,n}_{p,q}\left(\begin{matrix}a_1,\ldots,a_n;\, a_{n+1},\ldots,a_p \\ b_1,\ldots,b_m;\, b_{m+1},\ldots,b_q\end{matrix}\middle| x\right) \\ &= \frac{1}{2\pi\imath}\int_{\mathcal{L}} \frac{\prod_{j=1}^m \Gamma(b_j+s)\prod_{j=1}^n \Gamma(1-a_j-s)x^{-s}}{\prod_{j=n+1}^p \Gamma(a_j+s)\prod_{j=m+1}^q \Gamma(1-b_j-s)}\,ds, \end{aligned} \tag{43}$$

with the contour $\mathcal{L}$ separating the poles of $\Gamma(1-a_j-s)$ from the poles of $\Gamma(b_j+s)$. In addition to the summation form (38), the kernels also admit integral representation [13,14],

$$K_{00}(x,y) = \int_0^1 t^{2\alpha+1}H_\alpha(tx)H_{\alpha+1}(ty)\,dt, \tag{44a}$$

$$K_{01}(x,y) = x^{2\alpha+1}\int_0^1 t^{2\alpha+1}H_\alpha(ty)G_{\alpha+1}(tx)\,dt, \tag{44b}$$

$$K_{10}(x,y) = y^{2\alpha+1}\int_0^1 t^{2\alpha+1}H_{\alpha+1}(tx)G_\alpha(ty)\,dt, \tag{44c}$$

$$K_{11}(x,y) = (xy)^{2\alpha+1}\int_0^1 t^{2\alpha+1}G_{\alpha+1}(tx)G_\alpha(ty)\,dt - \frac{x^\alpha y^{\alpha+1}}{x+y}, \tag{44d}$$

where we denote

$$H_q(x) = G^{1,1}_{2,3}\left(\begin{matrix}-m-2\alpha-1;\, m \\ 0;\, -q,\, -2\alpha-1\end{matrix}\middle| x\right), \tag{45a}$$

$$G_q(x) = G^{2,1}_{2,3}\left(\begin{matrix}-m-2\alpha-1;\, m \\ 0,\, -q;\, -2\alpha-1\end{matrix}\middle| x\right). \tag{45b}$$

Finally, inserting Eqs. (36) and (37) into Eq. (35), the induced variance is represented as

$$\mathbb{V}_h[T] = \tfrac{1}{2}(I_A - I_B - I_C + 2I_D), \tag{46}$$

where

$$I_A = \int_0^\infty x^2\ln^2 x[K_{01}(x,x) + K_{10}(x,x)]dx, \tag{47}$$

$$I_B = \int_0^\infty\int_0^\infty xy\ln x\ln y\, K_{01}(x,y)K_{01}(y,x)\,dx\,dy, \tag{48}$$

$$I_C = \int_0^\infty\int_0^\infty xy\ln x\ln y\, K_{10}(x,y)K_{10}(y,x)\,dx\,dy, \tag{49}$$

$$I_D = \int_0^\infty\int_0^\infty xy\ln x\ln y\, K_{00}(x,y)K_{11}(x,y)\,dx\,dy, \tag{50}$$

and we have used the fact [cf. Eq. (32)]

$$\int_0^\infty x\ln x[K_{01}(x,x) + K_{10}(x,x)]dx = 2\mathbb{E}_h[T]. \tag{51}$$

To show Eq. (33), the remaining task is to compute the four integrals (47)–(50).

C. Computing the integrals I_A, I_B, I_C, and I_D

1. Computation of I_A

The evaluation of I_A follows a similar procedure as in Sec. 2.2 of Ref. [5]. The key is to compute the integral

$$A_q(t) = \int_0^\infty x^\beta (tx)^{2\alpha+1} H_{2\alpha+1-q}(tx) G_q(tx)\, dx \tag{52}$$

and its derivatives with respect to β for $q = \alpha, \alpha + 1$. This integral has been obtained in Ref. [5] by using the Mellin transform of the Meijer G-function [19],

$$\int_0^\infty x^{s-1} G^{m,n}_{p,q}\left(\begin{matrix} a_1, \dots, a_n; a_{n+1}, \dots, a_p \\ b_1, \dots, b_m; b_{m+1}, \dots, b_q \end{matrix} \middle| \eta x \right) dx = \frac{\eta^{-s} \prod_{j=1}^m \Gamma(b_j + s) \prod_{j=1}^n \Gamma(1 - a_j - s)}{\prod_{j=n+1}^p \Gamma(a_j + s) \prod_{j=m+1}^q \Gamma(1 - b_j - s)}, \tag{53}$$

and the fact that the Meijer G-function $G^{1,1}_{2,3}$ of a negative parameter a_i $(i \leqslant n)$ is a terminating hypergeometric function [5,14,32] as

$$A_q(t) = t^{-\beta-1} A_q, \tag{54}$$

where

$$A_q = \sum_{k=0}^{m-1} \frac{(-1)^{k+m} \Gamma(k + 2\alpha + m + 2) \Gamma(k + \beta + 1)}{\Gamma(k + 2\alpha + 2) \Gamma(k + 2\alpha + 2 - q) \Gamma(m - k) k!} \frac{\Gamma(k + \beta + 2\alpha + 2) \Gamma(k + \beta + 2\alpha + 2 - q)}{\Gamma(k + \beta + 2\alpha + m + 2) \Gamma(k + \beta - m + 1)}. \tag{55}$$

Using a different representation of Eqs. (44b) and (44c) obtained in Ref. [5],

$$\begin{aligned} K_{01}(x, y) &= -x^{2\alpha+1} \int_1^\infty t^{2\alpha+1} H_\alpha(tx) G_{\alpha+1}(ty)\, dt, \\ K_{10}(x, y) &= -y^{2\alpha+1} \int_1^\infty t^{2\alpha+1} G_\alpha(tx) H_{\alpha+1}(ty)\, dt, \end{aligned} \tag{56}$$

and changing the order of integrations, I_A is calculated as

$$I_A = -\tfrac{1}{4}(A_\alpha + A_{\alpha+1})\big|_{\beta=2} + \tfrac{1}{2}\big(H^{(1)}_\alpha + H^{(1)}_{\alpha+1}\big) - \tfrac{1}{2}\big(H^{(2)}_\alpha + H^{(2)}_{\alpha+1}\big), \tag{57}$$

where we denote

$$H^{(1)}_q = \frac{d}{d\beta} A^{(\beta)}_q \bigg|_{\beta=2}, \quad H^{(2)}_q = \frac{d}{d\beta^2} A^{(\beta)}_q \bigg|_{\beta=2}, \tag{58}$$

and the integrals over t have been evaluated first by the fact

$$\int_1^\infty \frac{1}{t^3}\, dt = \frac{1}{2}, \quad \int_1^\infty \frac{\ln t}{t^3}\, dt = \frac{1}{4}, \quad \int_1^\infty \frac{\ln^2 t}{t^3}\, dt = \frac{1}{4}.$$

The first term $A_\alpha + A_{\alpha+1}$ in Eq. (57) for $\beta = 2$ has been obtained in Eq. (50) of Ref. [5]. By resolving indeterminacy in the limit $\epsilon \to 0$,

$$\Gamma(-l + \epsilon) = \frac{(-1)^l}{l!\epsilon}[1 + \psi_0(l + 1)\epsilon + o(\epsilon^2)], \tag{59a}$$

$$\psi_0(-l + \epsilon) = -\frac{1}{\epsilon}[1 - \psi_0(l + 1)\epsilon + o(\epsilon^2)], \tag{59b}$$

$$\psi_1(-l + \epsilon) = \frac{1}{\epsilon^2}[1 + o(\epsilon^2)], \tag{59c}$$

the terms (58) are evaluated into finite sums involving polygamma functions. Computing these summations by the identities in Appendix A, we obtain I_A as shown in Eq. (60), where the list of coefficients can be found in Table I of Appendix B. Note that as a result of employing the semi-closed-form identities (A6)–(A8), the obtained I_A expression (60) still contains five summations that may not be further simplified. These unsimplifiable sums eventually cancel with the ones in I_B and I_C as will be seen. Similar phenomena have also been observed in the higher order moment computations over the Hilbert-Schmidt measure [25,26],

$$I_A = \frac{1}{36\alpha(m+\alpha)(m+2\alpha)(2m+2\alpha+1)^3}\Bigg\{-2a_0\Bigg[\sum_{k=1}^{m}\frac{\psi_0(k+\alpha)}{k}+\sum_{k=1}^{m}\frac{\psi_0(k+2\alpha)}{k}-\sum_{k=1}^{m}\frac{\psi_0(k+m+2\alpha)}{k}$$
$$+\sum_{k=1}^{m}\frac{\psi_0(k+m+2\alpha)}{k+\alpha}+\sum_{k=1}^{m}\frac{\psi_0(k+m+2\alpha)}{k+2\alpha}\Bigg]+a_1+a_2[\psi_0(1)-\psi_0(m+1)]+a_3\psi_0(\alpha+1)+a_4\psi_0(2\alpha+1)$$
$$+a_5\psi_0(m+\alpha+1)+a_6\psi_0(m+2\alpha+1)+a_7\psi_0(2m+2\alpha+1)+a_0\big[-2\psi_0(1)\psi_0(\alpha+1)-2\psi_0(1)\psi_0(2\alpha+1)$$
$$+2\psi_0(1)\psi_0(m+2\alpha+1)+\psi_0^2(\alpha+1)+\psi_0^2(2\alpha+1)+2\psi_0(\alpha+1)\psi_0(m+1)-2\psi_0(\alpha+1)\psi_0(m+\alpha+1)$$
$$-2\psi_0(\alpha+1)\psi_0(m+2\alpha+1)+2\psi_0(2\alpha+1)\psi_0(m+1)-4\psi_0(2\alpha+1)\psi_0(m+2\alpha+1)-2\psi_0(m+1)$$
$$\times\psi_0(m+2\alpha+1)-2\psi_0(m+\alpha+1)\psi_0(m+2\alpha+1)+4\psi_0(m+\alpha+1)\psi_0(2m+2\alpha+1)-\psi_0^2(m+2\alpha+1)$$
$$+8\psi_0(m+2\alpha+1)\psi_0(2m+2\alpha+1)-4\psi_0^2(2m+2\alpha+1)-\psi_1(\alpha+1)-\psi_1(2\alpha+1)+\psi_1(m+2\alpha+1)\big]\Bigg\}. \tag{60}$$

2. *Computation of I_B and I_C*

The steps in calculating I_B and I_C are identical. The starting point is the integral form of the kernels (44b) and (44c) as well as finite sum representation [5,14] of the Meijer G-functions $G_{2,3}^{1,1}$. Instead of changing the order of summations as in I_A, here we directly evaluate the integrals over t by the identity [19]

$$\int_0^1 x^{a-1}G_{p,q}^{m,n}\left(\begin{matrix}a_1,\dots,a_n;a_{n+1},\dots,a_p\\ b_1,\dots,b_m;b_{m+1},\dots,b_q\end{matrix}\middle|\eta x\right)dx = G_{p+1,q+1}^{m,n+1}\left(\begin{matrix}1-a,a_1,\dots,a_n;a_{n+1},\dots,a_p\\ b_1,\dots,b_m;b_{m+1},\dots,b_q,-a\end{matrix}\middle|\eta\right). \tag{61}$$

This leads I_B to

$$I_B = \sum_{j,k=0}^{m-1} f_{j,k}f_{k,j}, \tag{62}$$

where we denote

$$f_{j,k} = \frac{(-1)^j\Gamma(m+2\alpha+j+2)}{\Gamma(j+1)\Gamma(\alpha+j+1)\Gamma(2\alpha+j+2)\Gamma(m-j)}\int_0^\infty x\ln x\, G_{3,4}^{2,2}\left(\begin{matrix}j-k,\,j-m;m+2\alpha+j+1\\ 2\alpha+j+1,\,\alpha+j;\,j,\,j-k-1\end{matrix}\middle|x\right)dx.$$

The above integral can be similarly evaluated as in Eq. (52) by first utilizing Eq. (53) before taking the derivative with respect to β. We then set $\beta=1$ and resolve the resulting indeterminacy by Eq. (59), and I_B becomes a double sum involving polygamma functions. The summations are evaluated with the help of the identities in Appendix A, which completes the calculation of I_B. Since I_C is computed to the same form as I_B, for convenience we provide the corresponding result of I_B+I_C as shown in Eq. (63), where the coefficients can be found in Table II,

$$I_B+I_C = \frac{1}{36\alpha(m+\alpha)(m+\alpha+1)(m+2\alpha)(2m+2\alpha+1)^4}\Bigg\{2(b_0+c_0)\Bigg[\sum_{k=1}^{m}\frac{\psi_0(k+\alpha)}{k}+\sum_{k=1}^{m}\frac{\psi_0(k+2\alpha)}{k}$$
$$-\sum_{k=1}^{m}\frac{\psi_0(k+m+2\alpha)}{k}+\sum_{k=1}^{m}\frac{\psi_0(k+m+2\alpha)}{k+\alpha}+\sum_{k=1}^{m}\frac{\psi_0(k+m+2\alpha)}{k+2\alpha}\Bigg]+b_1+c_1+(b_2+c_2)$$
$$\times[\psi_0(1)-\psi_0(m+1)]+(b_3+c_3)\psi_0(\alpha+1)+(b_4+c_4)\psi_0(2\alpha+1)+(b_5+c_5)\psi_0(m+\alpha+1)$$
$$+(b_6+c_6)\psi_0(m+2\alpha+1)+(b_7+c_7)\psi_0(2m+2\alpha+1)+(b_0+c_0)\big[2\psi_0(1)\psi_0(\alpha+1)+2\psi_0(1)\psi_0(2\alpha+1)$$
$$-2\psi_0(1)\psi_0(m+2\alpha+1)-\psi_0^2(\alpha+1)-\psi_0^2(2\alpha+1)-2\psi_0(\alpha+1)\psi_0(m+1)+2\psi_0(\alpha+1)\psi_0(m+\alpha+1)$$
$$+2\psi_0(\alpha+1)\psi_0(m+2\alpha+1)-2\psi_0(2\alpha+1)\psi_0(m+1)+4\psi_0(2\alpha+1)\psi_0(m+2\alpha+1)+2\psi_0(m+1)$$
$$\times\psi_0(m+2\alpha+1)+\psi_1(\alpha+1)+\psi_1(2\alpha+1)-\psi_1(m+\alpha+1)-3\psi_1(m+2\alpha+1)+2\psi_1(2m+2\alpha+1)\big]$$
$$+(b_8+c_8)\psi_0^2(m+\alpha+1)+(b_9+c_9)\psi_0(m+\alpha+1)\psi_0(m+2\alpha+1)+(b_{10}+c_{10})\big[\psi_0(m+\alpha+1)$$
$$\times\psi_0(2m+2\alpha+1)+2\psi_0(m+2\alpha+1)\psi_0(2m+2\alpha+1)-\psi_0^2(2m+2\alpha+1)\big]+(b_{11}+c_{11})$$
$$\times\psi_0^2(m+2\alpha+1)\Bigg\}. \tag{63}$$

3. Computation of I_D

We define the integral

$$D(\beta_1, \beta_2) = \int_0^\infty \int_0^\infty x^{\beta_1} y^{\beta_2} K_{00}(x, y) K_{11}(x, y)\, dx\, dy \tag{64}$$

so that the desired I_D integral (50) can be obtained as

$$I_D = \frac{\partial^2}{\partial \beta_1 \partial \beta_2} D(\beta_1, \beta_2)\Big|_{\beta_1=1, \beta_2=1}. \tag{65}$$

To compute the integral (64), one uses the summation form of the kernels (38) instead of the integral representation (44). The corresponding integrals over x and y can then be separately evaluated by the formula (53) and explicit expressions of the polynomials $p_k(x)$ and $q_k(y)$ in Ref. [14]. Now taking the partial derivatives (65) gives

$$I_D = I_{D1} - I_{D2}, \tag{66}$$

where

$$I_{D1} = \lim_{\beta \to 1} \sum_{j=0}^{m-1} \sum_{k=0}^{m-1} \sum_{i=0}^{j} \sum_{s=0}^{j} (j + \alpha + 1)(k + \alpha + 1) g_{\alpha,i}\, g_{\alpha+1,s}, \tag{67}$$

$$I_{D2} = \sum_{j=0}^{m-1} \sum_{i=0}^{j} \sum_{s=0}^{j} \frac{2(i + \alpha + 1)(s + \alpha + 2) h_i h_s}{(j + \alpha + 1)^{-1}(i + s + 2\alpha + 4)} \bigg[\psi_0(i + \alpha + 2)\psi_0(s + \alpha + 3) + \frac{2}{(i + s + 2\alpha + 4)^2} - \frac{\psi_0(i + \alpha + 2) + \psi_0(s + \alpha + 3)}{i + s + 2\alpha + 4} \bigg], \tag{68}$$

with the shorthand notations

$$h_r = \frac{(-1)^r \Gamma(r + j + 2\alpha + 2)}{\Gamma(r + 1)\Gamma(j - r + 1)\Gamma(r + 2\alpha + 2)}, \tag{69}$$

$$g_{p,r} = \frac{\Gamma(r + \beta + 1)\Gamma(p + r + \beta + 1)\Gamma(r + 2\alpha + \beta + 2)}{\Gamma(r - k + \beta + 1)\Gamma(r + k + 2\alpha + \beta + 3)} \frac{2h_r}{\Gamma(p + r + 1)} [\psi_0(p + r + \beta + 1) + \psi_0(r + 2\alpha + \beta + 2) - \psi_0(r - k + \beta + 1) + \psi_0(r + \beta + 1) - \psi_0(r + k + 2\alpha + \beta + 3)]. \tag{70}$$

For the I_{D1} sums (67), the summation over j is evaluated first by the identity Lemma 4.1 in Ref. [14],

$$\sum_{j=i}^{m-1} (j + \alpha + 1) \frac{\Gamma(j + i + 2\alpha + 2)\Gamma(j + s + 2\alpha + 2)}{\Gamma(j - i + 1)\Gamma(j - s + 1)} = \frac{\Gamma(i + m + 2\alpha + 2)\Gamma(s + m + 2\alpha + 2)}{2(i + s + 2\alpha + 2)\Gamma(m - i)\Gamma(m - s)}. \tag{71}$$

After determining the limits when $\beta \to 1$, the summation over k is evaluated next by the identity

$$\sum_{k=0}^{m} (k + \alpha + 1) \frac{[\Gamma(s - k + 2)\Gamma(k + s + 2\alpha + 4)]^{-1}}{\Gamma(i - k + 2)\Gamma(k + i + 2\alpha + 4)} = \frac{[\Gamma(i + 2\alpha + 3)\Gamma(s + 2\alpha + 3)]^{-1}}{2\Gamma(i + 2)\Gamma(s + 2)(i + s + 2\alpha + 4)}, \tag{72}$$

as well as three additional identities obtained by taking the derivatives of Eq. (72) with respect to i, s, and both i and s. Now the I_{D1} quadruple sum (67) reduces to double sums in i and s. Similarly, for the I_{D2} sums (68), we evaluate the summation over j first by using Eq. (71), which also leads to a double sum form for I_{D2}. We observe substantial cancellations among the obtained double sums of I_{D1} and I_{D2}. With the remaining sums evaluated by the formulas in Appendix A, we arrive at a closed-form expression of I_D as shown in (73), where the coefficients are listed in Table III,

$$I_D = \frac{m}{8(2m + 2\alpha + 1)^4} \big\{ d_0 + d_1 \psi_0(m + \alpha + 1) + d_2 \psi_0(m + 2\alpha + 1) + d_3 \psi_0(2m + 2\alpha + 1) + d_4 [\psi_0(m + 2\alpha + 1) - \psi_0(2m + 2\alpha + 1)][\psi_0(m + \alpha + 1) + \psi_0(m + 2\alpha + 1) - \psi_0(2m + 2\alpha + 1)] + d_5 \psi_0^2(m + \alpha + 1) + d_6 [\psi_1(m + 2\alpha + 1) - \psi_1(2m + 2\alpha + 1)] \big\}. \tag{73}$$

Finally, inserting Eqs. (60), (63), and (73) into Eq. (46), one observes cancellations of all but three terms,

$$\mathbb{V}_h[T] = m(m + 2\alpha + 1)\psi_0(m + \alpha + 1) + \frac{m(m + 2\alpha + 1)}{2} \psi_0^2(m + \alpha + 1) + \frac{m(m + 2\alpha + 1)}{4(2m + 2\alpha + 1)} (5m^2 + 5m + 10\alpha m + 4\alpha^2 + 4\alpha + 2)\psi_1(m + \alpha + 1). \tag{74}$$

Specializing the above expression with the α value in Eq. (7) establishes the induced variance formula (33). This completes the proof of the main result (12).

III. SUMMARY AND OUTLOOK

As an important step towards quantifying the statistical performance of bipartite systems, we derived the exact variance of von Neumann entanglement entropy over the Bures-Hall measure in this work. The result is based on recent progress in understanding the correlation functions of the Bures-Hall random matrix ensemble.

Although the Bures-Hall ensemble attains a more involved functional form, the expressions of its first two moments turn out to be simpler than the ones over the Hilbert-Schmidt ensemble. Further understanding of this counterintuitive fact requires the higher order moments of the von Neumann entropy of both ensembles. The results may also help prove the conjectured Gaussian limit for large dimensional quantum systems. Future work may also include the study of other performance indicators relevant for quantum information processing, such as the fidelity and volumes, over the Bures-Hall measure.

ACKNOWLEDGMENTS

The author thanks Shi-Hao Li and Peter Forrester for correspondence. The author also wishes to thank Santosh Kumar for providing the simulation codes. This work is supported in part by the US National Science Foundation (Grant No. CNS-2006612).

APPENDIX A: LIST OF SUMMATION IDENTITIES

In this Appendix, we list the closed-form (A1)–(A5) and semi-closed-form (A6)–(A8) finite sum identities that are useful in simplifying the summations in Sec. II C. The identities (A1)–(A3) and (A4)–(A8) can be found in Refs. [26,33], respectively. Note that it is sufficient to assume $a, b \geqslant 0$, $a \neq b$ in Eqs. (A1)–(A4) and (A6), and $a > m$ in Eqs. (A7) and (A8).

$$\sum_{k=1}^{m}\psi_0(k+a) = (m+a)\psi_0(m+a+1) - a\psi_0(a+1) - m, \tag{A1}$$

$$\sum_{k=1}^{m}k\psi_0(k+a) = \frac{1}{2}(m^2+m-a^2+a)\psi_0(m+a+1) + \frac{1}{2}(a-1)a\psi_0(a+1) + \frac{1}{4}m(2a-m-3), \tag{A2}$$

$$\begin{aligned}\sum_{k=1}^{m}k^2\psi_0(k+a) =& \frac{1}{6}(2m^3+3m^2+m+2a^3-3a^2+a)\psi_0(m+a+1) - \frac{1}{6}a(2a^2-3a+1)\psi_0(a+1)\\ &- \frac{1}{36}m(4m^2+15m-6ma+12a^2-24a+17),\end{aligned} \tag{A3}$$

$$\sum_{k=1}^{m}\frac{\psi_0(k+a)}{k+a} = \frac{1}{2}\big[-\psi_0^2(a+1) + \psi_0^2(m+a+1) - \psi_1(a+1) + \psi_1(m+a+1)\big], \tag{A4}$$

$$\sum_{k=1}^{m}\frac{\psi_0(m+1-k)}{k} = -\psi_0(1)\psi_0(m+1) + \psi_0^2(m+1) - \psi_1(1) + \psi_1(m+1), \tag{A5}$$

$$\begin{aligned}\sum_{k=1}^{m}\frac{\psi_0(k+b)}{k+a} =& -\sum_{k=1}^{m}\frac{\psi_0(k+a)}{k+b} + \psi_0(m+a+1)\psi_0(m+b+1) - \psi_0(a+1)\psi_0(b+1)\\ &+ \frac{1}{a-b}[\psi_0(m+a+1) - \psi_0(m+b+1) - \psi_0(a+1) + \psi_0(b+1)],\end{aligned} \tag{A6}$$

$$\sum_{k=1}^{m}\frac{\psi_0(k)}{a+1-k} = \sum_{k=1}^{m}\frac{\psi_0(k)}{k+a-m} + \frac{1}{2}[\psi_0(a-m+1) - \psi_0(a+1)]^2 - \frac{1}{2}[\psi_1(a-m+1) - \psi_1(a+1)], \tag{A7}$$

$$\begin{aligned}\sum_{k=1}^{m}\frac{\psi_0(a+1-k)}{k} =& -\sum_{k=1}^{m}\frac{\psi_0(k+a-m)}{k} + [\psi_0(m+1) - \psi_0(1)][\psi_0(a-m) + \psi(a+1)]\\ &+ \frac{1}{2}\{[\psi_0(a-m) - \psi_0(a+1)]^2 + \psi_1(a+1) - \psi_1(a-m)\}.\end{aligned} \tag{A8}$$

APPENDIX B: COEFFICIENT LISTS OF I_A, I_B, I_C, AND I_D

TABLE I. Coefficients of I_A in Eq. (60).

$$a_0 = -18\alpha m(m+\alpha)(m+2\alpha)(m+2\alpha+1)(2m+2\alpha+1)^2(5m^2+10\alpha m+5m+4\alpha^2+4\alpha+2)$$

$$a_1 = -\alpha m(m+\alpha)(m+2\alpha)(1756m^5+8760\alpha m^4+4464m^4+15900\alpha^2m^3+16152\alpha m^3+3941m^3+12736\alpha^3m^2+19320\alpha^2m^2 +9288\alpha m^2+1370m^2+4032\alpha^4m+8112\alpha^3m+5604\alpha^2m+1500\alpha m+147m+192\alpha^5+480\alpha^4+320\alpha^3-60\alpha-14)$$

$$a_2 = -18m(m+\alpha)(m+2\alpha+1)(2m+2\alpha+1)^2(3m+4\alpha)(5m^2+10\alpha m+5m+4\alpha^2+4\alpha+2)$$

$$a_3 = 12(m+2\alpha)(2m+2\alpha+1)^2(15m^5-30\alpha^2m^4+60\alpha m^4+30m^4-120\alpha^3m^3+27\alpha^2m^3+72\alpha m^3+21m^3-154\alpha^4m^2 -75\alpha^3m^2+19\alpha^2m^2+24\alpha m^2+6m^2-68\alpha^5m-50\alpha^4m-16\alpha^3m-\alpha^2m-4\alpha^6+4\alpha^5+\alpha^4-\alpha^3)$$

$$a_4 = 6(m+\alpha)(m+2\alpha)(2m+2\alpha+1)^2(15m^4-120\alpha^2m^3+30\alpha m^3+30m^3-360\alpha^3m^2-228\alpha^2m^2+12\alpha m^2+21m^2 -256\alpha^4m-336\alpha^3m-176\alpha^2m-18\alpha m+6m-16\alpha^5-32\alpha^4-68\alpha^3-52\alpha^2-12\alpha)$$

$$a_5 = 6(m+2\alpha)(2m+2\alpha+1)(106\alpha m^6-60m^6+636\alpha^2m^5-65\alpha m^5-150m^5+1506\alpha^3m^4+503\alpha^2m^4-324\alpha m^4-144m^4 +1784\alpha^4m^3+1300\alpha^3m^3+60\alpha^2m^3-235\alpha m^3-66m^3+1080\alpha^5m^2+1120\alpha^4m^2+468\alpha^3m^2+15\alpha^2m^2-58\alpha m^2 -12m^2+288\alpha^6m+320\alpha^5m+216\alpha^4m+96\alpha^3m+16\alpha^2m+16\alpha^7-8\alpha^6-12\alpha^5+2\alpha^4+2\alpha^3)$$

$$a_6 = 6(m+2\alpha+1)(2m+2\alpha+1)(212\alpha m^6-30m^6+1512\alpha^2m^5+198\alpha m^5-45m^5+4212\alpha^3m^4+1820\alpha^2m^4+111\alpha m^4 -27m^4+5760\alpha^4m^3+4344\alpha^3m^3+1238\alpha^2m^3+31\alpha m^3-6m^3+3936\alpha^5m^2+4304\alpha^4m^2+2296\alpha^3m^2+440\alpha^2m^2 +6\alpha m^2+1152\alpha^6m+1680\alpha^5m+1480\alpha^4m+584\alpha^3m+72\alpha^2m+64\alpha^7+128\alpha^6+272\alpha^5+208\alpha^4+48\alpha^3)$$

$$a_7 = -12\alpha(212m^8+1908\alpha m^7+636m^7+7252\alpha^2m^6+4920\alpha m^6+823m^6+15148\alpha^3m^5+15660\alpha^2m^5+5545\alpha m^5+601m^5 +18888\alpha^4m^4+26400\alpha^3m^4+14738\alpha^2m^4+3552\alpha m^4+255m^4+14192\alpha^5m^3+25104\alpha^4m^3+19608\alpha^3m^3+7724\alpha^2m^3 +1329\alpha m^3+59m^3+6080\alpha^6m^2+13056\alpha^5m^2+13480\alpha^4m^2+7696\alpha^3m^2+2250\alpha^2m^2+272\alpha m^2+6m^2+1248\alpha^7m +3168\alpha^6m+4304\alpha^5m+3432\alpha^4m+1494\alpha^3m+316\alpha^2m+24\alpha m+64\alpha^8+192\alpha^7+416\alpha^6+512\alpha^5+324\alpha^4+100\alpha^3+12\alpha^2)$$

TABLE II. Coefficients of I_B+I_C in Eq. (63).

$$b_0+c_0 = 18\alpha m(m+\alpha)(m+\alpha+1)(m+2\alpha)(m+2\alpha+1)(2m+2\alpha+1)^3(5m^2+10\alpha m+5m+4\alpha^2+4\alpha+2)$$

$$b_1+c_1 = -2\alpha m(m+\alpha)(m+\alpha+1)(m+2\alpha)(1756m^6+10516\alpha m^5+5504m^5+24660\alpha^2m^4+25608\alpha m^4+6479m^4+28636\alpha^3m^3 +44304\alpha^2m^3+22151\alpha m^3+3480m^3+16768\alpha^4m^2+34376\alpha^3m^2+25380\alpha^2m^2+7802\alpha m^2+805m^2+4224\alpha^5m +10752\alpha^4m+10268\alpha^3m+4464\alpha^2m+819\alpha m+37m+192\alpha^6+576\alpha^5+560\alpha^4+160\alpha^3-60\alpha^2-44\alpha-7)$$

$$b_2+c_2 = -18m(m+\alpha)(m+\alpha+1)(m+2\alpha+1)(2m+2\alpha+1)^3(3m+4\alpha)(5m^2+10\alpha m+5m+4\alpha^2+4\alpha+2)$$

$$b_3+c_3 = 12(m+\alpha+1)(m+2\alpha)(2m+2\alpha+1)^3(15m^5-30\alpha^2m^4+60\alpha m^4+30m^4-120\alpha^3m^3+27\alpha^2m^3+72\alpha m^3+21m^3 -154\alpha^4m^2-75\alpha^3m^2+19\alpha^2m^2+24\alpha m^2+6m^2-68\alpha^5m-50\alpha^4m-16\alpha^3m-\alpha^2m-4\alpha^6+4\alpha^5+\alpha^4-\alpha^3)$$

$$b_4+c_4 = 6(m+\alpha)(m+\alpha+1)(m+2\alpha)(2m+2\alpha+1)^3(15m^4-120\alpha^2m^3+30\alpha m^3+30m^3-360\alpha^3m^2-228\alpha^2m^2 +12\alpha m^2+21m^2-256\alpha^4m-336\alpha^3m-176\alpha^2m-18\alpha m+6m-16\alpha^5-32\alpha^4-68\alpha^3-52\alpha^2-12\alpha)$$

$$b_5+c_5 = 12(m+\alpha+1)(m+2\alpha)(2m+2\alpha+1)(106\alpha m^7-60m^7+742\alpha^2m^6-141\alpha m^6-180m^6+2142\alpha^3m^5+378\alpha^2m^5 -670\alpha m^5-219m^5+3290\alpha^4m^4+1743\alpha^3m^4-698\alpha^2m^4-679\alpha m^4-138m^4+2864\alpha^5m^3+2448\alpha^4m^3 +104\alpha^3m^3-613\alpha^2m^3-291\alpha m^3-45m^3+1368\alpha^6m^2+1524\alpha^5m^2+500\alpha^4m^2-84\alpha^3m^2-132\alpha^2m^2-47\alpha m^2 -6m^2+304\alpha^7m+360\alpha^6m+172\alpha^5m+62\alpha^4m+18\alpha^3m+2\alpha^2m+16\alpha^8-16\alpha^6-4\alpha^5+3\alpha^4+\alpha^3)$$

$$b_6+c_6 = 6(2m+2\alpha+1)(424\alpha m^9-60m^9+4720\alpha^2m^8+1036\alpha m^8-240m^8+22640\alpha^3m^7+13764\alpha^2m^7+716\alpha m^7 -399m^7+61184\alpha^4m^6+62444\alpha^3m^6+17280\alpha^2m^6-80\alpha m^6-357m^6+102120\alpha^5m^5+148488\alpha^4m^5+74852\alpha^3m^5 +13467\alpha^2m^5-265\alpha m^5-183m^5+108240\alpha^6m^4+206232\alpha^5m^4+152936\alpha^4m^4+53828\alpha^3m^4+7650\alpha^2m^4 -74\alpha m^4-51m^4+71744\alpha^7m^3+170208\alpha^6m^3+169160\alpha^5m^3+90024\alpha^4m^3+25492\alpha^3m^3+3057\alpha^2m^3+13\alpha m^3 -6m^3+27776\alpha^8m^2+79328\alpha^7m^2+100928\alpha^6m^2+74144\alpha^5m^2+32140\alpha^4m^2+7480\alpha^3m^2+722\alpha^2m^2+6\alpha m^2 +5248\alpha^9m+17664\alpha^8m+28608\alpha^7m+28512\alpha^6m+17544\alpha^5m+6216\alpha^4m+1112\alpha^3m+72\alpha^2m+256\alpha^{10} +1024\alpha^9+2432\alpha^8+3712\alpha^7+3344\alpha^6+1696\alpha^5+448\alpha^4+48\alpha^3)$$

$$b_7+c_7 = -12\alpha(2m+2\alpha+1)(212m^9+2120\alpha m^8+758m^8+9160\alpha^2m^7+6726\alpha m^7+1162m^7+22400\alpha^3m^6+25294\alpha^2m^6 +9200\alpha m^6+1049m^6+34036\alpha^4m^5+52450\alpha^3m^5+29970\alpha^2m^5+7475\alpha m^5+631m^5+33080\alpha^5m^4+65124\alpha^4m^4 +51812\alpha^3m^4+20908\alpha^2m^4+4038\alpha m^4+251m^4+20272\alpha^6m^3+48896\alpha^5m^3+50800\alpha^4m^3+29310\alpha^3m^3 +9392\alpha^2m^3+1417\alpha m^3+59m^3+7328\alpha^7m^2+21056\alpha^6m^2+27624\alpha^5m^2+21476\alpha^4m^2+10018\alpha^3m^2+2550\alpha^2m^2 +284\alpha m^2+6m^2+1312\alpha^8m+4416\alpha^7m+7312\alpha^6m+7576\alpha^5m+4866\alpha^4m+1802\alpha^3m+340\alpha^2m+24\alpha m +64\alpha^9+256\alpha^8+608\alpha^7+928\alpha^6+836\alpha^5+424\alpha^4+112\alpha^3+12\alpha^2)$$

$$b_8+c_8 = -18\alpha m(m+\alpha)(m+\alpha+1)(m+2\alpha)(m+2\alpha+1)(2m+2\alpha+1)^2(7m^2+14\alpha m+7m+8\alpha^2+8\alpha+2)$$

$$b_9+c_9 = 36\alpha m(m+\alpha)(m+\alpha+1)(m+2\alpha)(m+2\alpha+1)(2m+2\alpha+1)^2(10m^3+30\alpha m^2+9m^2+28\alpha^2m+8\alpha^3+4\alpha^2+16\alpha m+3m)$$

$$b_{10}+c_{10} = -72\alpha m(m+\alpha)(m+\alpha+1)(m+2\alpha)(m+2\alpha+1)(2m+2\alpha+1)^2(10m^3+30\alpha m^2+12m^2+28\alpha^2m+22\alpha m +6m+8\alpha^3+8\alpha^2+4\alpha+1)$$

$$b_{11}+c_{11} = 18\alpha m(m+\alpha)(m+\alpha+1)(m+2\alpha)(m+2\alpha+1)(2m+2\alpha+1)^2(10m^3+30\alpha m^2+3m^2+28\alpha^2m+4\alpha m-3m +8\alpha^3-4\alpha^2-8\alpha-2)$$

TABLE III. Coefficients of I_D in Eq. (73).

$d_0 = m(36m^4 + 136\alpha m^3 + 68m^3 + 196\alpha^2m^2 + 188\alpha m^2 + 31m^2 + 128\alpha^3m + 184\alpha^2m + 64\alpha m - 6m + 32\alpha^4 + 64\alpha^3 + 36\alpha^2 - 4\alpha - 5)$

$d_1 = 2m(2m + 2\alpha + 1)(14m^3 + 46\alpha m^2 + 29m^2 + 48\alpha^2m + 60\alpha m + 20m + 16\alpha^3 + 32\alpha^2 + 22\alpha + 5)$

$d_2 = 4(2m + 2\alpha + 1)(32\alpha^4 + 64\alpha^3 + 48\alpha^2 + 16\alpha + 30m^4 + 126\alpha m^3 + 69m^3 + 192\alpha^2m^2 + 204\alpha m^2 + 56m^2 + 128\alpha^3m + 200\alpha^2m + 106\alpha m + 19m + 2)$

$d_3 = -4(60m^5 + 312\alpha m^4 + 168m^4 + 636\alpha^2m^3 + 672\alpha m^3 + 181m^3 + 640\alpha^3m^2 + 1000\alpha^2m^2 + 528\alpha m^2 + 94m^2 + 320\alpha^4m + 656\alpha^3m + 508\alpha^2m + 176\alpha m + 23m + 64\alpha^5 + 160\alpha^4 + 160\alpha^3 + 80\alpha^2 + 20\alpha + 2)$

$d_4 = 8(m + 2\alpha + 1)(2m + 2\alpha + 1)^2(3m^2 + 6\alpha m + 3m + 4\alpha^2 + 4\alpha + 1)$

$d_5 = -2m(m + 2\alpha + 1)^2(2m + 2\alpha + 1)^2$

$d_6 = 4(m + 2\alpha + 1)(2m + 2\alpha + 1)^3(5m^2 + 10\alpha m + 5m + 4\alpha^2 + 4\alpha + 2)$

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