Performance bounds of nonadiabatic quantum harmonic Otto engine and refrigerator under a squeezed thermal reservoir

Varinder Singh * and Özgür E. Müstecaplıoğlu †
Department of Physics, Koç University, 34450 Sarıyer, Istanbul, Turkey

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We analyze the performance of a quantum Otto cycle, employing a time-dependent harmonic oscillator as the working fluid undergoing sudden expansion and compression strokes during the adiabatic stages, coupled to a squeezed reservoir. First, we show that the maximum efficiency that our engine can achieve is $1/2$ only, which is in contrast with earlier studies claiming unit efficiency under the effect of a squeezed reservoir. Then, in the high-temperature limit, we obtain analytic expressions for the upper bound on the efficiency as well as on the coefficient of performance of the Otto cycle. The obtained bounds are independent of the parameters of the system and depend on the reservoir parameters only. Additionally, with a hot squeezed thermal bath, we obtain an analytic expression for the efficiency at maximum work which satisfies the derived upper bound. Further, in the presence of squeezing in the cold reservoir, we specify an operational regime for the Otto refrigerator otherwise forbidden in the standard case. Finally, we find the cost of creating a squeezed state from the thermal state and show that in order to harvest the benefits of squeezing, it is sufficient to squeeze only one mode of the reservoir in resonance with the transition frequency of the working fluid. Further, we show that when the cost of squeezing is included in the definition of the operational efficiency of the engine, the advantages of squeezing fade away. Still, being purely quantum mechanical fuel in nature, squeezed reservoirs are beneficial in their own way by providing us with more compact energy storage medium or offering effectively high-temperature baths without being actually too hot.

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I. INTRODUCTION

The concept of Carnot efficiency (η_C) is one of the most important results in physics, which led to the formulation of the second law of thermodynamics [1]. It puts a theoretical upper bound on the efficiency of all macroscopic heat engines working between two thermal reservoirs at different temperatures. However, with the rise of quantum thermodynamics [2–5], many studies have showed that this sacred bound may be surpassed by quantum heat machines exploiting exotic quantum resources such as quantum coherence [6–10], quantum correlations [11–15], and squeezed reservoirs [16–27], among others [28]. In such cases, the second law of thermodynamics has to be modified to account for the quantum effects, and the notion of generalized Carnot bound is introduced, which is always satisfied [11,16,29,30]. In this context, different theoretical studies have been carried out to study the implications of work extraction when quantum heat machines are coupled to nonequilibrium stationary reservoirs [30–34]. In particular, it is instructive to look into the working of heat machines coupled to squeezed thermal reservoirs. The use of a squeezed thermal reservoir allows us to extract work from a single reservoir [20], operate thermal devices beyond the Carnot bound [16,20,22,23], and define multiple operational regimes [20,31] otherwise impossible for the standard case with two thermal reservoirs. Moreover, in Ref. [35], the idea of treating a squeezed thermal reservoir as a generalized equilibrium reservoir is explored. Recently, a nanomechanical engine consisting of a vibrating nanobeam coupled to squeezed thermal noise, operating beyond the standard Carnot efficiency, has been realized experimentally [23].

Over the past few years, there has been increasing interest in investigating the performance of a quantum Otto cycle [36–42], based on a time-dependent harmonic oscillator as the working fluid, coupled to squeezed thermal baths [16,20–23]. Due to its simplicity, a harmonic quantum Otto cycle (HQOC) serves as a paradigm model for quantum thermal devices. It consists of two adiabatic branches during which the frequency of the oscillator is varied, and two isochoric branches during which the system exchanges heat with the thermal baths at constant frequency. Roßnagel and coauthors optimized the work output of a HQOC in the presence of a hot squeezed thermal bath and obtained a generalized version of Curzon-Ahlborn efficiency [16]. Manzano *et al.* studied a modified version of HQOC and discussed the effect of a squeezed hot bath in different operational regimes [20]. Extending the analysis to quantum refrigerators, Long and Liu optimized the performance of a HQOC in contact with a low-temperature squeezed thermal bath and concluded that the coefficient of performance (COP) can be enhanced by squeezing [22].

With the exception of Refs. [21,25], all the above-mentioned studies involving squeezed reservoirs are confined to the study a of quasistatic Otto cycle in which adiabatic steps are performed quasistatically, thus producing vanishing power

*vsingh@ku.edu.tr
†omustecap@ku.edu.tr

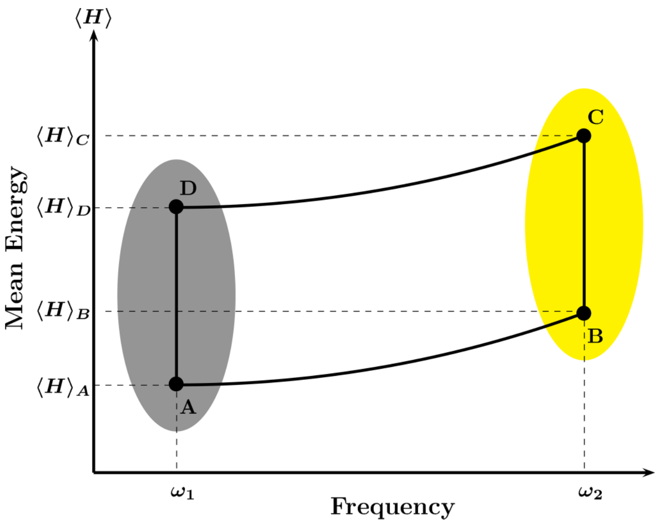

FIG. 1. Model of quantum Otto cycle employing a time-dependent harmonic oscillator as the working fluid.

output. In this work, we fill this gap by confining our focus to the highly nonadiabatic (dissipative) regime corresponding to the sudden switch of frequencies (sudden compression and expansion strokes) during the adiabatic stages of the Otto cycle. We obtain analytic expressions for the upper bounds on the efficiency and COP of the HQOC coupled to a squeezed thermal reservoir. We also calculate the cost of preparing a squeezed state by squeezing a thermal state and show that to exploit the advantages of a squeezed reservoir, it is sufficient to squeeze just one mode of the thermal reservoir in resonance with the frequency of the working fluid. Then, by including this cost into the definition of the operational efficiency of the engine, we analyze the behavior of the operational efficiency and net work extracted for a different amount of squeezing injected into the system.

The paper is organized as follows. In Sec. II we discuss the HQOC model coupled to a hot squeezed thermal reservoir. In Sec. III we obtain analytic expression for the upper bound on the efficiency of the engine operating in the sudden switch limit. We also obtain an analytic expression for the efficiency at maximum work and compare it with the derived upper bound. In Sec. IV we repeat our analysis for the Otto refrigerator coupled to a cold squeezed reservoir and obtain upper bound on the COP of the refrigerator. In Sec. V we calculate the cost of squeezing. We conclude in Sec. VI.

II. QUANTUM OTTO CYCLE WITH SQUEEZED RESERVOIR

We consider a quantum Otto cycle of a time-dependent harmonic oscillator coupled to a hot squeezed thermal bath while the cold bath is still purely thermal in nature. It consists of four stages: two adiabatic and two isochoric (see Fig. 1). These processes occur in the following order [39,43]: (1) Adiabatic compression $A \longrightarrow B$: To begin, the system is at inverse temperature β_1. The system is isolated and the frequency of the oscillator is increased from ω_1 to ω_2. Work is done on the system in this stage. The evolution is unitary, and von Neumann entropy of the system remains constant. (2) Hot isochore $B \longrightarrow C$: During this stage, the oscillator is coupled to the squeezed thermal heat reservoir at inverse temperature β_2 at fixed frequency (ω_2) and allowed to thermalize. No work is done in this stage; only heat exchange between the system and reservoir takes place. After the completion of the hot isochoric stage, the system relaxes to a squeezed thermal state [44,45] with mean photon number $\langle n(\beta_2, r)\rangle = \langle n\rangle + (2\langle n\rangle + 1)\sinh^2 r$, where r is the squeezing parameter and $\langle n\rangle = 1/(e^{\beta_2\omega_2} - 1)$ is the thermal occupation number (we have set $\hbar = k_B = 1$ for simplicity). (3) Adiabatic expansion $C \longrightarrow D$: The system is isolated, and the frequency of the oscillator is unitarily decreased back to its initial value ω_1. Work is done by the system in this stage. (4) Cold isochore $D \longrightarrow A$: To bring back the working fluid to its initial state, the system is coupled to the cold reservoir at inverse temperature β_1 ($\beta_1 > \beta_2$) and allowed to relax back to the initial thermal state A.

The average energies, $\langle H\rangle = (\langle n(\beta_i, r)\rangle + 1/2)\omega_i$ (where $i = 1, 2$ and $r = 0$ for $i = 1$), of the oscillator at the four stages of the cycle read as follows [16]:

$$\langle H\rangle_A = \frac{\omega_1}{2}\coth\left(\frac{\beta_1\omega_1}{2}\right), \tag{1}$$

$$\langle H\rangle_B = \frac{\omega_2}{2}\lambda\coth\left(\frac{\beta_1\omega_1}{2}\right), \tag{2}$$

$$\langle H_C\rangle = \frac{\omega_2}{2}\coth\left(\frac{\beta_2\omega_2}{2}\right)\Delta H(r), \tag{3}$$

$$\langle H_D\rangle = \frac{\omega_1}{2}\lambda\coth\left(\frac{\beta_2\omega_2}{2}\right)\Delta H(r), \tag{4}$$

where $\Delta H(r) = \cosh(2r)$ reflects the effect of the squeezed hot thermal bath on the mean energy of the oscillator, and λ is the dimensionless adiabaticity parameter [46]. For the adiabatic process, $\lambda = 1$; for nonadiabatic expansion and compression strokes, $\lambda > 1$. The expression for mean heat exchanged during the hot and cold isochores can be evaluated, respectively, as follows:

$$\begin{aligned}\langle Q_2\rangle &= \langle H\rangle_C - \langle H\rangle_B\\ &= \frac{\omega_2}{2}\left[\Delta H(r)\coth\left(\frac{\beta_2\omega_2}{2}\right) - \lambda\coth\left(\frac{\beta_1\omega_1}{2}\right)\right],\end{aligned} \tag{5}$$

$$\begin{aligned}\langle Q_4\rangle &= \langle H\rangle_A - \langle H\rangle_D\\ &= \frac{\omega_1}{2}\left[\coth\left(\frac{\beta_1\omega_1}{2}\right) - \lambda\Delta H(r)\coth\left(\frac{\beta_2\omega_2}{2}\right)\right].\end{aligned} \tag{6}$$

Here we are employing a sign convention in which heat absorbed (rejected) from (to) the reservoir is positive (negative). Since after one complete cycle the working fluid comes back to its initial state, the net work done on the system in a cycle is given by the first law of thermodynamics, $W_{\rm total} = -(\langle Q_2\rangle + \langle Q_4\rangle)$. Work is said to be extracted from the engine when $W_{\rm ext} = -W_{\rm total} = \langle Q_2\rangle + \langle Q_4\rangle > 0$. In this work, we are interested in the sudden switch case for which $\lambda = (\omega_1^2 + \omega_2^2)/2\omega_1\omega_2$ [46–48]. Substituting the above expression for λ in Eqs. (5) and (6), we obtain the following expressions for the extracted work, $\langle W_{\rm ext}\rangle$, and efficiency, $\eta = \langle W_{\rm ext}\rangle/\langle Q_2\rangle$, of the engine, respectively:

$$\langle W_{\rm ext}\rangle = \langle Q_2\rangle + \langle Q_4\rangle = \frac{\omega_2^2-\omega_1^2}{4\omega_1\omega_2}\left[\omega_1\Delta H(r)\coth\left(\frac{\beta_2\omega_2}{2}\right) - \omega_2\coth\left(\frac{\beta_1\omega_1}{2}\right)\right], \tag{7}$$

$$\eta = \frac{\langle W_{\rm ext}\rangle}{\langle Q_2\rangle} = \left[\frac{2}{1-\frac{\omega_1^2}{\omega_2^2}} + \frac{1}{\frac{\omega_1}{\omega_2}\Delta H(r)\coth\left(\frac{\beta_2\omega_2}{2}\right)\tanh\left(\frac{\beta_1\omega_1}{2}\right)-1}\right]^{-1} \equiv \left(\frac{2}{\Delta_1}+\frac{1}{\Delta_2}\right)^{-1}. \tag{8}$$

As ω_1 and ω_2 are always positive by construction and $\omega_2 > \omega_1$, $\Delta_1 = 1-\omega_1^2/\omega_2^2$ lies in the range (0,1), i.e., $0 < \Delta_1 < 1$. This result, together with the positive work condition (PWC), $\langle W_{\rm ext}\rangle > 0$ [see Eq. (7)], implies that $\Delta_2 \equiv \frac{\omega_1}{\omega_2}\Delta H(r)\coth\left(\frac{\beta_2\omega_2}{2}\right)\tanh\left(\frac{\beta_1\omega_1}{2}\right) - 1 > 0$ or $1/\Delta_2 > 0$. Consequently, from Eq. (8), we have

$$\eta < \frac{\Delta_1}{2} \quad \text{and} \quad \eta < \Delta_2. \tag{9}$$

Again using the condition $0 < \Delta_1 < 1$ in Eq. (9), we arrive at our first main result:

$$\eta < \tfrac{1}{2}. \tag{10}$$

The result is very interesting as it implies that even in the presence of very very large squeezing ($r \to \infty$), the efficiency of the engine can never surpass 1/2. This is in contrast with the previous studies, valid for the quasistatic regime, implying that the thermal engine fueled by a hot squeezed thermal reservoir asymptotically attains unit efficiency for a large squeezing parameter ($r \gg 1$) [16,20,29]. We attribute this to the highly frictional nature of the sudden switch regime as explained below. In the sudden switch regime, the sudden quench of the frequency of the harmonic oscillator induces nonadiabatic transitions between its energy levels and leaves the system in a nonequilibrium state. When written in terms of the energy eigenkets of the instantaneous Hamiltonian, the off-diagonal terms of the density matrix, known as coherences, are nonzero. Generating coherences give rise to extra energetic cost when compared to adiabatic driving, and an additional parasitic internal energy is stored in the working medium. This extra cost gets dissipated to the heat reservoirs during the proceeding isochoric stages of the cycle and is termed as quantum friction [18,49–53]. Inner friction limits the performance of the device under consideration.

III. UPPER BOUND ON THE EFFICIENCY

In order to obtain analytic expression in closed form for the efficiency, we will work in the high-temperature regime [54–56]. In this regime, we set $\coth(\beta_i\omega_i/2) \approx 2/(\beta_i\omega_i)$ $(i = 1, 2)$. Then the expressions for the extracted work $\langle W_{\rm ext}\rangle$ [Eq. (7)] and the efficiency [Eq. (8)] take the following forms:

$$\langle W_{\rm ext}\rangle = \frac{(1-z^2)[z^2\cosh(2r)-\tau]}{2z^2\beta_2}, \tag{11}$$

$$\eta = \frac{(z^2-1)[z^2\cosh(2r)-\tau]}{\tau - z^2[2\cosh(2r)-\tau]}, \tag{12}$$

where we have defined $z = \omega_1/\omega_2$ and $\tau = \beta_2/\beta_1 = 1-\eta_C$. From Eq. (11), the positive work condition, $\langle W_{\rm ext}\rangle > 0$, implies that

$$z^2\cosh(2r) > 1-\eta_C. \tag{13}$$

Using the expression for efficiency in Eq. (12), z^2 can be written in terms of η and η_C and is given by

$$z^2 = \tfrac{1}{2}\{(1-\eta_c)(1+\eta) + (1-2\eta)\cosh(2r) - \sqrt{[(1-\eta_c)(1+\eta)+(1-2\eta)\cosh(2r)]^2 - 4(1-\eta_c)(1+\eta)\cosh(2r)}\}. \tag{14}$$

Using the above expression for z in Eq. (13), we obtain following upper bound on the efficiency of the engine:

$$\eta < \frac{[1-\eta_C-\cosh(2r)][-1+\eta_C-2\cosh(2r)+2\sqrt{2(1-\eta_C)\cosh(2r)}]}{[1-\eta_C-2\cosh(2r)]^2} \equiv \eta_{\rm up}. \tag{15}$$

This is our second main result. Notice that the above derived bound is independent of the parameters of the model under consideration and depends on the reservoir parameters r and η_C (or τ) only. For $r\to\infty$, $\eta_{\rm up} \to 1/2$, which reconfirms our earlier result [Eq. (9)] that the maximum efficiency that our engine can attain is one-half the unit efficiency; it never reaches unit efficiency, unlike the engines operating in the quasistatic regime [16,20,29].

Further, we derive analytic expression for the efficiency at maximum work by optimizing Eq. (11) with respect to z, and it is given by

$$\eta_{\rm MW} = \frac{1-\sqrt{(1-\eta_C){\rm sech}(2r)}}{2+\sqrt{(1-\eta_C){\rm sech}(2r)}}. \tag{16}$$

We have plotted Eqs. (15) and (16) in Fig. 2 as a function of r for different fixed values of Carnot efficiency η_C. For the given values of η_C smaller than 1/2, both $\eta_{\rm up}$ (solid red and blue curves) and $\eta_{\rm MW}$ (dashed red and blue curves) can surpass corresponding Carnot efficiency (dotted curves with

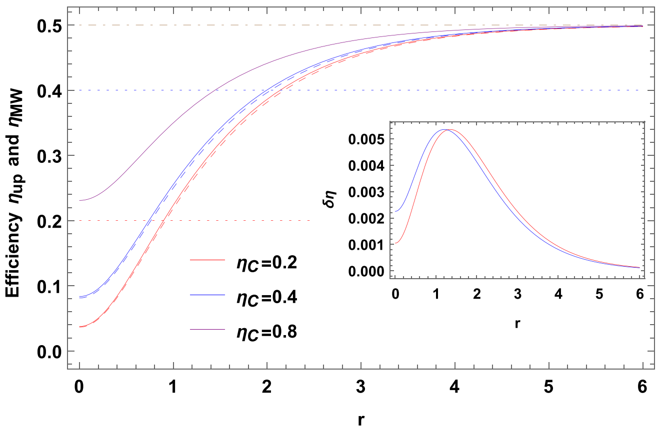

FIG. 2. Plots of $\eta_{\rm up}$ [Eq. (15)] and $\eta_{\rm MW}$ [Eq. (16)] as a function of squeezing parameter r. Solid red and blue curves represent $\eta_{\rm up}$ for $\eta_C = 0.2$ and $\eta_C = 0.4$, respectively. Dashed curves in the corresponding color represent $\eta_{\rm MW}$. Dotted red and blue curves denote the standard Carnot efficiency at values $\eta_C = 0.2$ and $\eta_C = 0.4$, respectively. Solid purple curve represents $\eta_{\rm up}$ for $\eta_C = 0.8$ and shows that for the given value of $\eta_C > 1/2$, $\eta_{\rm up}$ can never surpass Carnot efficiency η_C even in the presence of very large squeezing. In the inset, we have plotted the difference between $\eta_{\rm up}$ and $\eta_{\rm MW}$ ($\delta\eta = \eta_{\rm up} - \eta_{\rm MW}$) as a function of r for $\eta_C = 0.2$ (solid red curve) and $\eta_C = 0.4$ (solid blue curve). This difference is always positive, indicating that $\eta_{\rm up} > \eta_{\rm MW}$.

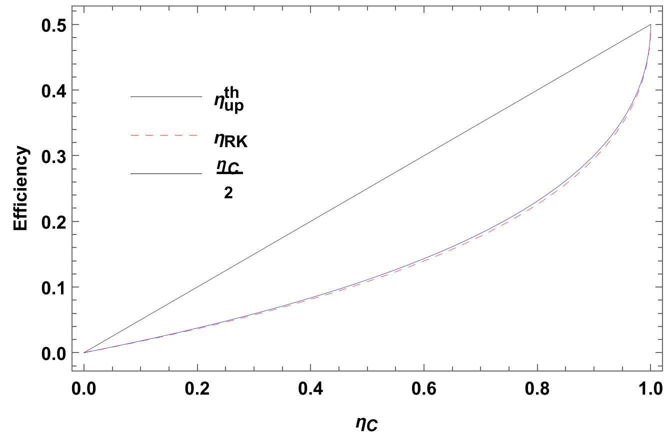

FIG. 3. Plots of $\eta^{\rm th}_{\rm up}$ [Eq. (20)] and $\eta_{\rm RK}$ [Eq. (21)] versus Carnot efficiency. We can see that $\eta_{\rm RK}$ (dashed red curve) lies below $\eta^{\rm th}_{\rm up}$ (solid blue curve). Both are bounded above by half the Carnot efficiency, $\eta_C/2$.

same color) for some value of squeezing parameter r and approach $1/2$ for relatively larger values of r ($r > 5$). From the inset of Fig. 2, it is clear that $\eta_{\rm MW}$ always lies below $\eta_{\rm up}$, which should be the case as for the given temperature ratio (η_C), $\eta_{\rm up}$ is the upper bound on the efficiency.

One more comment is in order here. Although, for given values of η_C ($\eta_C < 1/2$), $\eta_{\rm up}$ and $\eta_{\rm MW}$ may surpass standard Carnot efficiency, they can never surpass generalized Carnot efficiency (not shown in Fig. 2) [16,33],

$$\eta_C^{\rm gen} = 1 - \frac{\beta_2}{\beta_1\cosh(2r)} \equiv 1 - \frac{T_1}{T_2\cosh(2r)}, \tag{17}$$

which follows from the second law of thermodynamics applied to the nonequilibrium reservoirs [30]. The concept of generalized Carnot efficiency can be understood as follows. We can always assign a frequency dependent local temperature (or apparent temperature [57]) to a squeezed thermal reservoir characterized by its genuine temperature T and squeezing parameter r [32,33]. The expression for this frequency dependent local temperature can be obtained from the following relation [32,33]:

$$\exp\left(-\frac{\omega}{T(\omega,r)}\right) = \frac{\langle n\rangle + (2\langle n\rangle+1)\sinh^2 r}{1+\langle n\rangle + (2\langle n\rangle+1)\sinh^2 r}. \tag{18}$$

In the high-temperature limit, the effective temperature of the squeezed hot bath reads as

$$T_2^{\rm eff}(r) = T_2(1+2\sinh^2 r) = T_2\cosh(2r). \tag{19}$$

Hence, for positive values of r, the engine may be assumed to be operating between temperatures T_1 and $T_2^{\rm eff}(r)$. The actual (generalized) Carnot efficiency should then be given by Eq. (17).

Finally, we discuss the special case when $r \to 0$. This corresponds to the case in which our harmonic quantum engine is working between two purely thermal reservoirs. Thus, for $r \to 0$, Eqs. (15) and (16) reduce to the following forms, respectively:

$$\eta < \frac{[3 - 2\sqrt{2(1-\eta_C)} - \eta_C]\eta_C}{(1+\eta_C)^2} \equiv \eta^{\rm th}_{\rm up}, \tag{20}$$

$$\eta_{\rm RK} = \frac{1-\sqrt{1-\eta_C}}{2+\sqrt{1-\eta_C}}. \tag{21}$$

The above bound, $\eta^{\rm th}_{\rm up}$, is much tighter than the classical Carnot bound, even tighter than $\eta_C/2$ (see Fig. 3) [58]. Equation (21), which we derived as a special case of our more general result Eq. (15), was first derived by Rezek and Kosloff (RK) for the optimization of a harmonic quantum Otto engine undergoing sudden switch of frequencies in the adiabatic stages [38]. Again, it is clear from Fig. 3 that $\eta_{\rm RK}$ (dashed red curve) always lies below $\eta^{\rm th}_{\rm up}$ (solid blue curve), which should be the case.

IV. UPPER BOUND ON THE COEFFICIENT OF PERFORMANCE

Here we discuss the operation of QHOC as a refrigerator. In the refrigeration process, heat is absorbed from the cold bath, $\langle Q_4\rangle > 0$, and dumped into the hot bath, $\langle Q_2\rangle < 0$. The net work invested in the system is positive, $\langle W_{\rm total}\rangle = -(\langle Q_2\rangle + \langle Q_4\rangle) > 0$. The COP of the refrigerator is defined as $\zeta = \langle Q_4\rangle/\langle W_{\rm in}\rangle$. Here we will first discuss the case when a refrigerator is coupled to two purely thermal reservoirs. We follow the same procedure as done for the heat engine in Sec. III. Since the calculations are straightforward, we merely present our results here. For the refrigerator running between two purely thermal reservoirs, a positive cooling condition, $\langle Q_4\rangle > 0$, implies that

$$\zeta_C > 1 \quad \text{and} \quad \zeta \leqslant 1 + 3\zeta_C - 2\sqrt{2\zeta(1+\zeta_C)} \equiv \zeta^{\rm th}_{\rm up}, \tag{22}$$

where $\zeta_C = \beta_2/(\beta_1 - \beta_2)$ is the Carnot COP. The condition $\zeta_C > 1$ implies that $\tau > 1/2$, which in turns implies that the machine cannot work as a refrigerator when the cold reservoir temperature is below $T_2/2$. The upper bound $\zeta_{\rm up}$ derived here is independent of the parameters of the system and depends on ratio of the reservoir temperatures only, which makes it quite general in nature. Similar to the heat engine case, the obtained upper bound is much tighter than the corresponding Carnot bound.

Now we will discuss the effect of coupling the refrigerator to the cold squeezed reservoir. In the high-temperature regime, the mean energies at points A, B, C, and D are given by $\langle H\rangle_A = \omega_1 \coth(\beta_1\omega_1/2)\cosh(2r)/2$, $\langle H\rangle_B = \omega_2\lambda \coth(\beta_1\omega_1/2)\cosh(2r)/2$, $\langle H\rangle_C = \omega_2\coth(\beta_2\omega_2/2)$, $\langle H\rangle_D = \omega_1\lambda\coth(\beta_2\omega_2/2)/2$. The positive cooling condition, $\langle Q_4\rangle > 0$, yields the following expressions:

$$\frac{1}{2}\text{sech}(2r) < \tau < \text{sech}(2r) \quad \text{and} \quad \zeta < \frac{3}{1-\tau\cosh(2r)} - 2 - 2\sqrt{2}\sqrt{\frac{\tau\cosh(2r)}{[\tau\cosh(2r)-1]^2}} \equiv \zeta_{up}. \tag{23}$$

Equation (23) along with Eq. (22) is our third main result. As expected, $\zeta_{\rm up}$ reduces to $\zeta_{up}^{\rm th}$ for the vanishing squeezing parameter, $r = 0$. To discuss the physical significance of condition given in Eq. (23), we invert it in terms of lower and upper limits on squeezing parameter r:

$$0 < \tau < \frac{1}{2}, \quad \frac{1}{2}\cosh^{-1}\left(\frac{1}{2\tau}\right) < r < \frac{1}{2}\cosh^{-1}\left(\frac{1}{\tau}\right), \quad \text{or} \quad \frac{1}{2} < \tau < 1, \quad 0 < r < \frac{1}{2}\cosh^{-1}\left(\frac{1}{\tau}\right). \tag{24}$$

It is clear from the above equation that we can extract heat from squeezed cold reservoir even for $\tau < 1/2$, which is otherwise impossible with the refrigeration operation with purely thermal reservoirs. Again this can be explained on the basis of effective temperature of the cold reservoir [see Eq. (19)]. For $r = \frac{1}{2}\cosh^{-1}(\frac{1}{2\tau})$ and $r = \frac{1}{2}\cosh^{-1}(\frac{1}{\tau})$, the effective temperatures of the cold reservoir become $T_2/2$ and T_2, respectively. As per the original positive work condition $(1/2 < \tau)$ without a cold squeezed reservoir, $T_1 > T_2/2$, and hence in the case of a cold squeezed reservoir this condition is satisfied for the given range of squeezing parameter r in Eq. (24). Eventually, the refrigeration stops when the effective temperature of cold squeezed reservoir approaches T_2, which is the temperature of the thermal hot reservoir. Finally, for $\tau = 1/2$ or $T_2 = 2T_1$, the allowed range of r is $0 < r < \frac{1}{2}\cosh^{-1}(2)$, which implies that effective temperature of cold reservoir should be smaller than $2T_1$, which is natural.

V. COST OF SQUEEZING

In order to calculate the cost of squeezing, we will use the formalism developed in Refs. [32,33]. First, we outline a general scheme for defining the local temperature of a stationary nonequilibrium reservoir and then apply it to the case of a squeezed thermal reservoir. Thermal baths can be considered as a collection of independent oscillators with a quasicontinuum spectrum of frequencies. Hence, the free Hamiltonian of a bosonic bath is given by $H_B = \sum_k \omega_k \hat{b}_k^\dagger \hat{b}_k$ (ignoring the zero-point energy). In most applications (including our case), the bath operator that weakly couples to the system is linear in annihilation and creation operators: $\hat{B} = \sum_k (g_k\hat{b}_k + \bar{g}_k\hat{b}_k^\dagger)$. The nonequilibrium state of the bath, ρ_B, can be made stationary by ergodic averaging and is given by $\bar{\rho}_B = \sum_k |k\rangle\langle k|\rho_B|k\rangle\langle k|$, where $\{|k\rangle\}$ is the energy basis of the free Hamiltonian of the bath. Consider a quantum harmonic oscillator with frequency ω weakly coupled to a nonequilibrium stationary bath by means of the interaction Hamiltonian $H_{\rm int} = (\hat{a} + \hat{a}^\dagger)\otimes\hat{B}$. Then the bath coupling spectrum, $G_B(\omega) = \int e^{it\omega}\langle B(t)B\rangle_B\, dt$, yields [32,33]

$$G_B(\omega) = \begin{Bmatrix} \sum_k |g_k|^2(n_k+1)\delta(\omega_k-\omega), & \omega > 0 \\ \sum_k |g_k|^2 n_k\delta(\omega_k-\omega), & \omega < 0 \end{Bmatrix}, \tag{25}$$

where $n_k = \text{Tr}(\bar{\rho}_B\hat{b}_k^\dagger\hat{b}_k)$ is the k-mode population. The lower (upper) line in Eq. (25) is the absorption (emission) rate. By virtue of Eq. (25), the frequency-dependent local temperature of a stationary nonequilibrium reservoir can be defined by the following equation:

$$e^{-\omega/T_B(\omega)} = \frac{G_B(-\omega)}{G_B(\omega)} = \frac{n(\omega)}{n(\omega)+1}, \tag{26}$$

where

$$n(\omega) = \frac{\sum_k |g_k|^2 n_k\delta(\omega_k-\omega)}{\sum_k |g_k|^2\delta(\omega_k-\omega)} \tag{27}$$

denotes the average population number for the frequency ω.

Now, consider a hot thermal bath at temperature T_2 weakly coupled to a harmonic oscillator of frequency ω_2. If we selectively squeeze the mode of thermal reservoir in resonance with the working medium (thermal mode of frequency ω_2) only, then by virtue of Eqs. (25) and (26), the bath will drive the working fluid (harmonic oscillator) to a Gibbs state with the local temperature [32,33]

$$T_2^{\rm eff}(\omega_2) = \frac{\omega_2}{\ln\{1 + [\langle n\rangle + (2\langle n\rangle+1)\sinh^2 r]^{-1}\}}, \tag{28}$$

where we have used $n(\omega_2) = \langle n(\beta_2, r)\rangle = \langle n\rangle + (2\langle n\rangle + 1)\sinh^2 r$. In the above, we showed that in order to take the advantage of squeezing, it is sufficient to squeeze just one resonant mode of the thermal reservoir. Now we will turn to the question of determining the cost of squeezing.

The generation of reservoir squeezing is not part of the engine cycle, and the energetic cost of their creation does not enter into thermodynamic efficiency of the heat engine. On the other hand, one can talk about an operational efficiency of the heat engine by including such energetic costs, which may be relevant, especially for resources such as squeezed thermal reservoirs, if the squeezing is produced by artificial means and not natural ones. For such a case, the cost may be defined as the energetic difference between the free, natural, thermal state and the squeezed thermal state. This energy difference corresponds to an additional work done, denoted by $W_{\rm cost} = {\rm Tr}(H_R\rho_{\rm sq}) - {\rm Tr}(H_R\rho_{\rm th})$, and the operational efficiency of the thermal machine is introduced to be $\eta_{\rm op} = (\langle W_{\rm ext}\rangle - W_{\rm cost})/\langle Q_2\rangle$. Since in our case, the advantage of squeezing can be exploited by squeezing only the mode of frequency ω_2 (resonant with system frequency), the cost of squeezing is calculated as the energy difference between the single-mode squeezed thermal state of frequency ω_2, $\rho_{\rm sq}(\omega_2)$, and corresponding single-mode thermal state, $\rho_{\rm th}(\omega_2)$. Hence, $W_{\rm cost}$ is given by

$$W_{\rm cost} = {\rm Tr}[H(\omega_2)\rho_{\rm sq}(\omega_2)] - {\rm Tr}[H(\omega_2)\rho_{\rm th}(\omega_2)], \quad (29)$$

where $H(\omega_2) = (\hat{b}^\dagger\hat{b} + 1/2)\omega_2$ is the Hamiltonian corresponding to the mode frequency ω_2. For the case of a heat engine, when the hot reservoir is taken to be squeezed, Eq. (29) can be solved as follows:

$$W_{\rm cost} = \left[\langle n(\beta_2, r)\rangle + \tfrac{1}{2}\right]\omega_2 - \left(\langle n\rangle + \tfrac{1}{2}\right)\omega_2 \quad (30)$$

$$= (2\langle n\rangle + 1)\omega_2. \quad (31)$$

The above equation can be further simplified by taking the high-temperature limit. In this limit, $\langle n\rangle \gg 1$ and can be approximated by $\langle n\rangle \approx 1/\beta_2\omega_2$. Thus, Eq. (31) becomes

$$W_{\rm cost} = \frac{2\sinh^2 r}{\beta_2}. \quad (32)$$

From the experimental and operational point of view, it is better to mimic the action of the squeezed reservoir by replacing the nonthermal Otto cycle by an equivalent cycle involving a hot thermal reservoir at inverse temperature β_2 and an external squeezing source [31,59]. The action of the hot squeezed reservoir can be mimicked by first placing the system in contact with the hot thermal reservoir and then applying the unitary transformation (squeezing operator) on the thermal state of the working fluid. In this way, the work invested ($W_{\rm cost}$) to create the thermal squeezed state of the working fluid from the standard thermal state is simply given by the energy difference between them, which is the same as given by Eq. (30).

Taking into account this cost, the net work extracted ($\langle W'_{\rm ext}\rangle = \langle W_{\rm ext}\rangle - W_{\rm cost}$) is given by subtracting Eq. (32) from Eq. (11) and is given by

$$\langle W'_{\rm ext}\rangle = \frac{1}{\beta_2}\left\{\frac{(1-z^2)[z^2\cosh(2r) - \tau]}{2z^2} - 2\sinh^2 r\right\}, \quad (33)$$

and the expression for the operational efficiency takes the form

$$\eta_{\rm op} = \frac{\langle W'_{\rm ext}\rangle}{\langle Q_2\rangle} = \frac{\tau - z^2(2+\tau) + z^2(1+z^2)\cosh(2r)}{\tau(1+z^2) - 2z^2\cosh(2r)}. \quad (34)$$

We can check analytically that Eq. (33) is a monotonically decreasing function of r. Similarly, for the PWC, we have checked numerically that $\eta_{\rm op}$ is also a monotonically decreasing function of r. Now we present an example plot describing the reduced efficiency ($\eta_{\rm op}$) of the machine when the squeezing generation cost is taken into account in Fig. 4 for a different amount of squeezing injected into the same thermal state.

From Fig. 4 it is clear that both $\eta_{\rm op}$ and $W'_{\rm ext}$ are deceasing for increasing r. Hence, we can conclude that when accounting for the cost of squeezing, squeezed reservoirs can be costly as synthetic quantum fuels. But they may offer advantages by offering effectively high-temperature baths without being actually too hot, and by potentially compact and direct integration to quantum working systems.

There is a limitation in our approach of calculating the cost of squeezing. We assume that the squeezing parameter r goes to 0 after each cycle. In practice, the bath's quantum state may still possess some squeezing or quantumness after the interaction with the system. Further cost reduction can be possible by choosing optimum bath-system coupling, parameters, and durations. We could start with the initial state of the total working system and bath, evolve the cycle stages where the working system couples to the bath, and then determine the bath's final state by tracing out the working system degrees of freedom. Then to reset the bath state, only the energetic difference between the initial squeezed state and the state of the bath after the cycle would need to be provided. This cost can be estimated by an upper bound if the final state has no squeezing at all. Accordingly, our cost estimation is the worst-case scenario. Unfortunately, in the present paper, we do not model the baths and their coupling to the working resonator microscopically. We do not perform finite-time studies either; hence, we cannot address cost optimization with sufficient detail. We leave this exciting point for future research. In general, quantum coherence is an expensive resource as studied in the literature in different quantum heat engine settings [8,60–63]. Discussions of the cost of quantum coherent fuels and their various advantages in storage, compactness, integration, speed, and scaling have been made for various physical systems [8,60–63].

VI. CONCLUSIONS

We have investigated the performance of a HQOC, operating in the sudden switch limit, coupled to a squeezed thermal reservoir. First, we showed that even in the presence of very large squeezing ($r \to \infty$), the maximum efficiency of the engine is $1/2$ only. This is due to the frictional effects caused by the nonadiabatic transitions when we operate in the sudden switch regime. Our study is in contrast with previous studies which claim that the efficiency can reach unity for large squeezing. Then we obtained a closed-form expression for the upper bound on the efficiency of the engine operating in the high-temperature regime. The result is interesting in the sense that the obtained bound is independent of the parameters of the model under consideration and depends on the ratio of the reservoir temperatures and squeezing parameter r only. Additionally, we also derive the analytic expression for the efficiency at maximum work and showed that it satisfies the derived upper bound. As a special case of our more general setup, when squeezing parameter $r \to 0$, our results correspond to the case in which an engine is running between two purely thermal reservoirs. Further, we have also obtained upper bounds for the Otto refrigerator working between two

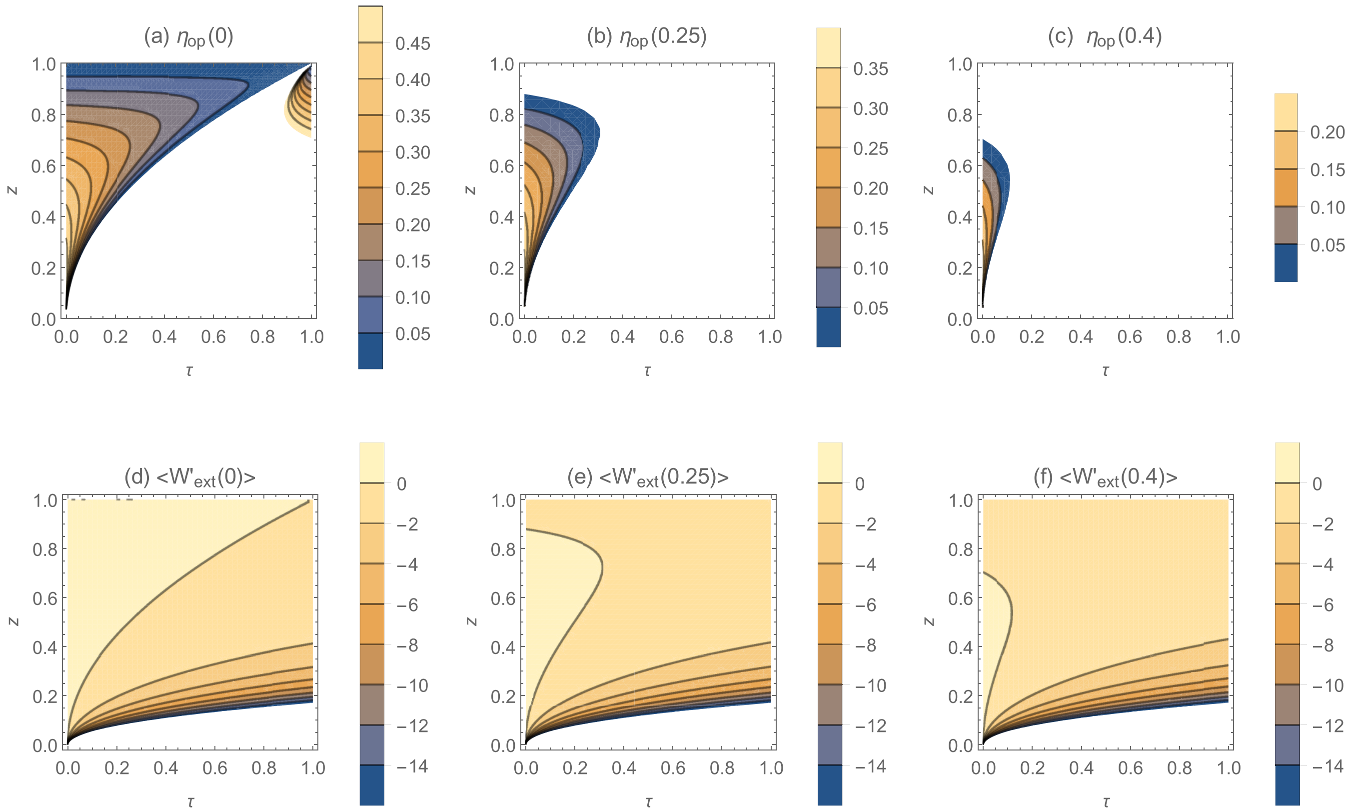

FIG. 4. Plots of operational efficiency [Eq. (34)] and net extracted work [Eq. (33)] as functions of $\tau = \beta_2/\beta_1$ and $z = \omega_1/\omega_2$ for different fixed values of squeezing parameter r. Squeezing parameter r takes the values $r = 0$, $r = 0.25$ and $r = 0.4$ going from left to right. We have chosen the range of efficiency from 0 to $1/2$ as we have already shown that the efficiency cannot surpass half the unit efficiency [see Eq. (10)]. It is clear from the plots that as r increases, both $\eta_{op}(r)$ and $W'_{\rm ext}(r)$ decrease. The white region in the upper set of graphs corresponds to the engine efficiency greater than $1/2$, and, thus, it does not correspond to the engine operation. Further, the efficiency contours covering the small colored region at the top right side of Fig. 4(a) also does not represent the operational regime of the engine as it does not correspond to the PWC (see Fig. 4(d)). Similarly, the white region in the lower set of graphs represents the numerical values of $W'_{\rm ext}(r)$ less than -14, which again does not correspond to engine operation.

purely thermal reservoirs as well as for the case when the cold reservoir is taken to be a squeezed thermal reservoir. Furthermore, we showed that squeezing can help in a cooling process otherwise impossible in a standard setup with thermal reservoirs. Finally, we showed that it is sufficient to squeeze only one mode, in resonance with the transition frequency of the working fluid, of the thermal reservoir to harvest the benefits of squeezing. Then we showed that when the cost of generating a squeezed state is also included in the definition of the operational efficiency of the engine, the squeezed reservoirs are costly as synthetic quantum fuels. However, they may present us with other advantages from a practical point of view, such as by serving as effectively high-temperature baths without being actually too hot, and by offering a potentially compact and dense energy storage medium.

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