Anomalous persistence exponents for normal yet aging diffusion

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The persistence exponent θ, which characterizes the long-time decay of the survival probability of stochastic processes in the presence of an absorbing target, plays a key role in quantifying the dynamics of fluctuating systems. So far, anomalous values of the persistence exponent ($\theta \neq 1/2$) were obtained, but only for anomalous processes (i.e., with Hurst exponent $H \neq 1/2$). Here we exhibit examples of ageing processes which, even if they display asymptotically a normal diffusive scaling ($H = 1/2$), are characterized by anomalous persistent exponents that we determine analytically. Based on this analysis, we propose the following general criterion: The persistence exponent of asymptotically diffusive processes is anomalous if the increments display ageing and depend on the observation time T at *all* timescales.

I. INTRODUCTION

The survival probability $S(t)$ of a one-dimensional (1D) unbounded stochastic process $x(t)$ is defined as the probability that $x(t)$ has not reached a threshold value up to time t. This observable has proved to be very useful to quantify the dynamics of a broad range of complex systems in contexts as varied as diffusion controlled reactions, finance, search processes, or biophysics [1–7]. In many examples of symmetric stochastic processes, the large-time behavior of the survival probability is characterized by a power-law decay $S(t) \propto t^{-\theta}$ that defines the persistence exponent θ.

Because of its importance to characterize the dynamics of various systems, the determination of θ has been the focus of a vast amount of works in the fields of stochastic processes and nonequilibrium statistical mechanics, and general results have been obtained in particular for processes without memory, i.e., Markovian processes [3,4]. More precisely, in the case of 1D translation-invariant symmetric Markovian jump processes, for which the random walker position $x(t)$ can be expressed as a sum of independent identically distributed variables, the Sparre-Andersen theorem yields a robust universal result $\theta = 1/2$, independently of the value of the Hurst exponent H that characterizes the large-time behavior of the mean-square displacement $\langle x^2(t) \rangle \propto t^{2H}$ [8]. In the case of processes with memory, i.e., non-Markovian processes, θ has been determined exactly only for specific examples of processes such as fractional Brownian motion [9,10] or the random acceleration process [11], and perturbatively for weakly non-Markovian Gaussian processes [12–14]. In all the examples studied so far, anomalous values of the persistence exponent (i.e., $\theta \neq 1/2$) were obtained but only for anomalous processes (i.e., $H \neq 1/2$). Note, however, that no general result rules out the possibility of anomalous persistence also for non-Markovian diffusive processes ($H = 1/2$), even if no such examples have been proposed to the best our knowledge. In fact, all

available examples of processes that are diffusive ($H = 1/2$), even if only asymptotically for $t \to \infty$, display the universal exponent $\theta = 1/2$. This is illustrated by the example where $x(t)$ is the position of a given monomer of a finite 1D ideal Rouse chain of N monomers; this non-Markovian process is diffusive at times larger than the Rouse time (that is the slowest relaxation time of the internal degrees of freedom of the chain) and is shown to be characterized by $\theta = 1/2$ [7]; note that in this example the increments $x(t + T) - x(T)$ are stationary at long times, i.e., have statistics independent of the observation time T, for T larger than the Rouse time. Other examples include persistent random walks and Levy walks and are discussed in Appendix H.

In this paper, we demonstrate that anomalous persistence exponents can be obtained even for asymptotically normal diffusive processes. We argue that this results from the ageing properties of the process and propose the following general criterion: The persistence exponent of asymptotically diffusive processes is anomalous if the increments are nonstationary and have statistics that depend explicitly on the observation time T at all timescales. Our claim is based on the analysis of a range of examples of strongly non-Markovian processes and in particular of the so-called self-attracting walk (SATW) [15] and the elephant random walk (ERW) [16]. These have been studied in the context of random search processes as prototypical examples of processes with long-range memory and have important potential applications in the theoretical description of the trajectories of living organisms such as cells, insects, or larger animals [17–19]. For both processes, we derive analytically the persistence exponent and show that it is anomalous, while we demonstrate that the processes are asymptotically diffusive, with, however, ageing increments at all timescales. Our general criterion is confirmed by the analysis of further examples such as the persistent self-attracting walk and the polynomial self-repelling walk.
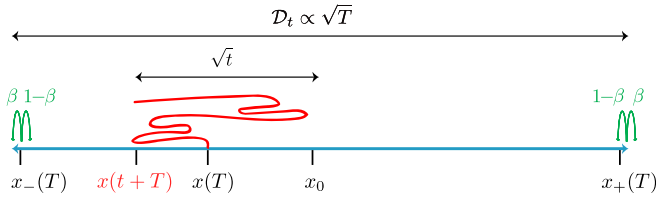
FIG. 1. Definition of the SATW. The process is a simple nearest-neighbor random walk in the bulk of the visited territory $\mathcal{D}_t$ and is modified only at the edges of $\mathcal{D}_t$; we show that increments $\langle [x(t + T) - x(T)]^2 \rangle$ are diffusive at short and long timescales and display ageing.

II. PERSISTENCE EXPONENT OF THE SELF-ATTRACTING WALK

We consider first the SATW, or one-step reinforced walk, which was introduced as a natural example of random walk that interacts fully (either attractively or repulsively) with its own path [15,20–26]. This model can therefore be related to the class of self-avoiding random walks, which have played a crucial role in physics [27] and have applications in the modeling of trajectories of living systems [28–30]; in essence, such models consider a random walker that deposits a signal at each visited site, which alters the future dynamics of the walker on its next visits. Due to the long-range memory effect of this class of processes, their properties, and in particular their persistence exponents, are notoriously difficult to characterize analytically. More precisely, the SATW is defined here on the $1d$ lattice as a jump process in discrete time t where the probability that the random walker jumps to a neighboring site i is proportional to $\exp(-u\, n_i)$, where $n_i = 0$ if the site i has never been visited up to time t and $n_i = 1$ otherwise. For $u > 0$ the random walker is thus repelled by its own path, while it is attracted for $u < 0$.

This process is strongly non-Markovian since the definition of its evolution at time t requires the knowledge of the full territory $\mathcal{D}_t$ visited by the random walker up to time t (see Fig. 1). However, in $1d$, the knowledge of the two extrema $x_- = \inf_{0 \leqslant t' \leqslant t} x(t')$ and $x_+ = \sup_{0 \leqslant t' \leqslant t} x(t')$ of the trajectory at t is sufficient to fully determine the visited territory $\mathcal{D}_t = [x_-, x_+]$ and thus the evolution of the process. When in the bulk of the visited territory $[x_- < x(t) < x_+]$, the random walker performs a classical symmetric nearest-neighbor random walk. The dynamics is modified only for $x = x_-, x_+$ and is then conveniently encoded locally by the probability $\beta = 1/[1 + \exp(-u)]$ to visit the yet unvisited neighboring site ($\notin \mathcal{D}_t$) rather than the previously visited one ($\in \mathcal{D}_t$). Only a few results are available to describe this non-Markovian process [15,31–34], which has been shown in particular to be asymptotically diffusive at large times $\langle x^2(t) \rangle \propto t$, with, however, a large time diffusion coefficient that has not been determined analytically.

In view of determining the persistence exponent of this process, it is convenient to parametrize the dynamics in terms of the number $N = x_+ - x_- + 1$ of distinct sites visited up to time t. We define by $S(N)$ the corresponding survival probability that the target, located at $x = 0$, has not been reached when N sites have been visited; in addition, we denote by $x_0 > 0$ the initial position of the random walker. We first show
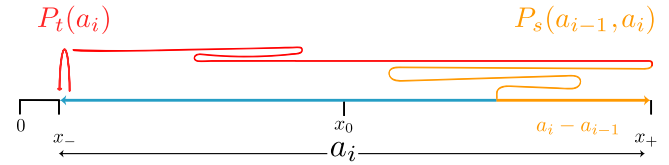

FIG. 2. Side changes and increments of the visited territory. Between successive side changes, the random walker discovers $a_i - a_{i-1} - 1$ new sites on the same side of the visited territory with a probability $P_s(a_{i-1}, a_i)$; the side change i occurs for $a = a_i$ with probability $P_t(a_i)$.

below that $S(N)$ satisfies the following scaling in the large-N limit:

$$S(N) \propto N^{-\frac{1-\beta}{\beta}}, \tag{1}$$

by providing lower and upper bounds for $S(N)$.

To this end, we note that a lower bound for $S(N)$ is straightforwardly given by the probability $P_s(N)$ that $\mathcal{D}_t = [x_0, x_0 + N - 1]$, i.e., the probability that all of the N visited sites have been discovered from the x_+ edge of $\mathcal{D}_t$. Adapting classical results for splitting probabilities of 1D random walks [1,35] (see Appendix A for details), an exact recurrence relation for $P_s(N)$ can be obtained and reads:

$$P_s(N+1) = \left[1 - \frac{1-\beta}{2 + \beta(N-3)} \right] P_s(N). \tag{2}$$

Taking the log of Eq. (2) then yields in the large-N limit the following scaling of the lower bound:

$$S(N) > P_s(N) \propto N^{-\frac{1-\beta}{\beta}}. \tag{3}$$

We now turn to the determination of an upper bound to $S(N)$. We define a side change as the visit of a new site on the side opposite to the last newly visited site. With this definition, it is easy to see that all trajectories involving more than $2x_0$ side changes cross the target at $x = 0$. As a consequence, by defining $P(N, k)$ as the probability that exactly k side changes occurred until N sites have been visited in the presence of the target at $x = 0$, one can write $S(N) = \sum_{k=0}^{2x_0} P(N, k)$. In order to determine the scaling of $P(N, k)$, we introduce k random variables $\{a_1, \ldots, a_k\}$, where a_i is the total number of distinct visited sites when the ith side change occurs. Hence, between side changes i and $i + 1$, the random walker visits $a_{i+1} - a_i$ new sites on the same side of the visited territory $\mathcal{D}$. Correspondingly, we denote by $P_s(a_i, a)$ the probability that $a - a_i - 1$ new sites have been visited on the same side of $\mathcal{D}$ after side change i. Note in particular that $P_s(1, N) = 2P_s(N)$ introduced above. We also introduce $P_t(a)$ as the probability that a side change occurs when the number of distinct visited sites is a (see Fig. 2); it is the probability that the random walker, starting from one edge of $\mathcal{D}$ discovers the next new site on the opposite edge.

The joint probability of the event $\{a_1, \ldots, a_k\}$ given a specific first step can then be written

$$P(\{a_1, \ldots, a_k\}) = 2^{-1} P_s(1, a_1) P_t(a_1) \left[\prod_{i=2}^{k} P_s(a_{i-1}, a_i) P_t(a_i) \right]$$

$$\times P_s(a_k, N). \tag{4}$$

Both P_s and P_t can be straightforwardly expressed in terms of splitting probabilities introduced above to determine $P_s(N)$, and exact recurrence relations similar to Eq. (2) can be obtained (see Appendix B). This yields in particular the following asymptotics ($i > 1$):

$$P_s(a_i, a_{i+1}) \underset{a_i \to \infty}{\sim} \frac{a_i^{\frac{1-\beta}{\beta}}}{a_{i+1}^{\frac{1-\beta}{\beta}}}; \quad P_t(a_i) \underset{a_i \to \infty}{\sim} \frac{1-\beta}{\beta}\frac{1}{a_i}, \tag{5}$$

which will be useful below. In order to express $P(N, k)$, we must condition on the first step and distinguish odd and even values of k. Assuming without loss of generality that the first step is made in the $+$ direction, and that $k = 2p$ is even, we obtain in the large-N limit where sums can be approximated by integrals:

$$P(N, 2p) \leqslant \int_1^N \int_{a_1}^{\min(x_0+a_1,N)} \int_{a_2}^N \int_{a_3}^{\min(x_0+a_3,N)} .. \int_{a_{2p-2}}^N \int_{a_{2p-1}}^{\min(x_0+a_{2p-1},N)} P(\{a_1, \ldots, a_k\}) \times da_1..da_{2p}. \tag{6}$$

Upper bounds in the integrals take into account the absorbing target at $x = 0$. Next, using the definition of Eq. (4) and the asymptotics of Eq. (5), one obtains in the large-a_i limit

$$P(\{a_1, \ldots, a_k\}) \underset{a_i \to \infty}{\propto} N^{-\frac{1-\beta}{\beta}}(a_1 a_2 \times .. \times a_k)^{-1}. \tag{7}$$

We then make use of the following estimate in the large-N limit derived in (C 2):

$$\int_1^N \int_{a_1}^{\min(x_0+a_1,N)} \int_{a_2}^N .. \int_{a_{2p-2}}^N \int_{a_{2p-1}}^{\min(x_0+a_{2p-1},N)} (a_1 a_2 \times .. \times a_k)^{-1} da_1..da_{2p} = O(1), \tag{8}$$

to finally obtain

$$P_+(N, 2p) = O\left(N^{-\frac{1-\beta}{\beta}}\right). \tag{9}$$

The same $O\left(N^{-\frac{1-\beta}{\beta}}\right)$ estimate is obtained for $P_\pm(N, k)$ for all combinations of the first step direction $\pm$ and parity of k as shown in Appendix C. This finally yields

$$S(N) = \sum_{k=0}^{2x_0} P(N, k) = O\left(N^{-\frac{1-\beta}{\beta}}\right), \tag{10}$$

which, together with the lower bound of Eq. (3), proves the scaling of Eq. (1). We finally conclude on the scaling of the survival probability $S(t)$ as a function of the elapsed time t. To do so, we make use of the ansatz that the conditional probability density $\Phi_N(t)$ that the target is found at time t knowing that $N - 1$ sites have been visited is a scaling function of the variable t/N^2: $\Phi_N(t) = f(t/N^2)/N^2$ (see Appendix D for numerical check). This, together with Eq. (1), yields the following analytical determination of θ:

$$\theta = \frac{1-\beta}{2\beta}. \tag{11}$$

First, we underline that this result, which is expected to be exact (see numerical simulations, Fig. 3), shows that the SATW provides a simple model for which the persistence exponent is determined analytically and can be varied over a broad range of values by tuning a simple microscopic parameter (β). Second, this reveals that, despite the fact that the process is asymptotically diffusive $\langle x^2(t) \rangle \propto t$ at large times ($H = 1/2$) for all values of β [15,31], the persistence exponent θ is anomalous ($\theta \neq 1/2$) for $\beta \neq 1/2$.

III. SCALE-FREE AGEING OF INCREMENTS (SATW)

Based on this example, we now aim at determining a general criterion that allows us to identify the asymptotically diffusive processes that lead to anomalous persistence

exponents. It has been proposed, on the basis of general arguments and only minimal hypothesis, that for continuous scale-invariant processes with stationary increments the universal relation $\theta = 1 - H$ should hold [4]; this relation can in fact be extended to processes whose time-dependent increments $\langle [x(t + T) - x(T)]^2 \rangle$ are only asymptotically stationary after a finite characteristic timescale [7]. We review in Appendix H examples of such processes, which all lead to $\theta = 1/2$. Our finding that $\theta \neq 1 - H$ for the SATW indicates that its increments are not stationary after a finite timescale; this calls for a further analysis of the increments. These are known to be diffusive and to satisfy $\langle [x(t + T) - x(T)]^2 \rangle \sim 2D_L(\beta)t$ in the large-time limit $t \gg T$, where $D_L(\beta)$ has not been determined analytically. In turn it is easily seen that for $T \gg t$, one has $\langle [x(t + T) - x(T)]^2 \rangle \sim 2D_s t$ with $D_s = 1/2$, because the walker spends most of the time away from the boundaries of the visited territory $\mathcal{D}_T$ (see Fig. 1). Finally, because the process has no intrinsic timescale but the unit time step, dimensional analysis shows that in the limit $t, T \gg 1$ the increments can be written $\langle [x(t + T) - x(T)]^2 \rangle = 2D(t/T) t$. This behavior was confirmed by numerical simulations (see Fig. 3). Finally, the process is diffusive at both short and long timescales but displays ageing *at all observation timescales* T. We hypothesize that this behavior, which we call scale-free ageing, leads to anomalous persistence exponents for asymptotically diffusive processes. Below, we provide explicit examples supporting this hypothesis.

IV. PERSISTENT SATW MODEL

We first verify our hypothesis on a generalization of the SATW model to persistent random walks (see Appendix G for details). While the random walker is within the visited territory, it is now assumed to perform a classical persistent random walk of persistence probability (probability to reproduce the previous step) $p = \frac{e^k}{e^k + e^{-k}}$, where k is a free real parameter. When the random walker is at an edge $x_\pm$
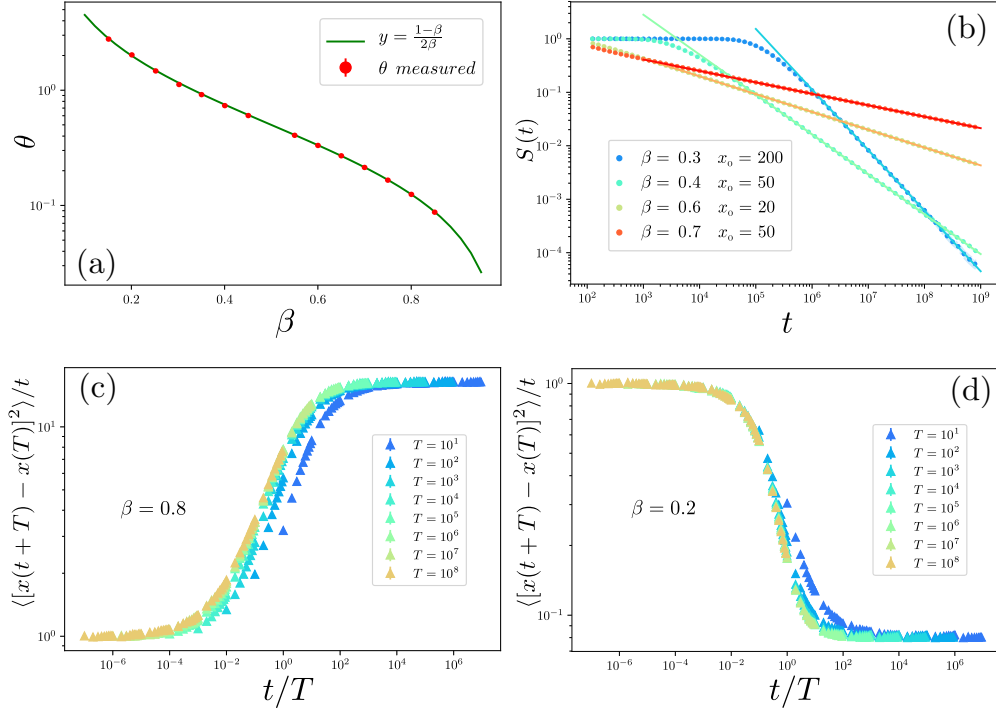
FIG. 3. Persistence and ageing of the SATW. (a) Persistence exponent: Numerical simulation and analytical prediction of Eq. (11). (b) Survival probability as a function of time (log scale) for different values of β. Numerical simulations (symbols) and power-law fit (plain line). [(c) and (d)] Ageing of the increments for the SATW (normalized by the expected diffusive scaling at long times). Each curve corresponds to a fixed value of T.

of the visited domain, reinforcement and persistence effects are coupled so that the probability to discover a new site is given by $\beta(k, u) = \frac{e^k}{e^{-k-u} + e^k}$ (note that the SATW model is recovered by taking $k = 0$). It is shown in Appendix G that the persistence exponent can still be determined explicitly for this model; it is found to be independent of k:

$$\theta = \frac{1 - \beta(k = 0, u)}{2\beta(k = 0, u)}. \tag{12}$$

As in the SATW case ($k = 0$), the persistence exponent is thus anomalous for $\beta(k = 0, u) \neq 1/2$. The analysis of the increments reveals that motion is ballistic below the timescale e^{2k}. In turn, for $T, t \gg e^{2k}$, increments are characterized by a time-dependent diffusion coefficient $D(t/T)$ that ranges from $D_s = e^{2k}/2$ for $T \gg t$ to an undetermined (k-dependent) value D_L for $t \gg T$ (see Fig. 5). The process therefore displays scale-free ageing, and our hypothesis is verified. To further assess the generality of our criterion, we analyze below two other examples of ageing asymptotically diffusive non-Markovian processes with anomalous persistence exponents.

V. ELEPHANT RANDOM WALK

First, the ERW was introduced in Ref. [16] and can be defined as follows on $1d$ lattice: At each time step t a nearest-neighbor jump $\bar{\sigma}$ is drawn with uniform distribution from the full set of jumps performed in the past $\{\sigma_i\}_{i<t}$, where $\sigma_i = \pm 1$ (with a symmetric first jump). The random walker then performs the jump $\sigma_t = \bar{\sigma}$ with probability β, and the jump $\sigma_t = -\bar{\sigma}$ with probability $1 - \beta$. The position process

is then defined by $x(t) = \sigma_t + x(t - 1)$. The process is non-Markovian by construction and can be shown for $\beta < 3/4$ to be characterized by scale-free, diffusive, ageing increments $\langle [x(t + T) - x(T)]^2 \rangle \sim 2D(t/T) t$ in the limit $T, t \gg 1$, with an effective diffusion coefficient that we show can be written (see Appendix E)

$$D(t/T) \sim \frac{1}{6 - 8\beta} - \frac{1}{3 - 4\beta} \left\{ \frac{T}{t} \left[\left(1 + \frac{t}{T} \right)^{2\beta - 1} - 1 \right] \right\}. \tag{13}$$

This behavior of the increments (see Fig. 4) is similar to the case of the SATW, with in particular finite limits of the effective diffusion coefficient $D(t/T)$ in both regimes $t \ll T$ and $t \gg T$ (see Fig. 4). The limit diffusion coefficients can be expressed analytically from (13) and read $D_s = 1/2$ and $D_L = 1/(6 - 8\beta)$ (in agreement with Refs. [16,36] for $T = 0$). Similarly to the SATW, the ERW is therefore diffusive at both short and long timescales, and displays scale-free ageing. We now determine the persistence exponent of the ERW. In the diffusive regime that we consider here, it has been shown that in the large-t limit, the one-point probability density function $p(x, t)$ is Gaussian [36] and satisfies a Fokker-Planck equation with a time-dependent drift for $\beta < 3/4$; based on this formalism, the first-passage time distribution to a target was computed in Ref. [37]. We used this result to derive analytically the persistence exponent of the ERW:
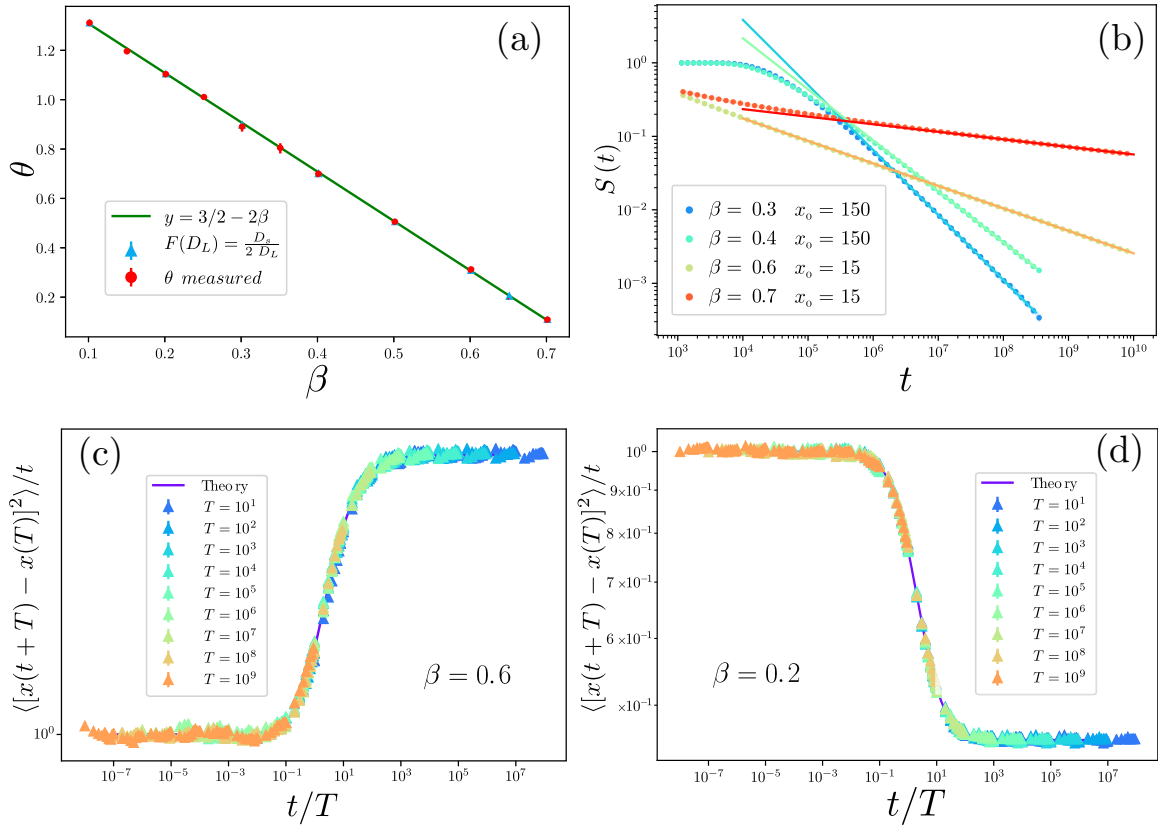
$$\theta = 3/2 - 2\beta, \tag{14}$$

FIG. 4. Persistence and ageing of the ERW. (a) Persistence exponent: Numerical simulations, analytical prediction of Eq. (14), and prediction based on an effective Fick's law derived in Appendix F. (b) Survival probability as a function of time (log scale) for different values of β. Numerical simulations (symbols) and power-law fit (plain line). [(c) and (d)] Ageing of the increments for the ERW (normalized by the expected diffusive scaling at long times) and analytical prediction of Eq. (13) (plain line). Each curve corresponds to a fixed value of T.

which was confirmed by numerical simulations (see Fig. 4). Again, we find that the persistence exponent θ is anomalous ($\theta \neq 1/2$) for $\beta \neq 1/2$ and takes a broad range of values when β is varied. The example of the ERW therefore supports our conjecture that scale-free ageing is responsible for the observed anomalous persistent exponent $\theta \neq 1/2$, even for asymptotically diffusive processes ($H = 1/2$).

VI. SELF-REPELLING RANDOM WALK

Last, we consider an example of self-repelling walk where the probability that the random walker jumps to a neighboring site i is proportional to $1/(1 + \tau_i^\beta)$, where τ_i is defined as the cumulative time spent by the random walker at the site i. This process has been shown to be asymptotically diffusive at large times [38,39]; however, its increments and persistence properties remain unexplored. Our numerical analysis reveals that the persistence exponent is anomalous $\theta \approx 0.25$ for all

values of β; in turn the increments are found to display scale-free ageing (see Fig. 5). Our hypothesis is therefore verified once again.

VII. CONCLUSION

We now conclude this paper by summarizing our results. We have analyzed the persistence exponents for several examples of non-Markovian random walks, which have been studied in the context of random search processes. These persistence exponents, some of which we determined analytically, were found to be anomalous and to vary over a broad range of values even if all processes remain asymptotically simply diffusive. Our analysis leads to the conjecture that the persistence exponent of asymptotically diffusive processes is anomalous if the increments are ageing and depend on the observation time T at all timescales. This conjecture is consistent with existing general results for processes with stationary increments and was checked on a broad range of examples of processes.

APPENDIX A: SCALING OF THE LOWER BOND

Here we derive the asymptotic scaling of the probability $P_s(N)$ that the set of visited sites is $\mathcal{D}_t = [x_0, x_0 + N - 1]$, i.e., the probability that all the N visited sites have been discovered from the same edge x_+ of the visited territory. We define $p_+(n, N)$ as the classical splitting probability of a walker which starts at site $n + x_0$ to hit site $N + x_0$ before hitting site $x_0 - 1$. It is then
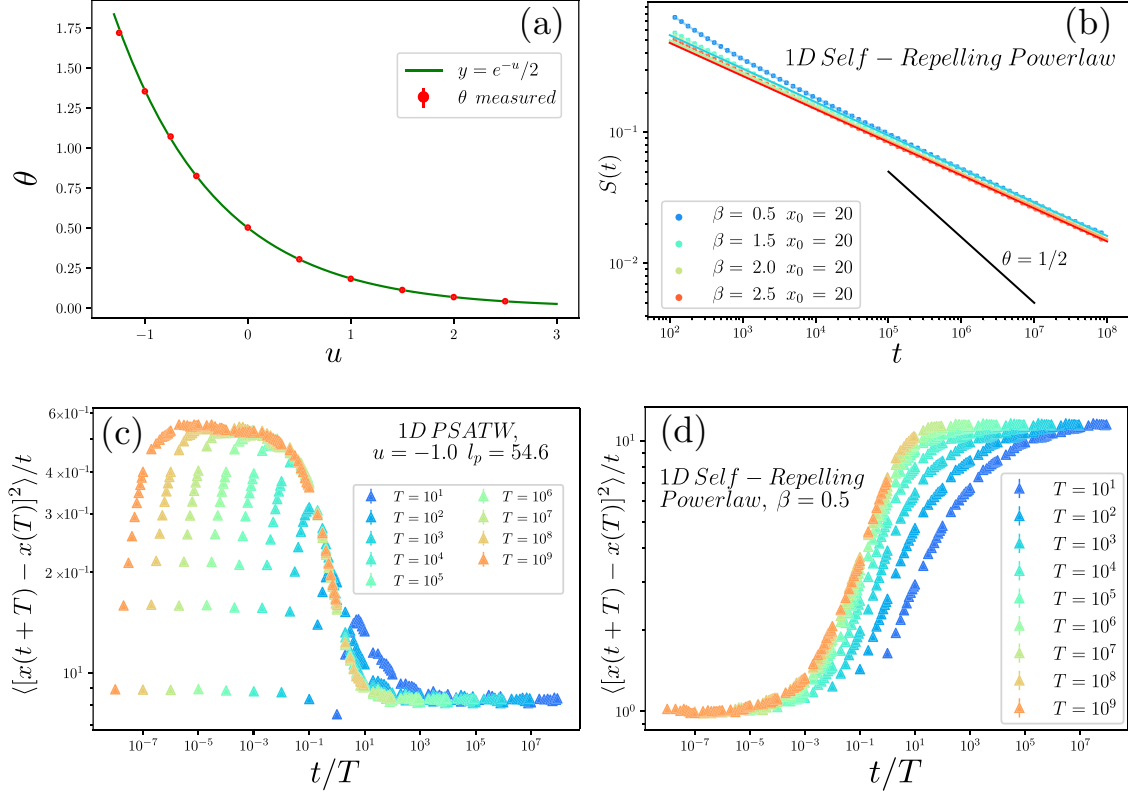
FIG. 5. Persistence and ageing of the persistent SATW model and the polynomial self-repelling walk. (a) Persistence exponent of the persistent SATW: Numerical simulation and exact prediction of Eq. (11) for $k \neq 0$. (b) Survival probability of polynomial self-repelling walk as a function of time (log scale) for different values of β. Numerical simulations (symbols) and power-law fit (plain line). (c) Ageing of the increments for the persistent SATW with $l_p = e^{2k}$ (normalized by the expected diffusive scaling at long times). Each curve corresponds to a fixed value of T. We observe a transient regime performed by the increments which disappears for $t \gg 1$. (d) Ageing of the increments of the polynomial self-repelling walk (normalized by the expected diffusive scaling at long times). Each curve corresponds to a fixed value of T.

clear that $P_s(N)$ verifies the following recurrence relation:

$$P_s(N + 1) = p_+(N - 1, N)P_s(N). \tag{A1}$$

The central observation underlying Eq. (A1) is that knowing the actual size of D_t, the future excursion inside the visited area is disconnected from the past. In the bulk of the visited territory the walker performs a classical symmetric nearest-neighbor random walk, thus for $0 < n < N$ one can write

$$p_+(n, N) = \tfrac{1}{2}p_+(n + 1, N) + \tfrac{1}{2}p_+(n - 1, N). \tag{A2}$$

The solution of (A2) can be written

$$p_+(n) = \lambda + \mu n, \tag{A3}$$

where λ and μ can be deduced from the boundary conditions

$$\begin{cases} p_+(0, N) = (1 - \beta) \, p_+(1, N) \\ p_+(N - 1, N) = \beta + (1 - \beta) \, p_+(N - 2, N) \end{cases}. \tag{A4}$$

This yields

$$\begin{cases} \lambda = \frac{1 - \beta}{2 + \beta(N - 3)} \\ \mu = \frac{\beta}{2 + \beta(N - 3)} \end{cases}. \tag{A5}$$

Combining the solution (A3) with (A5) and (A1) yields finally the exact recurrence relation given in the main text (1):

$$P_s(N + 1) = \left[1 - \frac{1 - \beta}{2 + \beta(N - 3)} \right] P_s(N). \tag{A6}$$

Let us notice that this relation is also valid for $N = 1$. Using the scaling of the harmonic series leads to the following asymptotics for $N \to \infty$, as is given in the main text:

$$P_s(N + 1) = 2^{-1} \exp \left\{ \sum_{i=2}^{N} \ln \left[1 - \frac{1 - \beta}{2 + \beta(i - 3)} \right] \right\} \underset{N \to \infty}{\sim} A_0 N^{-\frac{1-\beta}{\beta}}, \tag{A7}$$

where A_0 is a constant.

APPENDIX B: RECURRENCE RELATION FOR THE EVENTS P_s AND P_N

We define $P_s(N, M)$ as the probability that $M - N - 1$ sites have been visited successively from the same side starting with a visited domain of size $N + 1$. Similarly to (A1), one can write the following recurrence relation for $P_s(N, M)$:

$$P_s(N, M + 1) = \left[1 - \frac{1 - \beta}{2 + \beta(M - 3)} \right] P_s(N, M). \tag{B1}$$

Making use again of the $N \to \infty$ asymptotics of the harmonic series, we finally obtain from (B1)

$$P_s(N, M) = \exp \left[-\frac{1 - \beta}{\beta} \sum_{i=N+1}^{M-1} \frac{1}{j} + \sum_{i=N+1}^{M-1} O\left(\frac{1}{j^2} \right) \right] \underset{N \to \infty}{\sim} \frac{N^{\frac{1-\beta}{\beta}}}{M^{\frac{1-\beta}{\beta}}}. \tag{B2}$$

Defining $P_t(N)$ as the probability that a side change occurs when the number of distinct visited sites is N, we note that $P_t(N)$ is directly given by the splitting probability of (A6), which finally leads to

$$P_t(N) = \frac{1 - \beta}{2 + \beta(N - 3)} \underset{N \to \infty}{\sim} \frac{1 - \beta}{\beta N}. \tag{B3}$$

APPENDIX C: SCALING OF THE UPPER BOND

In order to extract the scaling of the upper bond we make a partition of all possible trajectories over the sequences of side changes. By definition, a side change occurs when the walker discovers a new site on the side opposite to the last newly visited site. The survival probability can then be written $S(N) = \sum_{k=0}^{2x_0} P(N, k)$, where $P(N, k)$ is the probability that k side changes occurred until N sites have been visited in the presence of the target at $x = 0$; it is therefore obtained by making a partition over the sequences of k side changes. To do so we introduce the variable a_i accounting for the total number of distinct visited sites when the ith side change occurs. Using the probability P_s and P_n defined in the previous section, the joint probability of a sequence $\{a_1, \ldots, a_k\}$ given a specific first step, knowing that N sites have been discovered can be written as

$$P(\{a_1, \ldots, a_k\}) = 2^{-1} P_s(1, a_1) P_t(a_1) \times \left[\prod_{i=2}^{k} P_s(a_{i-1}, a_i) P_t(a_i) \right] \times P_s(a_k, N). \tag{C1}$$

Making use of the scaling given by (B2) and (B3), and of Eq. (C1), we remark that all the a_i terms appearing in the $P_s(a_{i-1}, a_i)$ factors cancel each other, which leads to

$$P(\{a_1, \ldots, a_k\}) = A_K \left[1 + O\left(\frac{1}{a_1} \right) \right] N^{-\frac{1-\beta}{\beta}} (a_1 a_2 \times .. \times a_k)^{-1}, \tag{C2}$$

where we have defined the constant $A_K = 2^{-1} A_o (\frac{1-\beta}{\beta})^k$. In order to obtain $P(N, k)$, one then needs to sum over all sequences $\{a_1, \ldots, a_k\}$; in the limit $N \to \infty$, sums are approximated by integrals. In addition, one must enforce the constraint that the target is left unvisited. Because of this constraint, the walker cannot visit more than x_0 sites on the side (denoted as $-$) of the target; this will restrict the integration domains of the variables a_i and lead to an upper bound of $P(N, k)$ and therefore of the survival probability. In practice, four cases need to be analyzed depending on the direction of the first step (denoted as $\pm$) and the parity of k.

1. Scaling of $P_-(N, 2p)$

Summing over all sequences $\{a_1, \ldots, a_k\}$ that satisfy the constraint that the target is left unvisited yields:

$$P_-(N, 2p) < \frac{A_{2p}}{N^{\frac{1-\beta}{\beta}}} \int_1^{x_0} \int_{a_1}^N \int_{a_2}^{\min(x_0+a_2,N)} \int_{a_3}^N .. \int_{a_{2p-2}}^{\min(x_0+a_{2p-2},N)} \int_{N-x_0}^N \left[1 + O(a_1^{-1}) \right] \left(\prod_{i=1}^{2p} a_i \right)^{-1} da_1 .. da_{2p}. \tag{C3}$$

Let us notice that the lower bound of the last integral is given by $N - x_0$, because after the last change of side, the walker cannot discover more than x_0 successive sites without having crossed the target. This limits the integration range of the variable a_{2p}.

To obtain a useful upper bound for $P_-(N, 2p)$, it is in fact possible to relax all the constraints on variables a_i for $i \leqslant 2p - 1$, which leads to

$$P_-(N, 2p) < \frac{A_{2p}}{N^{\frac{1-\beta}{\beta}}} \int_1^N \int_1^N .. \int_1^N \int_{N-x_0}^N \left[1 + O\left(a_1^{-1}\right)\right] \left(\prod_{i=1}^{2p} a_i\right)^{-1} da_1 da_2 .. da_{2p}. \tag{C4}$$

All the integrals are now decoupled, which yields

$$P_-(N, 2p) < \frac{A_{2p}[1 + o(1)]}{N^{\frac{1-\beta}{\beta}}} \ln\left(\frac{N}{N - x_0}\right) \ln(N)^{2p-1}, \tag{C5}$$

from the which we finally deduce the upper bound,

$$P_-(N, 2p) = O\left(N^{-\frac{1-\beta}{\beta}}\right). \tag{C6}$$

2. Scaling of $P_+(N, k = 2p)$

Summing over all sequences $\{a_1, \ldots, a_k\}$ that satisfy the constraint that the target is left unvisited yields in this case:

$$P_+(N, 2p) < \frac{A_{2p}}{N^{\frac{1-\beta}{\beta}}} \int_1^N \int_{a_1}^{\min(a_1+x_0,N)} \int_{a_2}^N \int_{a_{2p-2}}^N .. \int_{a_{2p-1}}^{\min(a_{2p-1}+x_0,N)} \left[1 + O\left(a_1^{-1}\right)\right] \left(\prod_{i=1}^{2p} a_i\right)^{-1} da_1 da_2 .. da_{2p}. \tag{C7}$$

We next relax the min conditions in the upper bounds of the integrals to get

$$P_+(N, 2p) < \frac{A_{2p}}{N^{\frac{1-\beta}{\beta}}} \int_{1/N}^1 \int_{a_1}^{a_1+x_0/N} \int_{a_2}^1 \int_{a_{2p-2}}^1 ..$$

$$\int_{a_{2p-1}}^{a_{2p-1}+x_0/N} \left[1 + O\left(N^{-1} a_1^{-1}\right)\right] \left(\prod_{i=1}^{2p} a_i\right)^{-1} da_1 da_2 .. da_{2p}. \tag{C8}$$

We next perform iteratively the integrals two by two and make use of ($p > 1$):

$$\int_{a_{2p-2}}^1 \int_{a_{2p-1}}^{a_{2p-1}+x_0/N} \frac{1}{a_{2p-1} a_{2p}} da_{2p-1} da_{2p} < \int_{a_{2p-2}}^1 \frac{x_0}{N a_{2p-1}^2} da_{2p-1} < \frac{x_0}{N a_{2p-2}}. \tag{C9}$$

Applying these steps to Eq. (C8) leads to

$$P_+(N, 2p) < \frac{A_{2p}}{N^{\frac{1-\beta}{\beta}}} \int_{1/N}^1 \int_{a_1}^{a_1+x_0/N} .. \int_{a_{2p-3}}^{a_{2p-3}+x_0/N} \left[1 + O\left(\frac{1}{N^{-1} a_1^{-1}}\right)\right] \left(\prod_{i=1}^{2p-3} a_i\right)^{-1} \frac{x_0}{N a_{2p-2}^2} da_1 .. da_{2p-2}. \tag{C10}$$

Iterating these steps finally yields:

$$P_+(N, 2p) < \frac{A_{2p}}{N^{\frac{1-\beta}{\beta}}} \int_{1/N}^1 \frac{1 + O\left(\frac{1}{N a_1}\right)}{p - 1!} \frac{x_0^p}{N^p a_1^{p+1}} da_1, \tag{C11}$$

which satisfies

$$P_+(N, 2p) < \frac{A_{2p}}{N^{\frac{1-\beta}{\beta}}} \int_1^N \frac{1 + O\left(\frac{1}{a_1}\right)}{p - 1!} \frac{x_0^p}{a_1^{p+1}} da_1 = O\left(N^{-\frac{1-\beta}{\beta}}\right). \tag{C12}$$

Of note, the term $p = 0$ is exactly the lower bond $P_s(N)$, whose scaling is given by (A6).

3. Scaling of $P_+(N, 2p + 1)$

We reproduce the steps described above and obtain:

$$P_+(N, 2p + 1) < \frac{A_{2p+1}}{N^{\frac{1-\beta}{\beta}}} \int_1^N \int_{a_1}^{\min(a_1+x_0,N)} .. \int_{N-x_0}^N \left[1 + O\left(a_1^{-1}\right)\right] \left(\prod_{i=1}^{2p+1} a_i\right)^{-1} da_1 .. da_{2p+1}. \tag{C13}$$

Similarly to (C4) the lower bond of last integral is given by $N - x_0$. Applying the same steps (C5) and (C6) yields the expected scaling,

$$P_+(N, 2p + 1) = O\left(N^{\frac{1-\beta}{\beta}}\right). \tag{C14}$$

4. Scaling of $P_-(N, 2p + 1)$

We reproduce the steps described above and obtain:

$$P_-(N, 2p+1) < \frac{A_{2p+1}}{N^{\frac{1-\beta}{\beta}}} \int_1^{x_0} \int_{a_1}^N \int_{a_2}^{\min(a_2+x_0,N)} .. \int_{a_{2p}}^{\min(a_{2p}+x_0,N)} \left[1 + O\left(\frac{1}{a_1}\right)\right] \left(\prod_{i=1}^{2p+1} a_i\right)^{-1} da_1..da_{2p+1}. \qquad (C15)$$

We derive the same recurrence relation as obtained in (C8), which leads to

$$P_-(N, 2p+1) < \frac{A_{2p+1}}{N^{\frac{1-\beta}{\beta}}} \int_{1/N}^{x_0/N} \frac{1 + O\left(\frac{1}{Na_1}\right)}{p!} \frac{x_0^p}{N^p a_1^{p+1}} da_1. \qquad (C16)$$

For $p \geqslant 1$ we finally find

$$P_-(N, 2p+1) = O\left(N^{-\frac{1-\beta}{\beta}}\right). \qquad (C17)$$

We finally conclude this section by analyzing the odd case $p = 0$. One finds

$$P_+(N, 1) = A_1 \frac{1}{N^{\frac{1-\beta}{\beta}}} \int_{N-N_o}^N \left[1 + O\left(\frac{1}{a_1}\right)\right] \frac{1}{a_1} da_1 = o\left(N^{-\frac{1-\beta}{\beta}}\right), \qquad (C18)$$

and

$$P_-(N, 1) = \frac{A_1}{N^{\frac{1-\beta}{\beta}}} \int_1^{N_o} \left[1 + O\left(\frac{1}{a_1}\right)\right] \frac{1}{a_1} da_1 = O\left(N^{-\frac{1-\beta}{\beta}}\right). \qquad (C19)$$

Finally, all $P(N, k)$ are bounded by $O(N^{-\frac{1-\beta}{\beta}})$, and so is $S(N)$, as claimed in the main text.

APPENDIX D: SCALING OF THE SURVIVAL PROBABILITY AS A FUNCTION OF TIME

Here we extract the scaling with t of the survival probability $S(t)$, making use of the scaling with N derived for $S(N)$. To do so we consider the first passage time (FPT) distribution, defined as the probability to find the target exactly at time t. For a continuous time variable t, it satisfies:

$$F(t) = -\frac{dS}{dt} \underset{t \to \infty}{\sim} t^{-(1+\theta)}. \qquad (D1)$$

We rewrite $F(t)$ as a partition over the number of sites visited by the walker

$$F(t) = \sum_{N=0}^{\infty} F(N, t) = \sum_{N=0}^{\infty} \Phi_N(t) F(N), \qquad (D2)$$

where $\Phi_N(t)$ is the probability that the target is found at time t knowing that $N - 1$ sites have been visited before and $F(N)$ is the probability that the target is the Nth site discovered. In the large-N limit considering (D1) for $F(N)$ leads to

$$F(N) \underset{N \to \infty}{\sim} N^{-(\frac{1-\beta}{\beta}+1)}. \qquad (D3)$$

Making use of the diffusive scaling $\langle N \rangle \propto \sqrt{t}$ of the SATW [31], we assume that $\Phi_N(t)$ is a scaling function of the variable t/N^2. Taking into account normalization, we write

$$\Phi_N(t) = \frac{1}{N^2} f\left(\frac{t}{N^2}\right), \qquad (D4)$$

which is confirmed by numerical simulation (Fig. 6).

Rewriting in (D2) the sum as an integral in the limit $N \to \infty$, and making use of the change of variable $N \sim \sqrt{t}$, one obtains:

$$F(t) \sim t^{-1-\frac{1-\beta}{2\beta}}, \qquad (D5)$$

from the which one deduces using (D1) an analytical determination of θ.

APPENDIX E: INCREMENTS OF THE ERW IN DIMENSION 1

We start by reminding two key relations that will support the following analysis of the process [16]:

$$\langle \sigma_{t+1} | \sigma_1..\sigma_t \rangle = \frac{2\beta - 1}{t}[X(t) - X(0)], \qquad (E1)$$

$$\langle X(t)^2 \rangle = \frac{t}{3 - 4\beta} \left[1 - \frac{\Gamma(t + 4\beta - 2)}{\Gamma(t + 1)\Gamma(4\beta - 2)}\right], \qquad (E2)$$
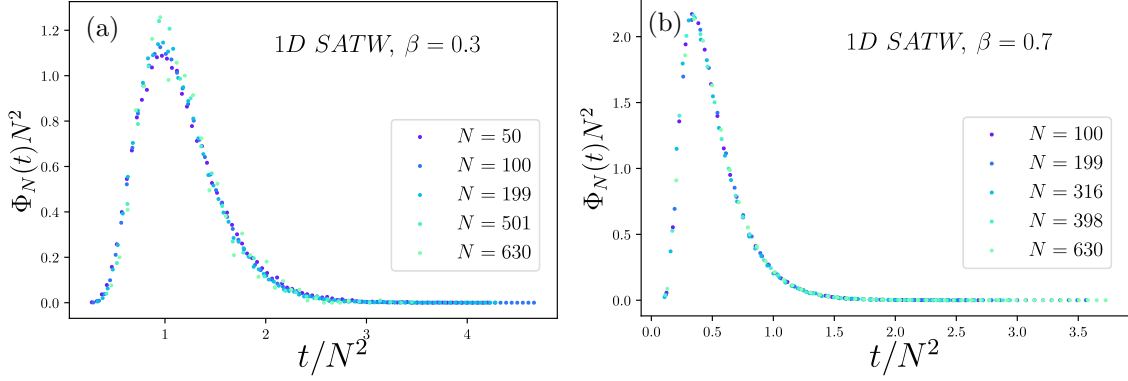
FIG. 6. Conditional pdf's of the FPT knowing that's the target is the Nth sites discovered for $\beta = 0.3$ (a) and $\beta = 0.7$ (b). The pdf's are generated along the normalized variable t/N^2 and are obtained with walkers starting at an initial distance $x_0 = 20$). The expected scaling in N^2 is obtained in the limit $N \gg x_0$.

where we have kept the notations of the main text: $\{\sigma_1, \sigma_2, .., \sigma_t\}$ denotes a sequence of steps up to time t. We remark from Eq. (E1) that the knowledge of the current position of the walker and time t are sufficient to determine the future evolution of the process. The process is non-Markovian because of the dependence on time t of the transition probabilities.

In this section, we determine a recurrence relation for the increments of the process. We denote by $p(\sigma_1, \sigma_2, .., \sigma_{T+t+1})$ the probability of a path $\{\sigma_1, \sigma_2, .., \sigma_{T+t+1}\}$. Making use of

$$X(t + T + 1) = X(t + T) + \sigma_{T+t+1}, \tag{E3}$$

yields

$$\langle [X(t + 1 + T) - X(T)]^2 \rangle = \langle \sigma_{T+t+1}^2 + [X(t + T) - X(T)]^2 + 2\sigma_{T+t+1}[X(t + T) - X(T)] \rangle$$

$$= 1 + \langle [X(t + T) - X(T)]^2 \rangle + 2 \left\langle \sigma_{T+t+1} \sum_{t'=T+1}^{t+T} \sigma_{t}' \right\rangle. \tag{E4}$$

We notice that the last term in bracket called (a) in the following encodes the correlations between the last step and the full trajectory after time T. We rewrite (a) by making explicit the partition over the past events $\{\sigma_1, .., \sigma_{t+T}\}$:

$$(a) = 2 \sum_{\{\sigma_1,...,\sigma_{t+T}\}} p(\sigma_1, .., \sigma_{t+T}) \left(\sum_{t'=T+1}^{t+T} \sigma_{t}' \right) \langle \sigma_{T+t+1} | \sigma_1 .. \sigma_{T+t} \rangle. \tag{E5}$$

Using (E1) and assuming with no loss of generality that the walker starts at 0 leads to

$$(a) = 2 \frac{2\beta - 1}{t + T} \sum_{\sigma_1,...,\sigma_{t+T}} p(\sigma_1, .., \sigma_{t+T})[X(t + T) - X(T)]X(t + T)$$

$$= 2 \frac{2\beta - 1}{t + T}[\langle X(t + T)^2 \rangle - \langle X(t + T)X(T) \rangle]. \tag{E6}$$

We next determine $\langle X(t + T)X(T) \rangle$ by using a second recurrence relation, conditionally to the position at time T:

$$\langle X(t + T + 1)|X(T) \rangle = \sum_{\sigma_{T+1},...,\sigma_{t+T}} p(\sigma_{T+1}, .., \sigma_{t+T}|X(T)) \langle \sigma_{t+T+1} + X(T + t)|X(T), \sigma_{T+1}.. \rangle$$

$$= \sum_{\sigma_{T+1},...,\sigma_{t+T}} p(\sigma_{T+1}, .., \sigma_{t+T}|X(T))[X(T + t) + \langle \sigma_{t+T+1}|X(T), \sigma_{T+1}.. \rangle]. \tag{E7}$$

Using (E1) for the second term inside the sum leads to

$$\langle X(t + T + 1)X(T)|X(T) \rangle = \sum_{\sigma_{T+1},...,\sigma_{t+T}} X(T)X(T + t) \left(1 + \frac{2\beta - 1}{T + t} \right) p[\sigma_{T+1}, .., \sigma_{t+T}|X(T)]$$

$$= \left(1 + \frac{2\beta - 1}{T + t} \right) \langle X(t + T)X(T)|X(T) \rangle. \tag{E8}$$

Iterating this relation we finally get after averaging over $X(T)$:

$$\langle X(t+T)X(T)\rangle = \langle X(T)^2\rangle \prod_{i=0}^{t-1}\left(1+\frac{2\beta-1}{T+i}\right).$$

(E9)

We now come back to Eq. (E4). Injecting (E9) into (E6) and finally replacing the squared position using (E2) leads to

$$\langle [X(t+1+T)-X(T)]^2\rangle = 1 + \langle [X(t+T)-X(T)]^2\rangle + 2\frac{2\beta-1}{3-4\beta}\left[1-\frac{\Gamma(t+T+4\beta-2)}{\Gamma(t+T+1)\Gamma(4\beta-2)}\right]$$

$$-2\frac{2\beta-1}{3-4\beta}\frac{T}{t+T}\left[1-\frac{\Gamma(T+4\beta-2)}{\Gamma(T+1)\Gamma(4\beta-2)}\right]\prod_{i=0}^{t-1}\left(1+\frac{2\beta-1}{T+i}\right).$$

(E10)

Iterating (E10) leads finally to an explicit expression of the increments:

$$\langle [X(t+T)-X(T)]^2\rangle = \frac{t}{3-4\beta} - \frac{2(2\beta-1)}{3-4\beta}\sum_{t'=1}^{t-1}\frac{\Gamma(T+t'+4\beta-2)}{\Gamma(T+t'+1)\Gamma(4\beta-2)}..$$

$$-\frac{2(2\beta-1)T}{3-4\beta}\left[1-\frac{\Gamma(T+4\beta-2)}{\Gamma(T+1)\Gamma(4\beta-2)}\right]\sum_{t'=1}^{t-1}\left[\frac{1}{T+t'}\prod_{i=0}^{t'-1}\left(1+\frac{2\beta-1}{T+i}\right)\right] - \frac{2(2\beta-1)}{3-4\beta}.$$

(E11)

Of note, the increments do not depend on the initial condition chosen for the first step. We will now exhibit the dominant behavior for $\beta < 3/4$. Under this hypothesis the ratio of the two Gamma functions decays and the second term of (E10) is negligible compared to t, which leads to

$$\frac{1}{t}\sum_{t'=1}^{t-1}\frac{2(2\beta-1)}{3-4\beta}\frac{\Gamma(T+t'+4\beta-2)}{\Gamma(T+t'+1)} = o_t(1).$$

(E12)

In turn the prefactor of the third term increases asymptotically linearly in T:

$$\frac{T}{3-4\beta}\left[1-\frac{\Gamma(T+4\beta-2)}{\Gamma(T+1)\Gamma(4\beta-2)}\right] \underset{T\to\infty}{\sim} \frac{T}{3-4\beta}.$$

(E13)

We next derive the asymptotics of the sum inside the third term. As a first step we derive the asymptotics of the product

$$\prod_{i=0}^{t'-1}\left(1+\frac{2\beta-1}{T+i}\right) = \exp\left[\sum_{i=0}^{t'-1}\ln\left(1+\frac{2\beta-1}{T+i}\right)\right]$$

$$= \left(\frac{T+t'}{T}\right)^{2\beta-1}[1+O(T^{-1})].$$

(E14)

Making use of (E14) then leads to the following asymptotics of third term:

$$\sum_{t'=1}^{t-1}\frac{\prod_{i=0}^{t'-1}\left(1+\frac{2\beta-1}{T+i}\right)}{T+t'} = \frac{(T+t)^{2\beta-1}-T^{2\beta-1}}{(2\beta-1)T^{2\beta-1}}[1+O(T^{-1})]$$

$$= \frac{1}{2\beta-1}\left[\left(1+\frac{t}{T}\right)^{2\beta-1}-1\right][1+O(T^{-1})].$$

(E15)

Finally, gathering the different results (E15), (E13), and (E12) together with (E11) leads to the final expression,

$$\langle [X(t+T)-X(T)]^2\rangle = \frac{t}{3-4\beta} - \frac{2t}{3-4\beta}\frac{T}{t}\left[\left(1+\frac{t}{T}\right)^{2\beta-1}-1\right] + o(t).$$

(E16)

APPENDIX F: CONDITIONAL PROBABILITY DENSITY FUNCTION OF THE ELEPHANT RANDOM WALK

In an attempt to make explicit the dependence of θ on the dynamics of the increments of the process, we remark, using the explicit form of $D_L(\beta)$ and Eq. (E16), that in the case of the ERW, θ can be reexpressed in terms of D_s, D_L only:

$$\theta = \frac{D_s}{2D_L(\beta)}.$$

(F1)

In the case of the ERW it suggests that θ could be determined only by the limits D_s, D_L of the time-dependent diffusion coefficient $D(t/T)$. We propose the following handwaving argument. We first write the 1 point pdf of the ERW started at $x_0 > 0$ with
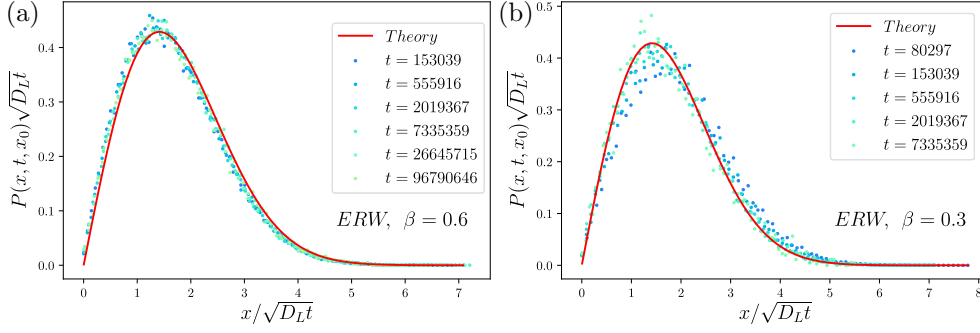
FIG. 7. Conditional pdf's taken at different times t for $\beta = 0.6$ (a) and $\beta = 0.3$ (b). The curves are both plotted along the normalized variable $x/\sqrt{D_L t}$. The conditional pdf is generated with the surviving walkers (starting with an initial number $N = 4, 5.10^5$). The solution of (F2) is plotted in red.

an absorbing target at $x = 0$ as $S(t, x_0)P(x, t, x_0)$, where $S(t, x_0)$ is the survival probability; this defines the conditional pdf $P(x, t, x_0)$, which is thus normalized. From the very definition of the ERW, it is clear that for $t \gg x_0^2/D_L$ and $x \to 0$, memory effects are irrelevant for all β: A step σ_i drawn from the past trajectory is $+1$ or -1 with probability $\sim 1/2$, because as many $+1$ and -1 steps have been made to ensure that $x \to 0$. In this regime, the random walk is therefore locally for $x \to 0$ a regular random walk of diffusion coefficient D_s. This implies in particular that Fick's law applies:

$$\frac{\partial S(t, x_0)}{\partial t} = -S(t, x_0)D_s \frac{\partial P(t, x, x_0)}{\partial x}\bigg|_{x=0}. \tag{F2}$$

To determine the conditional pdf $P(x, t, x_0)$ we make use of the fact that in the large-t limit the process behaves as a regular random walk of diffusion coefficient D_L, as shown by the Gaussian 1 point pdf in absence of absorbing target [36]. Assuming that the image method applies, one then obtains in the large t,

$$P(t, x, x_0) \simeq \frac{x}{2D_L t}e^{-\frac{x^2}{4D_L t}}, \tag{F3}$$

which was checked by numerical simulations in Fig. 7. Together with $S(t, x_0) \propto t^{-\theta}$, this result yields, making use of Eq. (F2), the exact expression of θ given in Eq. (F1). In order to assess the generality of Eq. (F1), we computed numerically D_L in the case of the SATW. Our results show that Eq. (F1) does not apply in this example.

APPENDIX G: PERSISTENCE EXPONENT OF THE PERSISTENT SATW

Here we derive analytically the persistence exponent of a generalization of the SATW to the case of a persistent random walker. The model is defined as follows: When the walker is inside the visited area, it performs a simple persistent walk characterized by two states $\pm$ that denote the direction of the last step. At each time step t the walker has a probability $Q = e^k/(e^{-k} + e^k)$ to reproduce the step performed at time $t - 1$. When the walker is at one of the edges of the visited domain, this probability is modified and reads $Q_1 = e^k/(e^{-k-u} + e^k)$.

Similarly to the case of the SATW, we define $p_\pm(n, N)$ as the splitting probability for a walker which starts at site n to hit site -1 before hitting site N knowing that N sites have been already visited and that the walker is initially in the state $\pm$ respectively. For $n \leqslant N - 2$ and $n \geqslant 2$ it is clear that $p_\pm(n, N)$ verifies the twofold recurrence relation:

$$\begin{cases} p_-(n, N) = Qp_-(n-1, N) + (1-Q)p_+(n+1, N) \\ p_+(n, N) = Qp_+(n+1, N) + (1-Q)p_-(n-1, N) \end{cases}. \tag{G1}$$

Combining (G1) leads to the second-order difference equations

$$\begin{cases} \frac{p_-(n-1,N)-Qp_-(n-2,N)}{1-Q} = \frac{Q}{1-Q}(p_-(n, N).. \\ \quad - Qp_-(n-1, N)) + (1-Q)p_-(n-1, N) \\ \frac{p_+(n+1,N)-Qp_+(n+2,N)}{1-Q} = \frac{Q}{1-Q}(p_+(n, N).. \\ \quad - Qp_+(n+1, N)) + (1-Q)p_+(n+1, N). \end{cases} \tag{G2}$$

The solutions of (G2) can be written as

$$p_\pm(n, N) = \lambda_\pm + \mu_\pm n. \tag{G3}$$

Equation (G2) then leads to

$$\mu_- n + \lambda_- = Q[\mu_-(n-1) + \lambda_-] + (1-Q)[\mu_+(n+1) + \lambda_+]. \tag{G4}$$

We deduce from (G4)

$$\begin{cases} \mu_- = \mu_+ \\ \lambda_- = -\frac{Q}{1-Q}\mu_- + \mu_+ + \lambda_+ \end{cases}. \tag{G5}$$

We finally use the conditions at the edges $(0, N-1)$ to derive $\{\mu_\pm, \lambda_\pm\}$

$$\begin{cases} p_-(0, N) = Q_1 + (1-Q_1)p_+(1, N) \\ p_+(N-1, N) = (1-Q_1)p_-(N-2, N) \end{cases}, \tag{G6}$$

which leads to

$$\begin{cases} \lambda_- = Q_1 + (1-Q_1)(\lambda_+ + \mu_+) \\ \lambda_+ + (N-1)\mu_+ = (1-Q_1)[\lambda_- + \mu_-(N-2)]. \end{cases} \tag{G7}$$

We finally obtain

$$\begin{cases} \mu_\pm = \frac{Q_1(1-Q)}{2Q_1 - 2Q - Q_1 Q - Q_1(1-Q)N} \\ \lambda_- = 1 + \frac{Q(1-Q_1)}{2Q_1 - 2Q - Q_1 Q - Q_1(1-Q)N} \\ \lambda_+ = 1 - \frac{Q_1 - Q - Q_1 Q}{2Q_1 - 2Q - Q_1 Q - Q_1(1-Q)N} \end{cases}, \tag{G8}$$

which fully determines the splitting probability for all n. The large N asymptotics of $p_-(0, N)$ is then given by

$$p_-(0, N) = 1 + \frac{Q(1-Q_1)}{2Q_1 - 2Q - Q_1 Q - Q_1(1-Q)N} \sim 1 - \frac{Q(1-Q_1)}{Q_1(1-Q)N}, \tag{G9}$$

which can be written in terms of u:

$$1 - p_-(0, N) \sim \frac{e^{-u}}{N}. \tag{G10}$$

Let us notice that the asymptotic regime is independent from the persistent parameter k. This expression parallels Eq. (B3) obtained from the SATW if one sets $\beta = 1/[1 + \exp(-u)]$. All technical steps can then be reproduced and lead to the following determination of the persistent exponent, which is independent of k:

$$\theta = \frac{e^{-u}}{2} = \frac{1-\beta}{2\beta}. \tag{G11}$$

APPENDIX H: OVERVIEW OF 1D PROCESSES WITH $\theta = 1/2$

Here we briefly review several processes that are known to be asymptotically diffusive. We check that in all cases where increments are stationary or stationary after a finite timescale, the persistent exponent is $1/2$ in agreement with our general criterion.

1. One-dimensional Rouse chain

We consider the Rouse model of a free polymer chain evolving in a 1D space. The chain is formed by N diffusing monomers connected by linear springs. It can be shown that the position process of a tagged monomer is diffusive at large times, ($t \gg \tau_0 N^2$, where τ_0 is a microscopic timescale) and that in this regime the increments are stationary [40]. Recently the exact algebraic decay of the survival probability has been obtained in Ref. [7], reporting the expected value $\theta = 1/2$.

2. Lévy flights and Lévy walks

We now consider the case of Lévy flights, where at each step a 1D walker performs in a random direction a jump whose length l is drawn from a distribution with the power-law tail given by

$$p(l) \sim \frac{1}{l^{1+\beta}}. \tag{H1}$$

By definition of the process the increments are stationary (the process is Markovian) and for $\beta \geqslant 2$ the process is diffusive. In previous studies two definitions of the first-passage time to the target have been used. In the first case, the target can be detected only when the walker ends a jump (arrival prescription). In the second case, the target can be detected as soon as it is crossed by the walker (crossing prescription). Let us note that both definitions lead to a persistent exponent $\theta = 1/2$ [8,41]. We now turn to Lévy walks. A Lévy walker performs a series of independent and randomly oriented ballistic jump at a constant
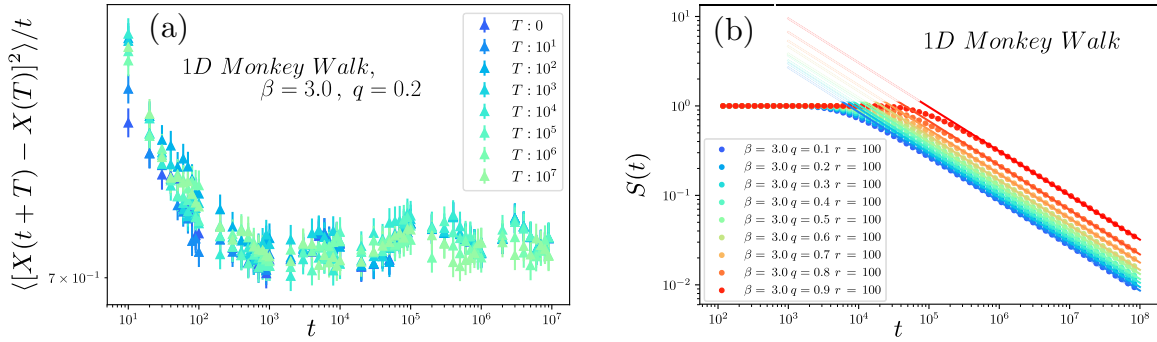
FIG. 8. (a) Ageing of the increments for the Monkey walk (normalized by the expected diffusive scaling at long times). Each curve corresponds to a fixed value of T. Even if the process is non-Markovian, the increments are stationary (respectively independent from T). (b) Survival probability as a function of time (log scale) with $\beta = 3.0$ for different values of q. Numerical simulations (symbols) and power-law fit (plain line). All the curves perform a power-law decay with $\theta = 1/2$.

speed, where each excursion of length l is drawn from the power-law distribution (H1). As for the Lévy flights for $\beta \geqslant 2$ the process is diffusive, performs asymptotically stationary increments [42–44]. In this case we recover a persistent exponent $\theta = 1/2$ [4,45].

3. Continuous-time random walk

The continuous-time random walk is a generalization of a simple random walk for the which the walker waits a random time between each jump. The successive waiting times are independent of each other and are drawn from a power-law distribution,

$$p(\tau) \sim \frac{1}{\tau^{1+\beta}}. \tag{H2}$$

For $\beta > 1$, the process is known to be diffusive and performs asymptotically stationary increments [46–49] and the persistent exponent is also equal to 1/2 [4].

4. Simple persistent walk

We consider the following persistent random walk model in continuous time and space:

$$\frac{dx}{dt} = v\sigma(t) + 2D\eta(t), \tag{H3}$$

where the random variable $\sigma(t)$ switches between ± 1 at a constant rate γ and $\eta(t)$ is Gaussian white noise. It is easily seen that the process is diffusive and displays transient ageing. The survival probability has been derived for this model as well as for similar versions [50–52] and shows that the persistent exponent is equal to 1/2.

5. Random walks with preferential relocation to visited places

To confirm the validity of our criterion we consider an another reinforced random walk with long-range memory effects introduced in Ref. [29]. It can be defined as follows in 1D. The walker starts at the origin on a discrete lattice of unit spacing. At each time step the walker has a probability $1 - q$ to jump on a site chosen uniformly among the nearest neighbors or with a complementary probability q to jump to a site that has been previously visited at a time $t' \in [0, t]$ with a probability given by a time-dependent kernel $p_t(t') = N(t)\frac{1}{(t-t'+1)^\beta}$ where $N(t)$ is a normalization constant. For $\beta > 2$ it has been shown that the process is diffusive at large times [29]. Here we present numerical results for $\beta = 3$. The increments are found to be stationary. The value $\theta = 1/2$ is verified for any parameter $q \in]0, 1[$, in agreement with our criterion (see Fig. 8).

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