Fokker-Planck equation for Coulomb relaxation and wave-particle diffusion: Spectral solution and the stability of the Kappa distribution to Coulomb collisions

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(Received 15 September 2020; accepted 2 November 2020; published 2 December 2020)

The present paper considers the time evolution of a charged test particle of mass m in a constant temperature heat bath of a second charged particle of mass M. The time dependence of the distribution function of the test particles is given by a Fokker-Planck equation with a diffusion coefficient for Coulomb collisions as well as a diffusion coefficient for wave-particle interactions. For the mass ratio $m/M \to 0$, the steady distribution is a Kappa distribution which has been employed in space physics to fit observed particle energy spectra. The time dependence of the distribution functions with some initial value is expressed in terms of the eigenvalues and eigenfunctions of the linear Fokker-Planck operator and also interpreted with the transformation to a Schrödinger equation. We also consider the explicit time dependence of the distribution function with a discretization of the Fokker-Planck equation. We study the stability of the Kappa distribution to Coulomb collisions.

DOI: 10.1103/PhysRevE.102.062103

I. INTRODUCTION

The current paper is directed towards the development of a physical model to identify the processes that maintain steady nonequilibrium Kappa distributions that are observed via satellite diagnostics in a large number of space plasma environments [1–5]. In addition to the Kappa distribution, there is a very large number of different physical and chemical systems which are characterized by nonequilibrium distributions [6–10]. In many such situations, the processes that drive the system from equilibrium and those that restore the system to equilibrium can be identified. This aspect of the departure from equilibrium in space physics and astrophysics and the establishment of steady Kappa distributions have been discussed [11–18] with regards to the timescales for the relaxation to equilibrium for astrophysical plasmas. It is often possible to quantify the collisional processes leading to nonequilibrium states in terms of kinetic equations such as the Boltzmann equation [19], the Fokker-Planck equation [20], the Vlasov equation [21], or a master equation [22]. These have been employed for decades by many researchers in different fields to account for the departure from equilibrium distributions in diverse systems. Fokker-Planck equations arise as the limiting form of the Boltzmann equation for binary disparate mass systems [23], for systems of charged particles which experience Coulomb collisions [24,25], the continuum representation of discrete Master equations [26,27], stellar systems [28–31], for chemically reactive systems [32], and in economics [33].

Numerous researchers [2,4,34] suggested that the origin of the Kappa distribution can be rationalized with the nonextensive entropy formalism developed by Tsallis [35–37]. A complete bibliography can be found elsewhere [2,38]. The nonextensive entropy approach to describe nonequilibrium phenomenon remains controversial as noted by several authors [39–48]. In the kinetic theory applications discussed here, there is no need to introduce the concept of nonextensive thermodynamics and q-extensive distributions [49]. The steady nonequilibrium distributions for these systems depend on Kappa (κ) and the mass ratio m/M and a large number of different nonequilibrium steady states are calculated for which the Kappa distribution is found only for a particular choice of these variables [11,12].

The nonextensive approach by Tsallis is based on the definition of an entropy functional of the form

$$S(q) = \frac{1}{q-1}\left[1 - \int f^q(v)d\mathbf{v}\right], \quad (1)$$

parameterized with q. In the limit $q \to 1$, we have, with l'Hopital's rule, that $\lim_{q\to 1} S(q) \to -\int f \ln(f) d\mathbf{v}$ which is the basis for Boltzmann's H-theorem for a dilute monatomic gas and the approach to a Maxwellian at equilibrium [50]. For systems with discrete quantum energy levels, the integral is replaced by a summation and the equilibrium distribution is generally referred to as the Maxwell-Boltzmann distribution. The equilibrium distributions are obtained by finding the extremum of the Boltzmann-Gibbs entropy with the method of Lagrange multipliers subject to the known values of the density and average energy [50]. This procedure provides the well-known equilibrium Maxwell-Boltzmann, Fermi-Dirac, and Bose-Einstein distributions. This formalism does not provide any information regarding nonequilibrium distributions.

The Tsallis formalism has been widely adopted in space physics as a rationale for the many satellite verifications of particle energy distributions as the Kappa distribution

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[2,4,51,52] given by

$$f_\kappa(x) = C(\kappa)\left[\frac{1}{1+\frac{x^2}{\kappa+1}}\right]^{\kappa+1}, \tag{2}$$

where $x = v/v_{th}$ is the reduced particle speed and $v_{th} = \sqrt{2kT_b/m}$ is the thermal speed with k the Boltzmann constant and m the particle mass. The heat bath temperature is denoted by T_b. The extent of the departure of $f_\kappa(x)$ from a Maxwellian is determined by the value of κ and $f_\kappa(x)$ tends to a Maxwellian for $\kappa \to \infty$. The Kappa distribution is normalized according to $4\pi\int_0^\infty f_k(x)x^2dx = 1$ so that $C(\kappa) = \Gamma(\kappa+1)/\{\Gamma(\kappa-\frac{1}{2})[\sqrt{\pi(\kappa+1)}]^3\}$. It has an asymptotic power-law dependence for large speed x, and joins smoothly with a Maxwellian distribution at low speed.

The basis for the curent paper is the Fokker-Planck equation for a test particle of mass m in a heat bath of a second species of mass M as described in Sec. II [11,12]. There are two collisional processes included, namely Coulomb collisions which drive the system to equilibrium and wave particle interactions which are responsible for the departure from equilibrium. The steady distribution is a Kappa distribution defined by a Pearson ordinary differential equation [11,12]; see Eq. (8).

By contrast to the previous works [11,12] in which the Chang-Cooper finite difference solution of the Fokker-Planck equation was used, in this paper we also use a spectral solution of the Fokker-Planck equation to represent the time evolution of the distribution function in terms of the eigenvalues and eigenfunctions of the Fokker-Planck operator. The eigenvalue spectrum of the Fokker-Planck operator is interpreted in terms of the potential in the Schrödinger equation corresponding to the Fokker-Planck equation. This relationship arises owing to the SUPERSYMETRIC quantum mechanics for potential functions arising from a Fokker-Planck equation [12,20]. The numerical evaluation of the eigenvalue spectrum is provided with the approximation of the eigenfunctions expanded in the Maxwell polynomials orthogonal with respect to the weight function $w(x) = x^2e^{-x^2}$ [53,54].

The Fokker-Planck equation for Coulomb collisions and a collision term for wave particle interactions is discussed in Sec. II with the spectral solution provided in Sec. III. In Sec. IV, the eigenvalue problem for the Fokker-Planck equation is interpreted with the solution of the isospectral Schrödinger equation. In Sec. V, we present the time evolution of the distribution to equilibrium from an initial Kappa distribution in the absence of the wave particle interaction and the speed-dependent relaxation times are evaluated. Analogously the time evolution of the distribution from an initial Maxwellian to a Kappa distribution in the presence of the wave particle interaction is also presented and the speed-dependent relaxation times are evaluated for this process. An analysis is provided as to the physics of the creation of a Kappa distribution. A summary of the results is presented in Sec. VI.

II. FOKKER-PLANCK EQUATION

The Fokker-Planck equation for the relaxation of a charged test-particle of mass m interacting via Coulomb collisions with background charged particles of mass M at equilibrium is given by

$$\frac{\partial f_0(v,t')}{\partial t'} = \frac{A_1}{v^2}\frac{\partial}{\partial v}\left[D_1(v)\left(1+\frac{kT_b}{mv}\frac{\partial}{\partial v}\right)\right]f_0(v,t'), \tag{3}$$

where $A_1 = (4\pi Ne^4Z^2Z_b^2/mM)\ln\Lambda$ with the diffusion coefficient

$$D_1(v) = \text{erf}\left(\sqrt{\frac{Mv^2}{2kT_b}}\right) - \sqrt{\frac{2Mv^2}{\pi kT_b}}\exp\left(-\frac{Mv^2}{2kT_b}\right), \tag{4}$$

as discussed elsewhere [11,12,24,54,55]. Equation (4) arises from the Coulomb cross section for charged particle collisions and averaged over the Maxwellian distribution function of the background ions. The parameters in A_1 are the density N of the background species, the electronic charge e, the atomic weights of the background species Z_b, and the test particle Z, respectively. The Coulomb logarithm is $\ln\Lambda$. The steady-state solution of Eq. (3) is clearly a Maxwellian.

The system described by the Fokker-Planck equation Eq. (3) is perturbed with the introduction of an energization mechanism. In the space plasma environment, the energization mechanism could include quasilinear wave-particle interactions modeled by a diffusion in velocity space [13,14,56–58] and thus an appropriate Fokker-Planck equation is given by

$$\frac{\partial f(v,t')}{\partial t'} = \frac{A_1}{v^2}\frac{\partial}{\partial v}\left[D_1(v)\left(1+\frac{kT_b}{mv}\frac{\partial}{\partial v}\right)\right]f(v,t') + \frac{B_1}{v^2}\frac{\partial}{\partial v}\left[v^2D_2(v)\frac{\partial}{\partial v}f(v,t')\right], \tag{5}$$

where B_1 gives the strength of the wave-particle interaction as modeled with the diffusion coefficient $D_2(v)$.

With the introduction of the dimensionless time $t = t'/t_0$, where $t_0 = [N\sigma_{\text{eff}}\sqrt{2kT_b/M}]^{-1}$ with $\sigma_{\text{eff}} = [4\pi NZ^2Z_b^2e^4\ln\Lambda]/(2kT_b)^2$ the Fokker-Planck equation, Eq. (5), can be written in terms of $\alpha = 2B_1/A_1$ as a measure of the strength of the wave-particle interaction that energizes the particles relative to Coulomb collisions. The main objective of the present paper is the mechanism for the establishment of a steady Kappa distribution that arises from a competition of collisional processes [15,16].

The steady distribution obtained by setting $\partial f/\partial t' = 0$ in Eq. (5) is given by

$$\frac{df_{ss}(x)}{f_{ss}(x)} = -\left[\frac{2x}{1+\alpha v_{th}x^3\frac{D_2(v_{th}x)}{\hat{D}_1(z)}}\right]dx, \tag{6}$$

where

$$\hat{D}_1(z) = \text{erf}(z) - \frac{2z}{\sqrt{\pi}}e^{-z^2}, \tag{7}$$

$v_{th} = \sqrt{2kT_b/m}$, $z = \sqrt{\gamma}x$, and $\gamma = M/m$. It is this mass-dependent dimensionless diffusion coefficient that controls the Coulomb relaxation to equilibrium and the features of $f_{ss}(x)$. It is clear from Eq. (6) that the steady distribution is a Maxwellian for $\alpha = 0$, that is in the absence of wave-particle interactions. For $\alpha \neq 0$, the steady distribution is a non-Maxwellian distribution, the features of which also depend on the mass ratio, γ.

The Coulomb cross section that varies as $1/g^4$ where $\mathbf{g}$ is the relative velocity of two charged particles in a collision does not appear explicitly in Eq. (5). The velocity dependence of this steady-state distribution function depends on both $\hat{D}_1(z)$ and $D_2(v_{th}x)$. The inverse velocity dependence of the wave-particle diffusion coefficient $D_2(v) \propto 1/v$ arises from the analysis of weak turbulence theory; see Eq. (10.31) in Ref. [57].

It is useful to examine the dependence of $\hat{D}_1(\sqrt{\gamma}x)$ versus x. It is clear that for $x \to \infty$, $\hat{D}_1(\sqrt{\gamma}x) \to 1$, and for small x,

$$\lim_{z\to 0} \hat{D}_1(z) \approx \frac{4}{3\sqrt{\pi}}\gamma^{3/2}x^3.$$

A dimensionless collision frequency $\nu(x) = \frac{3}{4z^3\sqrt{\pi}}D_1(z)/(4z^3)$ can be defined. It is this strong mass dependence that controls both the approach to a steady state and the features of the steady-state distribution that are emphasized in the present paper. It is easy to see that in the limit, $\gamma \to \infty$, $\hat{D}_1(z) \to 1$ and with $D_2(v_{th}x) = 1/(v_{th}x)$, the steady distribution function is then defined by

$$\frac{df_\kappa(x)}{f_\kappa(x)} = -\frac{2x}{1+\alpha x^2}dx, \tag{8}$$

which can be recognized as the Pearson ordinary differential equation [59,60] that defines the Kappa distribution Eq. (2), with $\kappa = (1-\alpha)/\alpha$. This result arises owing to the particular speed dependence of the drift and diffusion coefficients in the Fokker-Planck equation that gives Eq. (8). For $x \to 0$, the distribution approaches a Maxwellian and for $x \to \infty$ the distribution is a power law. Equation (8) does not arise owing to long-range forces [61], a collisionless weakly coupled plasma far from equilibrium [2,38], multiplicative noise [35], nonextensive entropy [4,34,38,49,51,62], or Levy flights [63].

The analyses by Yoon *et al.* [64] and Kim *et al.* [65] are based on the detailed physics of the interaction of electrons with Whistler-type waves and yield a Kappa distribution defined by Eq. (8) owing to the particular v dependence of the diffusion coefficients. The establishment of the Kappa distribution is dependent on the specific speed dependence of $D_1(v)$ rather than on the wave spectrum. Equation (48) in Ref. [66] confirms the approach here and in the previous work [12].

The section that follows presents some numerical results that further demonstrate the range of nonequilibrium distributions with this simple model for which only a subset are Kappa distributions. The works by Ma and Summers [13] for Whistler waves and Hasegawa *et al.* [14] for a radiation field use similar Fokker-Planck equations with drift and diffusion coefficients that yield the same ordinary differential equation, Eq. (8), for the steady-state distribution Eq. (2). Similarly, the analysis of anomalous diffusion in an optical lattice by Lutz [67] gives a Kappa distribution, Eq. (2), with the appropriate ratio of the drift to diffusion coefficients as in Eq. (3) of Ref. [67] leading exactly to Eq. (8).

We substitute $f(x,t) = f_{ss}(x)g(x,t)$ in Eq. (5) and use $F(x)f_{ss} + G(x)\frac{\partial f_{ss}}{\partial x} = 0$, where $F(x) = D_1(v_{\rm th}x)$ and $G(x) = \frac{D_1(v_{\rm th}x)}{2x} + \frac{\alpha v_{\rm th}x^2 D_2(v_{\rm th}x)}{2}$. Thus we have that

$$\frac{\partial g(x,t)}{\partial t} = -\left[\frac{F(x)-G'(x)}{x^2\sqrt{\gamma}}\frac{\partial}{\partial x} - \frac{G(x)}{x^2\sqrt{\gamma}}\frac{\partial^2}{\partial x^2}\right]g(x,t). \tag{9}$$

We define $A(x) = \frac{F(x)-G'(x)}{x^2\sqrt{\gamma}}$ and $B(x) = \frac{G(x)}{x^2\sqrt{\gamma}}$. so that Eq. (5) can be written as

$$\frac{\partial g(x,t)}{\partial t} = -\left[A(x)\frac{\partial}{\partial x} - B(x)\frac{\partial^2}{\partial x^2}\right]g(x,t), \tag{10}$$

where we identify the linear operator, L, as

$$L = A(x)\frac{\partial}{\partial x} - B(x)\frac{\partial}{\partial x^2}, \tag{11}$$

which is Hermitian owing to the zero-flux boundary condition

$$x^2 f_{ss}(x)B(x)\frac{\partial g(x)}{\partial x}\bigg|_0^\infty = 0. \tag{12}$$

We consider a solution of the Fokker-Planck equation in terms of the eigenfunctions, $\phi_n(x)$, and eigenvalues, λ_n, of L defined by

$$L\phi_n(x) = \lambda_n\phi_n(x), \tag{13}$$

where the eigenfunctions are orthogonal with respect to the weight function $w(x) = x^2 f_{ss}(x)$, that is,

$$\int_0^\infty x^2 f_{ss}(x)\phi_n(x)\phi_m(x)\,dx = \delta_{nm}. \tag{14}$$

The solution of Eq. (13) gives the spectral solution of Eq. (10) so that the time-dependent distribution function is given by

$$f(x,t) = \sum_{n=0}^\infty c_n f_{ss}(x)\phi_n(x)e^{-\lambda_n t}, \tag{15}$$

where the coefficients

$$c_n = \int_0^\infty x^2\phi_n(x)f(x,0)\,dx, \tag{16}$$

are determined with the initial distribution $f(x,0)$. The main objectives of this analysis are to study the relaxation of the Kappa distribution to a Maxwellian owing to Coulomb collisions as well as the establishment of a Kappa distribution from a Maxwellian with the inclusion of the wave-particle diffusion in Eq. (5).

III. SPECTRAL SOLUTION OF FOKKER-PLANCK EQUATION

We represent the solution of the the Fokker-Planck equation, Eq. (10), in terms of the eigenvalues λ_n and eigenfunctions $\phi_n(x)$ of the Fokker-Planck operator, Eq. (13). We propose to use this representation of the solutions of the Fokker-Planck equation to explain the stability against collision of the Kappa distribution, especially the high-energy tail.

The numerical determination of the eigenvalues and eigenfunctions has been described in a previous publication [53] (and references therein) and we here provide a brief overview of the methodology. The Maxwell polynomial basis set, $\{M_n(x)\}$, orthonormal with respect to the weight function $w(x) = x^2e^{-x^2}$ is used to define a nonuniform grid $\{x_i\}$ that coincides with the points of the Gauss quadrature rule defined by this nonclassical basis set [53], that is,

$$\int_0^\infty w(x)f(x)dx \approx \sum_{i=1}^N w_i f(x_i). \tag{17}$$

TABLE I. Convergence of the eigenvalues of the Fokker-Planck operator L, Eq. (18), for $\gamma = 0.02$; N is the number of quadrature grid points.

N	λ_1	λ_4	λ_7	λ_{10}
2	0.06517			
3	0.05840			
4	0.05840			
5		0.4663		
8		0.2174	1.4509	
11		0.2159	0.3897	3.3142
15		0.2159	0.3476	0.5614
20			0.3464	0.4602
25			0.3464	0.4484
30				0.4478
35				0.4478

The quadrature points x_i and weights w_i are determined with the diagonalization of the Jacobi matrix defined in terms of the recurrence coefficients in the three term recurrence relation for the polynomials [53]. For many applications, the Maxwell polynomials provide a rapid convergence for such eigenvalue problems.

The matrix representative of the Fokker-Planck operator L_{ij} in the physical space of quadrature points $\{x_i\}$ is

$$L_{ij} = -\sum_{k=1}^{N} B(x_k)[D_{ki} + h(x_k)\delta_{ij}][D_{kj} + h(x_k)\delta_{ij}], \tag{18}$$

where

$$h(x) = \frac{w'(x)}{2w(x)} - \frac{[x^2 f_{ss}(x)]'}{2x^2 f_{ss}(x)}, \tag{19}$$

and the matrix $D_{ij} = \sqrt{w_i w_j}\sum_{n=0}^{N-1} P_n'(x_i)P_n(x_j)$ is the physical space representation of the derivative operator, that is,

$$\left.\frac{d\hat{f}(x)}{dx}\right|_{x=x_i} = \sum_{j=1}^{N} D_{ij}\hat{f}(x_j), \tag{20}$$

TABLE II. Convergence of the eigenvalues of the Fokker-Planck operator L, Eq. (18), for $\gamma = 0.05$; N is the number of quadrature grid points.

N	λ_1	λ_2	λ_3	λ_4	λ_5
2	0.1587				
4	0.1392	0.2713	0.6952		
6	0.1392	0.2602	0.3777	0.6081	1.772
8		0.2600	0.3629	0.4710	0.8441
10		0.2600	0.3617	0.4484	0.5533
15			0.3617	0.4430	0.5051
20				0.4430	0.5010
25					0.5008
30					0.5005
35					0.4933
40					0.4947
45					0.5005
50					0.5004

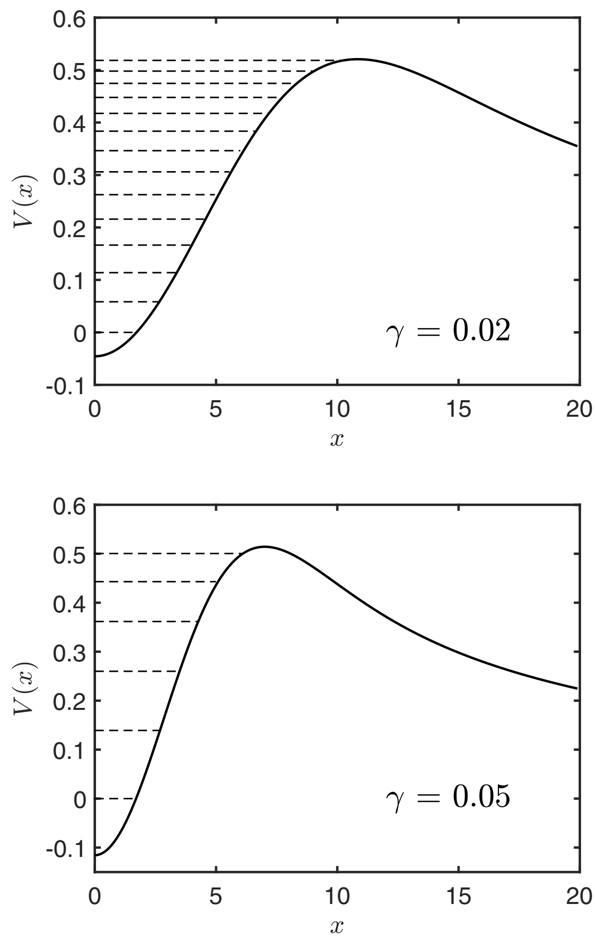

FIG. 1. Potential $V(x)$ of the Schrödinger equation corresponding to the Fokker-Planck equation with the Coulomb collision term $\alpha = 0$ and mass ratio $\gamma = 0.02,\ 0.05$, the horizontal lines denote the eigenvalues.

as discussed in Ref. [53]. The eigenvalues of the Fokker-Planck operator are approximated with the eigenvalues of the matrix **L** in Eq. (18).

The eigenvalues and eigenfunctions are calculated from the diagonalization of the N-by-N matrix representation of the Fokker-Planck operator in Eq. (18). For the small mass ratios and in the absence of the wave-particle interaction, the spectrum consists of a discrete spectrum and a continuum. We thus write the evolution of the distribution function as a sum over the discrete spectrum and an integral over the continuum states, that is,

$$f(x,t) = f_{ss}(x)\left[\sum_{n=0}^{N} c_n\psi_n(x)e^{-\lambda_n t} + \int_{\lambda_*}^{\infty} c(\lambda)\psi(\lambda,x)e^{-\lambda t}\,d\lambda\right], \tag{21}$$

where λ_* marks the boundary between the bound states and the continuum. Although the eigenvalues and eigenfunctions in the continuum are not convergent versus N, the integral over the continuum eigenstates in Eq. (21) does converge [54].

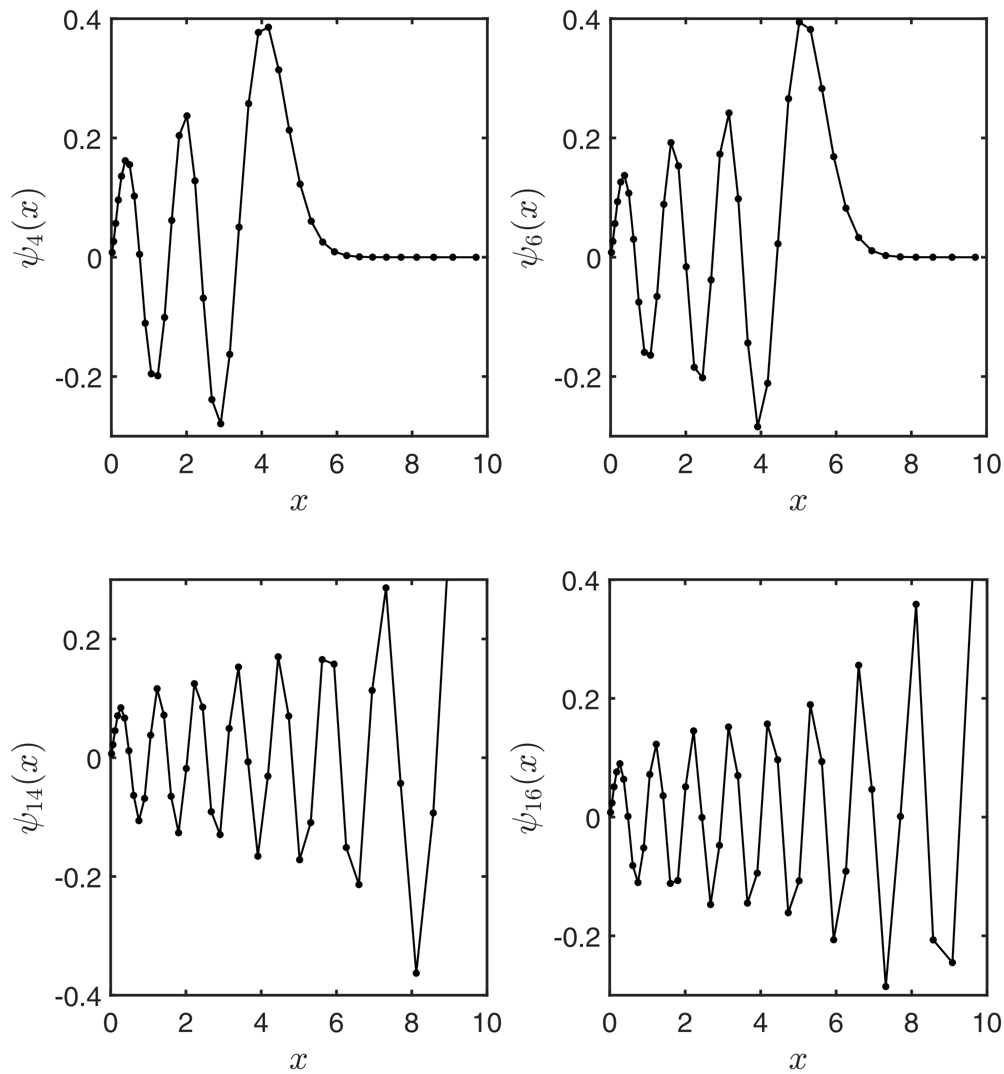

FIG. 2. Eigenfunctions for bound states (top) and for continuum states (bottom) for mass ratio $\gamma = 0.02$.

The presence of the continuum eigenstates is best understood with the transformation of the Fokker-Planck eigenvalue problem to the Schrödinger equation discussed in Sec. IV.

The convergence of the bound eigenvalues for $\gamma = 0.02$ is shown in Table I. For this mass ratio, the convergence of λ_1, λ_4, λ_7, and λ_{10} converge to four significant figures with 4, 15, 25, and 35 quadrature points, respectively. The convergence of the eigenvalues, λ_1 to λ_5 for $\gamma = 0.05$ is shown in Table II and 6, 10, 15, and 20 quadrature points are required for convergence to four significant figures. The convergence of λ_5 is not monotonic from above as it should be owing to the variational theorem. This slow convergence is the evidence that this state is very loosely bound.

The eigenfunctions for the bound and continuum states for $\gamma = 0.02$ are shown in Figs. 2(A) and 2(B), respectively. The eigenfunction $\psi_5(x)$ is presumably in the continuous spectrum of the Fokker-Planck operator and hence nonconvergent. For larger mass ratios ($\gamma > 0.3$) or with the wave-particle interaction, the spectrum is completely continuous except for $\lambda_0 = 0$ with $\psi_0(x) = f_{ss}(x)$.

IV. SPECTRUM OF THE FOKKER-PLANCK OPERATOR IN TERMS OF THE EQUIVALENT SCHRÖDINGER EQUATION

The eigenvalue spectrum of the Fokker-Planck operator can be understood with the transformation of the Fokker-Planck equation to a Schrödinger equation, $H\psi_n(y) = \lambda_n\psi_n(y)$, isospectral with the Fokker-Planck equation. This problem belongs to the class of problems in SUPERSYMMETRIC (SUSY) quantum mechanics [53]. With the change of variable

$$y(x) = \int_0^x \frac{1}{\sqrt{B(x')}}dx', \tag{22}$$

and the definition

$$C[y(x)] = \frac{1}{2}\int^y \frac{A(y')}{\sqrt{B(y')}}dy' + \frac{1}{4}\ln B(y), \tag{23}$$

we obtain the time-independent Schrödinger equation

$$\frac{d^2\psi_n}{dy^2} - [V(y) - \lambda_n]\psi_n = 0, \tag{24}$$

where the potential $V(y)$ is defined by

$$V(y) = C'^2(y) - C''(y), \tag{25}$$

where $C'(y) = dC/dy$ and $C''(y) = d^2C(y)/dy^2$. To compute the potential, we evaluate C' and C'' with Eq. (23), that is,

$$C'[y(x)] = [2A(x) + B'(x)]/4\sqrt{B(x)},$$

and

$$C''[y(x)] = [4A'(x)B(x) + 2B''(x)B(x) - 2A(x)B'(x) - B'^2(x)]/4B(x).$$

The potential is thus given by

$$V[y(x)] = \frac{4A^2(x) + 5B'^2(x) - 8A'(x)B(x) + 12A(x)B'(x) - 4B''(x)B(x)}{16B(x)}. \tag{26}$$

In the absence of wave-particle interactions, that is, with $\alpha = 0$ and $\kappa \to \infty$, the potential in Eq. (26) is given explicitly by [54]

$$V(x) = \frac{D_1(v_{\rm th}x)}{\sqrt{\gamma}x}\left(1 - \frac{9}{16x^4}\right) - 2\sqrt{\pi}\gamma\left[1 + \frac{\gamma^2}{2} - \frac{3}{8x^2}\right]e^{-\gamma^2x^2} - \frac{\gamma^{5/2}x}{2\pi D_1(v_{\rm th}x)}e^{-2\gamma^2x^2}, \tag{27}$$

with $V(0) = -2\gamma(1 + \gamma^2/2)/\sqrt{\pi}$. It is useful to note two limiting values of the potential, namely, $\gamma \to 0$, for which $V(x) = (2\gamma/\sqrt{\pi})(x^2 - 3)$ and the spectrum is discrete. For

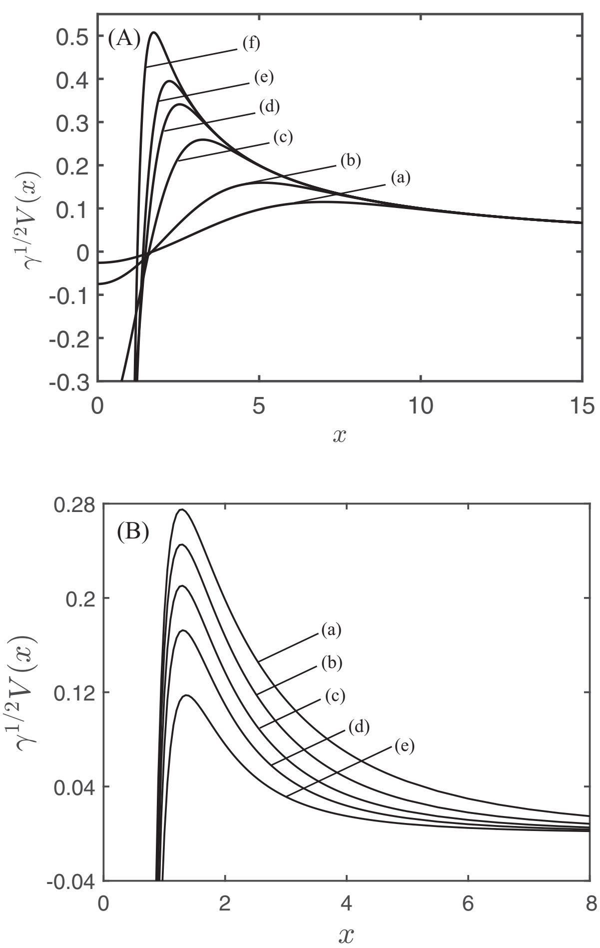

FIG. 3. Potential $\gamma^{1/2}V(x)$ of the Schrödinger equation corresponding to the Fokker-Planck equation; (A) $\alpha = 0$ and mass ratios $\gamma =$ (a) 0.05, (b) 0.1, (c) 0.3, (d) 0.6, (e) 0.9, and (f) 2. (B) (a) $\alpha = 1/20$, $\kappa = 19$, (b) $\alpha = 1/10$, $\kappa = 9$, (c) $\alpha = 1/6$, $\kappa = 5$, (d) $\alpha = 1/4$, $\kappa = 3$, and (e) $\alpha = 2/5$, $\kappa = 3/2$, and mass ratios $\gamma \to \infty$.

$\gamma \to \infty$, then $V(x) \sim (1 - 9/16x^4)/(\sqrt{\gamma}x)$. With the inclusion of wave-particle interactions, $\alpha \neq 0$ and $\gamma \to \infty$, the potential in Eq. (26) is of the form

$$V(x) = -\frac{9\alpha^2x^4 + 14\alpha x^2 - 16x^4 + 9}{32\sqrt{\gamma}(\alpha x^7 + x^5)}. \tag{28}$$

The potential functions for the Schrödinger equation corresponding to the Fokker-Planck equation for Coulomb collisions are shown in Fig. 1 for $\gamma = 0.02$ and $\gamma = 0.05$. The horizontal lines show the energy levels for each potential and there appears to be six bound states for $\gamma = 0.05$ and 14 bound states for $\gamma = 0.02$. Thus, we can better interpret the convergence of the eigenvalues versus the number N of quadrature points in the representation of the Fokker-Planck operator $\mathbf{L}$ shown in Tables I and II. As can be seen from the tables, the convergence of the eigenvalues is rapid except for the states near the top of the potential barriers such as for λ_1 in Table I and λ_5 in Table II. For $\gamma = 0.02$, the bound eigenfunctions $\psi_4(x)$ and $\psi_6(x)$ are shown in Fig. 2 together with the eigenfunctions $\psi_{14}(x)$ and $\psi_{16}(x)$ which are clearly in the continuum.

It is useful to note the behavior of the potential function versus $\kappa = (1 - \alpha)/\alpha$ and the mass ratio γ. The potential for $\alpha = 0$ ($\kappa \to \infty$) and different values of γ is shown in Fig. 3(A). As $\gamma \to \infty$, the maximum of the potential approaches the origin and $V(0)$ decreases rapidly. In this case, the spectrum is completely continuous, except for $\lambda = 0$. The potential for finite κ is shown in Fig. 3(B). The singularity at $x = 0$ arises from $V(x) \sim 1/x^3$ for small x. With different κ, the potentials have the same maximum and for larger κ exhibit a larger $V_{\rm max}$. For these potential parameters, the spectrum is continuous.

Mathematical treatments of the spectrum of the Fokker-Planck operator for Coulomb collisions [68–70] suggest that the only bound state is the ground state with $\lambda_0 = 0$. The other "states" can "tunnel" through the potential barrier and hence have finite lifetimes. Our view is that the convergent eigenvalues are discrete eigenvalues. The diagonalization of $\mathbf{L}$ also yields additional eigenvalues that do not converge

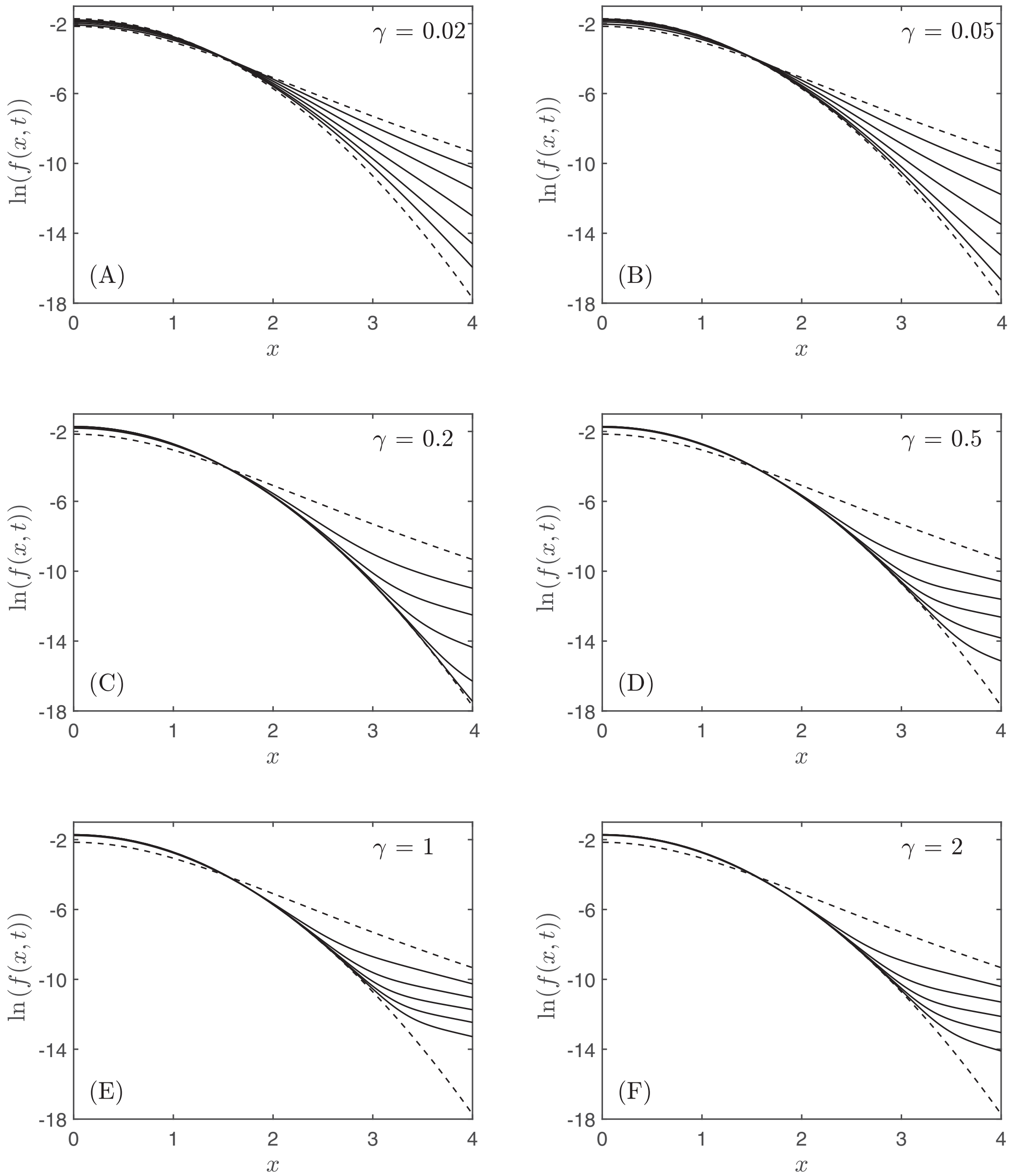

FIG. 4. Time dependence of the distribution from an initial kappa ($\kappa = 4$) distribution (upper dashed curve) to the steady Maxwellian distribution (lower dashed curve) with the Coulomb collision Fokker-Planck equation for mass ratios $\gamma = M/m$ as shown; distributions shown are for reduced times equal to (A) 10, 20, 30, 40 and 50; (B) 6, 12, 18, 24, and 30; (C), (D) and (E) same as panel (B); (F) same as panel (A).

that we consider as continuum eigenstates. The mathematical treatments cited also indicate an asymptotic dependence of the distribution of the form $\exp(-at^b)$ which is not strictly exponential. This is discussed further in the next section.

V. ANALYSIS OF THE TIME-DEPENDENT DISTRIBUTION FUNCTIONS

One of our main objectives is to study the filling and emptying of the heavy tail of the Kappa distribution. We first examine the relaxation of the Kappa distribution to a Maxwellian in the time-dependent solution of the Fokker-Planck equation in the absence of the wave-particle interaction. We consider the relaxation of the distribution function at a specific speed point x_i given by the eigenfunction expansion of the form

$$f(x_i, t) = \sum_{n=0}^{k} c_n f_{ss}(x_i)\psi_n(x_i)e^{-\lambda_n t} + \int_{\lambda_*}^{\infty} c(\lambda) f_{ss}(x_i)\psi(\lambda, x_i)e^{-\lambda t} d\lambda \qquad (29)$$

with contributions from the discrete as well as the continuous spectrum of the Fokker-Planck operator. A crude approximation of Eq. (29) is to consider the relaxation as represented by

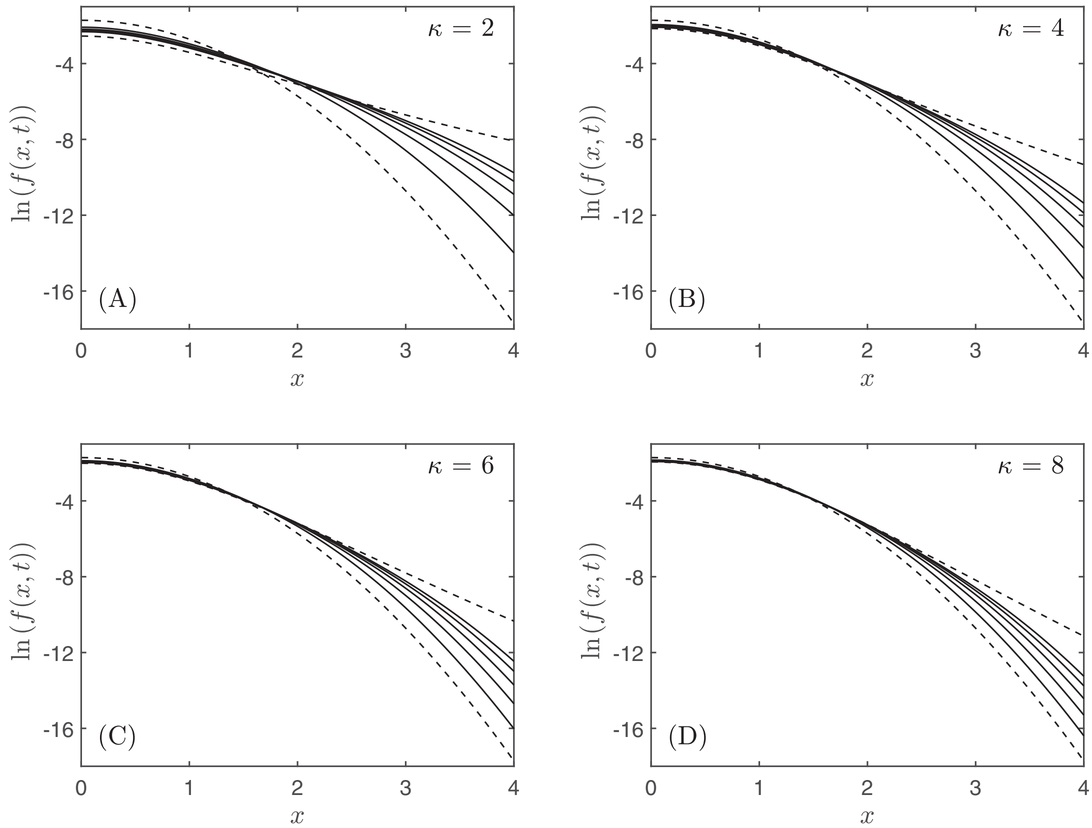

FIG. 5. Time dependence of the distribution from an initial Maxwellian (lower dashed curve) to the steady Kappa distribution (upper dashed curve) owing to Coulomb collisions and wave-particle interactions in the Fokker-Planck equation; $\gamma = 200$ and $\kappa = (1-\alpha)/\alpha$ as shown; reduced times are (A) $t = 1.6$, 3.2, 4.8, 6.4 and 8; (B), (C), and (D) successive reduced times are the same as panel (A).

a single relaxation time τ_i for each speed point defined as

$$f(x_i,t) = f_{ss}(x_i) + e^{-t/\tau_i}\left[\sum_{n=1}^{k} c_n f_{ss}(x_i)\psi_n(x_i) + \int_{\lambda_*}^{\infty} c(\lambda) f_{ss}(x_i)\psi(\lambda,x_i)d\lambda\right]. \quad (30)$$

With the initial condition, we have that

$$f(x_i,0) - f_{ss}(x_i) = \sum_{n=1}^{k} c_n f_{ss}(x_i)\psi_n(x_i) + \int_{\lambda_*}^{\infty} c(\lambda) f_{ss}(x_i)\psi(\lambda,x_i)d\lambda, \quad (31)$$

and thus the simple exponential decay

$$f(x_i,t) - f_{ss}(x_i) = e^{-t/\tau_i}[f(x_i,0) - f_{ss}(x_i)], \quad (32)$$

which is a gross approximation. We use the approximation Eq. (32) to define a single relaxation time τ for the time-dependent distribution.

The representation of the distribution function in terms of the eigenfunctions of the Fokker-Planck equation, as in Eq. (29), can lead to poor convergence especially for short times. Therefore, to determine the time-dependent distributions and in particular the speed-dependent relaxation time Eq. (32), we use instead the stable Chang-Cooper finite difference method of solution of the Fokker-Planck equation [12,71,72]. We use a uniform grid in the interval $x \in [0,6]$ with 600 grid points.

In Fig. 4, we show the relaxation of an initial Kappa distribution ($\kappa = 4$), Eq. (2), (upper dashed curve) to a Maxwellian distribution (lower dashed curve) for various mass ratios. The first observation is that the relaxation for $\gamma = 0.02$ in Fig. 4(A) is slower than the relaxation for $\gamma = 0.05$ in Fig. 4(B) consistent with the eigenvalues in Tables I and II. For Figs. 4(C) to 4(F), the high-energy tail of the initial Kappa distribution approaches the equilibrium Maxwellian on a longer timescale with increasing mass ratio. For these mass ratios, the eigenvalue spectrum of the Fokker-Planck operator is completely continuous. This is a classic relaxation to an equilibrium Maxwellian distribution from an initial nonequilibrium Kappa distribution function owing to Coulomb collisions. The relaxation from an initial Maxwellian or Gaussian distribution was considered in a previous paper [73] and the role of the continuous portion of the spectrum of the Fokker-Planck equation was also discussed. The appropriate entropy functional to describe these relaxation processes is the Kullbach-Leibler entropy [11,12].

In Fig. 5, we show the time-dependent approach to a steady Kappa distribution with $\kappa = 2$, 4, 6, and 8 (upper dashed curve) from an initial Maxwellian distribution (lower dashed curve) obtained with the solution of the Fokker-Planck equation, Eq. (5), with the inclusion of the wave-particle diffusion term. The mass ratio is chosen sufficiently large ($\gamma = 200$) so that the spectrum of the Fokker-Planck operator is completely

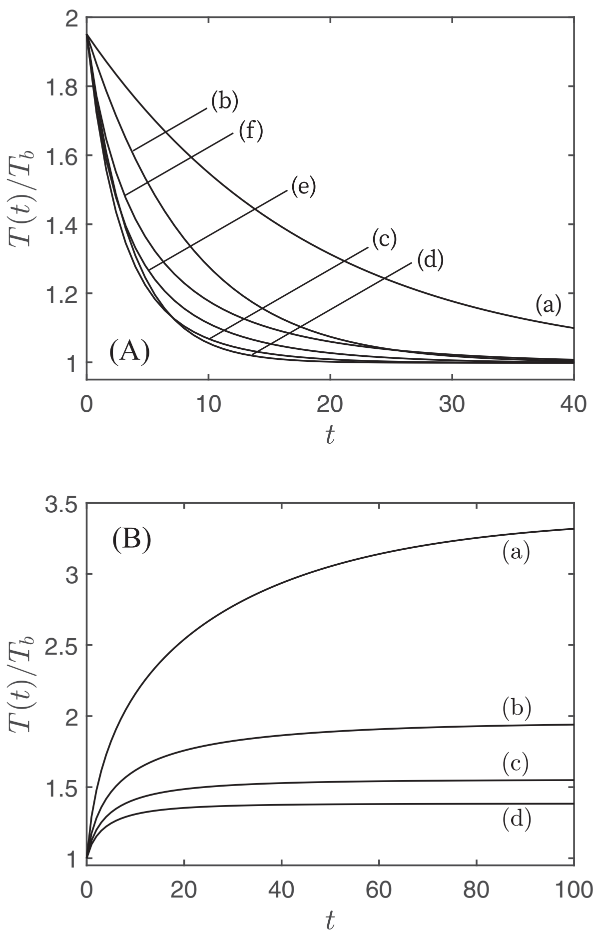

FIG. 6. (A) Variation of the ratio of the temperature $T(t)/T_b$ corresponding to distribution function in Fig. 4 with Coulomb collision Fokker-Planck equation; the curves (a)–(f) correspond to (A)–(F) in Fig. 4. (B) Variation of the ratio of the temperature $T(t)/T_b$ corresponding to distribution function in Fig. 5 with Coulomb collision and Wave-particle interaction Fokker-Planck equation; the curves (a)–(d) correspond to the (A)–(D) in Fig. 5.

continuous, except for $\lambda_0 = 0$. As can be seen from the figure, the distribution function approaches the Kappa distribution on a shorter timescale with increasing κ. As for the distributions in Fig. 4, the appropriate entropy functional to describe this time-dependent approach to the steady Kappa distribution is the Kullbach-Leibler entropy.

The cooling of the distributions in Fig. 4 is shown in Fig. 6(A) whereas the heating of the distributions in Fig. 5 is shown in Figs. 6(B). The relaxation timescales for the approach to a steady state in Figs. 4 and 5 are more clearly illustrated in Fig. 6. These results demonstrate the nonequilibrium statistical mechanical behavior of such systems. In view of the linearity of the governing Fokker-Planck equations, this relaxation appears to demonstrate an exponential approach to equilibrium. However, there have been recent analyses that suggest that owing to the continuous spectrum the approach to equilibrium is not strictly exponential [68–70].

Figure 7 shows the results of an analysis of the relaxation shown in Fig. 4 for the relaxation from an initial Kappa distribution with $\kappa = 4$ to a Maxwellian distribution for x_0 equal to Figs. 7(a) 0.8, 7(b) 1.6, 7(c) 2.4, and 7(d) 3.2 for each of six mass ratios γ. We consider the curves as approximately linear for sufficiently large times t, consistent with Eq. (32) and we consider a speed-independent relaxation time τ. The long time behavior is fit to an exponential so as to extract an approximate relaxation time. These relaxation times are extracted from the data in Fig. 7 for time intervals Δt equal to Figs. 7(A) 90–160, 7(B) 50–120, 7(C) 45–60, 7(D) 60–80, 7(E) 60–100, and 7(F) 90–120. The relaxation times τ extracted in this way are equal to Figs. 7(A) 17, 7(B) 7.1, 7(C) 2.7, 7(D) 3.7, 7(E) 5.1, and 7(F) 7.3. The λ_1 values in Tables I and II for $\gamma = 0.02$ and 0.05 are $\lambda_1 = 0.0584$ and $\lambda_1 = 0.1392$ and thus we confirm the relationship between the relaxation times extracted in this way with the discrete eigenvalues of the Fokker-Planck operator. We find that $\lambda_1\tau_A = 0.993$ for $\gamma = 0.02$ and $\lambda_1\tau_B = 0.988$ for $\gamma = 0.05$ giving good agreement between the eigenvalues and the computed relaxation times. For the other mass ratios, the eigenvalue spectrum is continuous and the relationship between the computed relaxation times from Fig. 7 and the eigenvalues is not apparent.

Analogously, Fig. 8 shows the results of the analysis of the relaxation shown in Fig. 5 for the relaxation of a Maxwellian distribution to a Kappa distribution, for $\gamma = 200$ and κ equal to 2, 4, 6, and 8. As done for Fig. 7, we consider the curves as approximately linear for sufficiently large times t consistent with Eq. (32) for x_0 equal to Figs. 8(a) 0.8, 8(b) 1.6, 8(c) 2.4, and 8(d) 3.2. The curves labeled (c) in Figs. 8(A) and 8(B) exhibit a kind of "singularity" for which $f(x,t) = f_{ss}(x_0)$ owing to the approximate nature of Eq. (32). The relaxation times are extracted from the data in Fig. 8 for time intervals Δt equal to Figs. 8(A) 80–120, 8(B) 120–160, 8(C) 140–220, and 8(D) 150–180. For the different values of κ in the figure, we find that τ is equal to Figs. 8(A) 31, 8(B) 30, 8(C) 25, and 8(D) 23. These results demonstrate large relaxation times for the larger speed points, consistent with the large $\gamma = 200$ chosen. The nature of the relaxation to equilibrium or to a steady nonequilibrium distribution for large mass ratios for which the spectrum of the Fokker-Planck operator is continuous involves a nontrivial mathematical problem [68–70].

VI. SUMMARY: PHYSICAL BASIS FOR THE FORMATION OF KAPPA DISTRIBUTIONS

In the absence of the wave-particle diffusion term in the Fokker-Planck equation, Eq. (5), that is for $B_1 = 0$, the steady distribution for Coulomb collisions is clearly a Maxwellian for arbitrary $D_1(v)$. This Fokker-Planck equation satisfies detailed balance at equilibrium. Since this is a two component system with one component acting as a constant temperature heat bath, the time-dependent approach to equilibrium of the test particle system is to the Maxwellian distribution of the heat bath. The Kullback-Leibler entropy [11,12]

$$\Sigma_{\mathrm{KL}}(t) = -4\pi \int_0^\infty f(v,t) \ln\left[\frac{f(v,t)}{f^{(\mathrm{Max})}(v)}\right] v^2 dv, \qquad (33)$$

increases monotonically with time until the distribution function attains the Maxwellian distribution of the background particles [11,12]. With the inclusion of the wave-particle diffusion term in Eq. (5) with $B_1 \neq 0$ the steady distribution is given by Eq. (6) which is the form of a Pearson differential

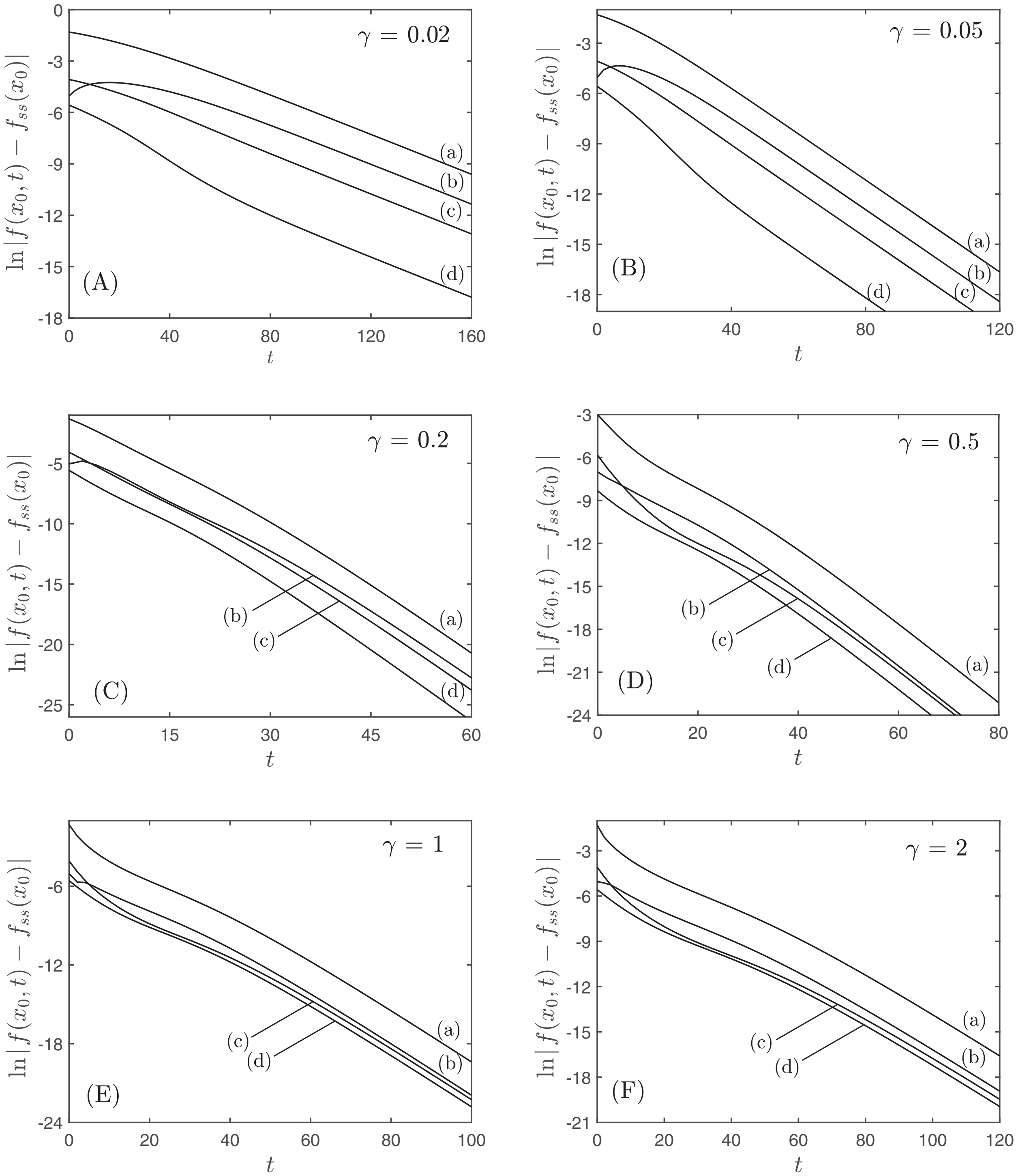

FIG. 7. Relaxation from a Kappa distribution ($\kappa = 4$) to a Maxwellian calculated with the Coulomb collision Fokker-Planck equation for speed points x_0 equal to (a) 0.8, (b)1.6, (c) 2.4, (d) 3.2, and γ as shown.

equation [59,60] and yields a nonequilibrium distribution function. This distribution is not necessarily a Kappa distribution and depends on the speed dependence of $D_2(v)$. This result provides a large class of nonequilibrium distributions [12] as based on the dynamical information in the two diffusion coefficients; see Eq. (6).

We presented a detailed analysis of the statistical mechanics of the formation of Kappa distributions as based on dynamical information. This approach is as well founded in nonequilibrium statistical mechanics as are other treatments of the departure from equilibrium in shock tubes [6], plasma sheaths [25], atmospheric science [7], and numerous other physical systems. In particular, the wave-particle diffusion term can be considered analogous to a reactive term as treated with the Boltzmann equation for chemical reactions [8,9]. If this term is evaluated with a Maxwellian distribution as a

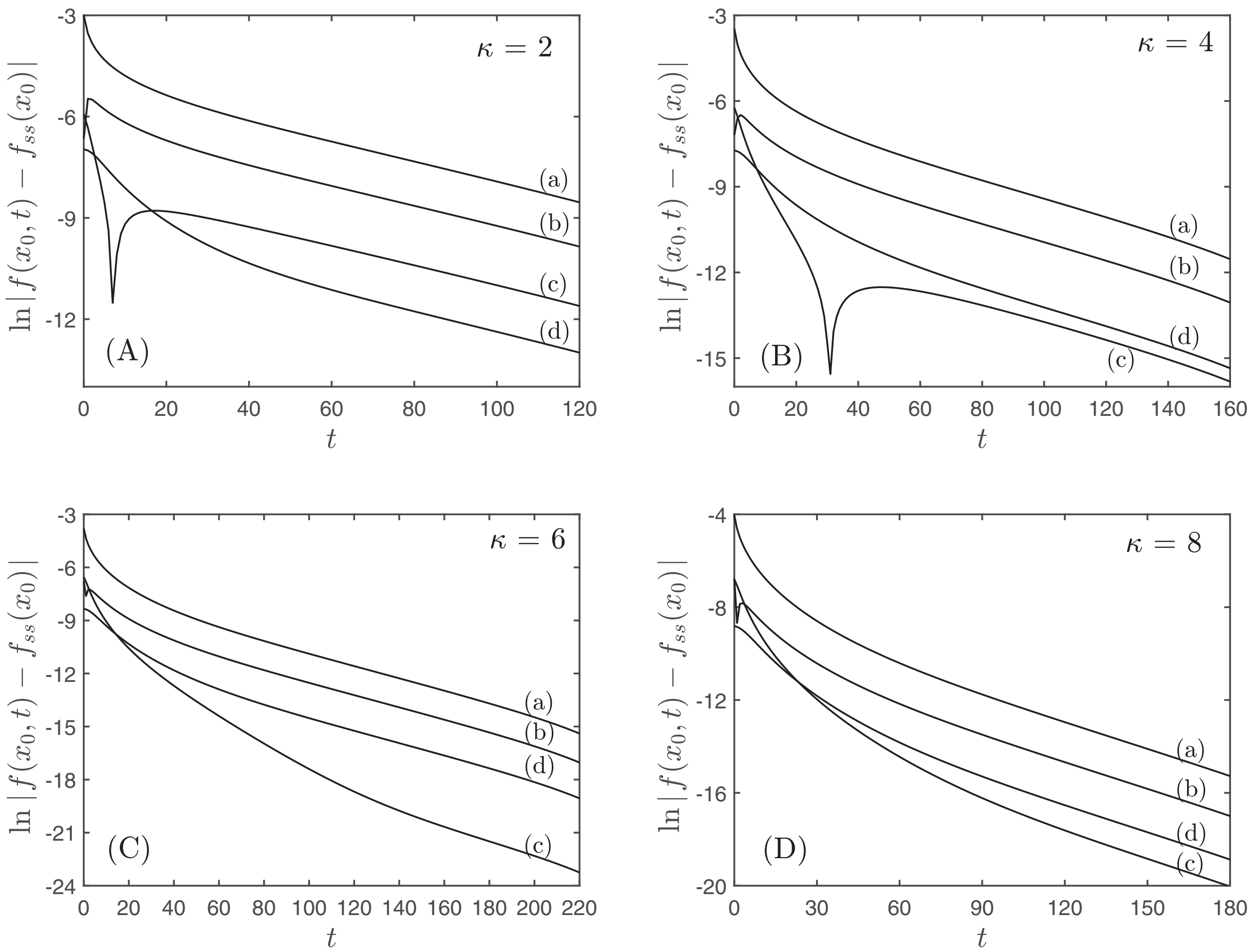

FIG. 8. Relaxation from a Maxwellian distribution to a Kappa distribution calculated with the Coulomb collision Fokker-Planck equation for $\gamma = 200$ at speed points x_0 equal to (a) 0.8, (b) 1.6, (c) 2.4, and (d) 3.2 and for κ as shown.

first-order estimate, the loss of particles in velocity space is very large for slow moving electrons and as a consequence the "gas" is heated analogous to the heating of electrons that accompanies electron attachment in electronegative gases [10]. We presented a detailed analysis of the spectrum of the Fokker-Planck equation in relation to the states in a related Schrödinger equation. The time-dependent evolution of this system was analyzed in terms of this spectral analysis.

ACKNOWLEDGMENTS

This research is supported by a grant to Bernie Shizgal from the Natural Science and Engineering Research Council of Canada (NSERC Grant No. 01390). W.Z. is supported in part by the UBC Work Learn International Undergraduate Research Award. We are grateful to Hrishikesh Patel for useful discussions.

[1] E. Marsch, Kinetic Physics of the Solar Corona and Solar Wind, Living Rev. Solar Phys. **3**, 1 (2006).

[2] V. Pierrard and V. Lazar, Kappa distributions: Theory and applications in space plasmas, Sol. Phys. **267**, 153 (2010).

[3] B. M. Randol and E. R. Christian, Simulations of plasma obeying Coulomb's law and the formation of suprathermal ion tails in the solar wind, J. Geophys. Res. **119**, 7025 (2014).

[4] G. Livadiotis, *Kappa Distributions* (Elsevier, Amsterdam, 2017).

[5] P. H. Yoon, Non-equilibrium statistical mechanical approach to the formation of non-Maxwellian electron distribution in space, Eur. Phys. J.: Spec. Top. **229**, 819 (2020).

[6] B. D. Shizgal and F. Lordet, Vibrational nonequilibrium in a supersonic expansion with reaction: Application to O_2-O, J. Chem. Phys. **104**, 3579 (1996).

[7] B. Shizgal and M. J. Lindenfeld, Energy distribution function of translationally hot $O(^3P)$ atoms in the atmosphere of Earth, Planet. Space Sci. **27**, 1321 (1979).

[8] J. Ross and P. Mazur, Some deductions from a formal statistical mechanical theory of chemical kinetics, J. Chem. Phys. **35**, 91 (1961).

[9] B. Shizgal and M. Karplus, Nonequilibrium contributions to the rate of reaction. I. Perturbation of the velocity distribution function, J. Chem. Phys. **52**, 4262 (1970).

[10] B. Shizgal, The coupling of electron thermalization and electron attachment; SF_6 and CCl_4 in rare gas moderators, J. Phys. B: At., Mol. Opt. Phys. **21**, 1699 (1988).
[11] B. D. Shizgal, Suprathermal particle distributions in space physics: Kappa distributions and entropy, Astrophys. Space Sci. **312**, 227 (2007).
[12] B. Shizgal, Kappa and other nonequilibrium distributions from the Fokker-Planck equation and the relationship to Tsallis entropy, Phys. Rev. E **97**, 052144 (2018).
[13] C.-Y. Ma and D. Summers, Formation of power-law energy spectra in space plasmas by stochastic acceleration due to whistler-mode waves, Geophys. Res. Lett. **26**, 1121 (1999).
[14] A. Hasegawa, K. Mima, and M. Duong-van, Plasma Distribution Function in a Superthermal Radiation Field, Phys. Rev. Lett. **54**, 2608 (1985).
[15] D. C. Nicholls, M. A. Dopita, and R. S. Sutherland, Resolving the electron temperature discrepancies in HII regions and planetary nebulae: κ - distributed electrons, ApJ **752**, 148 (2012).
[16] D. C. Nicholls, M. A. Dopita, R. S. Sutherland, L. J. Kewley, and E. Palay, Measuring nebular temperatures: The effect of new collision strengths with equilibrium and κ - distributed electron energies, ApJS **207**, 21 (2013).
[17] M. Rosenblum and I. Kelson, The slowing down of charged particles in non-Maxwellian laser produced plasmas, Nucl. Fusion, **20**, 1363 (1980).
[18] D. S. Oliveira and R. M. O. Galvao, Transport equations in magnetized plasmas for non-Maxwellian distribution, Phys. Plasmas, **25**, 102308 (2018).
[19] S. Chapman and T. G. Cowling, *The Mathematical Theory of Nonuniform Gases* (Cambridge University Press, Cambridge, England, 1970).
[20] H. Risken, *The Fokker-Planck Equation: Methods of Solution and Applications*, 2nd ed. (Springer, Berlin, 1996).
[21] P. M. Bellan, *Fundamentals of Plasma Physics* (Cambridge University Press, Cambridge, England, 2006).
[22] D. T. Gillespie, *Markov Processes: An Introduction for Physical Scientists* (Academic, San Diego, 1992).
[23] K. Andersen and K. E. Shuler, On the relaxation of hard sphere Rayleigh and Lorentz gas, J. Chem. Phys. **40**, 633 (1964).
[24] F. L. Hinton, Collisional transport in plasma, in *Handbook of Physics, Basic Plasma Physics*, edited by A. A. Galeev and R. Z. Sagdeev (Elsevier, Amsterdam, 1983), pp. 147–197.
[25] A. V. Vasenkov and B. D. Shizgal, Numerical study of a direct current plasma sheath based on kinetic theory, Phys. Plasmas **9**, 691 (2002).
[26] D. M. Busiello and A. Maritan, Entropy production in master equations and Fokker Planck equations: Facing the coarse-graining and recovering the information loss, J. Stat. Mech. (2019) 104013.
[27] B. Shizgal and J. C. Barrett, Time dependent nucleation, J. Chem. Phys. **91**, 6506 (1989).
[28] S. Chandrasekhar, Brownian motion, dynamical friction, and stellar dynamics, Rev. Mod. Phys. **21**, 383 (1949).
[29] J. Binney and S. Tremaine, *Galactic Dynamics*, 2nd ed. (Princeton University Press, Princeton, NJ, 2008).
[30] C. Hamilton, J.-B. Fouvy, J. Binney, and C. Pichon, Revisiting relaxation in globular clusters, MNRS **481**, 2041 (2018).
[31] E. Vasiliev, A new Fokker-Planck approach for the relaxation-driven evolution of Galactic nuclei, Astrophys. J. **848**, 10 (2017).
[32] P. Viegas, M. C. M. van Sanden, S. Longo, and P. Diomede, Validation of the Fokker-Planck approach to vibrational kinetics in CO_2 plasma, J. Phys. Chem. C **123**, 22823 (2019).
[33] G. Bonanno, D. Valenti, and B. Spagnolo, Mean escape time in a system with stochastic volatility, Phys. Rev. E **75**, 016106 (2007).
[34] G. Livadiotis, Derivation of the entropic formula for the statistical mechanics of space plasmas, Nonlin. Processes Geophys. **25**, 77 (2018).
[35] T. S. Biró and A. Jakovác, Power-Law Tails from Multiplicative Noise, Phys. Rev. Lett. **94**, 132302 (2005).
[36] J. P. Boon and C. Tsallis, Special issue overview: Nonextensive statistical mechanics: new trends, new perspectives, Europhys. News **36**, 185 (2005).
[37] C. Tsallis, The nonadditive entropy S_q and its application in physics and elsewhere: Some remarks, Entropy **13**, 1765 (2011).
[38] G. Livadiotis and D. J. McComas, Understanding Kappa distributions: A toolbox for space science and astrophysics, Space Sci. Rev. **175**, 183 (2013).
[39] B. R. La Cour and W. C. Schieve, Tsallis maximum entropy principle and the law of large numbers, Phys. Rev. E **62**, 7494 (2000).
[40] A. Cho, A fresh take on disorder, or disorderly science? Science **297**, 1268 (2002).
[41] M. Nauenberg, Critique of q–entropy for thermal statistics, Phys. Rev. E **67**, 036114 (2003).
[42] C. Tsallis, Comment on Critique of q-entropy for thermal statistics, Phys. Rev. E **69**, 038101 (2004).
[43] M. Nauenberg, Reply to comment on critique of q–entropy for thermal statistics, Phys. Rev. E **69**, 038102 (2004).
[44] D. H. Zanette and M. A. Montemurro, A note on the non–thermodynamical applications of non-extensive entropy, Phys. Lett. A **324**, 383 (2004).
[45] R. Balian and M. Nauenberg, Comment on the paper by A. Rapisarda and A. Pluchino, Europhys. News. **36**, 202 (2005), Letter to the editors, Europhys. News. 37, 9 (2006).
[46] F. Bouchet, T. Dauxois, and S. Ruffo, Comment on the paper by A. Rapisarda and A. Pluchino, Europhys. News. **36**, 202 (2005), Letter to the editors, Europhys. News 37, 9 (2006).
[47] J. F. Lutsko and J. P. Boon, Questioning the validity of non-extensive thermodynamics for classical Hamiltonian systems, Europhys. Lett. **95**, 20006 (2011).
[48] J. Cartwright, *Roll Over, Boltzmann*, Physics World (IOP, Bristol, England, 2014).
[49] C. Tsallis, Possible generalization of Boltzmann–Gibbs statistics, J. Stat. Phys. **52**, 479 (1988).
[50] H. Grad, On Boltzmann's H-theorem, J. Soc. Indust. Appl. Math **13**, 259 (1965).
[51] G. Livadiotis and D. J. McComas, Beyond Kappa distributions: Exploiting Tsallis statistical mechanics in space plasmas, J. Geophys. Res. **114**, A11105 (2009).
[52] N. Meyer-Vernet, Large scale structure of planetary environments: the importance of not being Maxwellian, Planet. Space Sci. **49**, 247 (2001).
[53] B. Shizgal, *Spectral Methods in Chemistry and Physics*, Applications to Kinetic Theory and Quantum Mechanics (Springer, New York, 2015).
[54] B. Shizgal, Relaxation in ionized gasses; the role of the spectrum of the collision operator, *Proceedings of the 17th In-*

ternational Symposium on Rarefied Gases, Aachen (Wiley-VCH Verlag, 1990).

[55] L. J. Spitzer, *Physics of Fully Ionized Gases* (Interscience, New York, 1962).

[56] T. H. Stix, *Waves in Plasmas* (Springer, New York, 1992).

[57] D. R. Nicholson, *Introduction to Plasma Theory* (Wiley, New York, 1983).

[58] R. M. Kulsrud, *Plasma Physics for Astrophysics* (Princeton University Press, Princeton, NJ, 2005).

[59] G. M. Leonenko and T. N. Phillips, Numerical approximation of high-dimensional Fokker-Planck equations with polynomial coefficients, J. Comput. Appl. Math. **273**, 296 (2015).

[60] N. N. Leonenko and N. Suvak, Statistical inference for student diffusion process, Stoch. Anal. Appl. **28**, 972 (2010).

[61] M. P. Leubner, Consequences of entropy bifurcation in non–Maxwellian astrophysical environments, Nonlin. Processes Geophys. **15**, 531 (2008).

[62] C. Tsallis, Non–extensive thermostatics: Brief review and comments, Physica A **221**, 277 (1995).

[63] M. R. Collier, Are magnetospheric suprathermal particle distributions (κ functions) inconsistent with maximum entropy considerations, Adv. Space Res. **33**, 2108 (2004).

[64] P. H. Yoon, S. Kim, G. S. Choe, and Y.-J. Moon, Revised model of the steady–state solar wind halo electron velocity distribution function, Astrophys. J. **826**, 204 (2016).

[65] S. Kim, P. H. Yoon, G. S. Choe, and L. Wang, Asymptotic theory of solar wind electrons, Astrophys. J. **806**, 32 (2015).

[66] P. H. Yoon, Thermodynamic, non-extensive or turbulent quasi-equilibrium for space physics environment, Entropy, **21**, 820 (2019).

[67] E. Lutz, Anomalous diffusion and Tsallis statistics in an optical lattice, Phys. Rev. A **67**, 051402(R) (2003).

[68] R. M. Strain and Y. Guo, Exponential decay for soft potentials near Maxwellian, Arch. Ration. Mech. Anal. **187**, 287 (2008).

[69] A. Bobylev, I. M. Gamba, and C. Zhang, On the rate of relaxation for the landau kinetic equation and related models, J. Stat. Phys. **168**, 535 (2017).

[70] N. Corngold, Kinetic equation for a weakly coupled test particle. II. Approach to equilibrium, Phys. Rev. A **24**, 656 (1981).

[71] J. C. Chang and G. Cooper, A practical difference scheme for Fokker-Planck equations, J. Comput. Phys. **6**, 1 (1970).

[72] K. Leung, B. D. Shizgal, and H. Chen, The quadrature discretization method QDM in comparison with other numerical methods of solution of the Fokker–Planck equation for electron thermalization, J. Math. Chem. **24**, 291 (1998).

[73] B. D. Shizgal, Coulomb collisional processes in space plasmas; relaxation of suprathermal particle distributions, Planet. Space Sci. **52**, 923 (2004).