Low-dimensional description for ensembles of identical phase oscillators subject to Cauchy noise

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We study ensembles of globally coupled or forced identical phase oscillators subject to independent white Cauchy noise. We demonstrate that if the oscillators are forced in several harmonics, stationary synchronous regimes can be exactly described with a finite number of complex order parameters. The corresponding distribution of phases is a product of wrapped Cauchy distributions. For sinusoidal forcing, the Ott-Antonsen low-dimensional reduction is recovered.

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I. INTRODUCTION

One of the central challenges in synchronization theory is finding a possibility to describe the dynamics of populations of globally coupled oscillators in terms of a few order parameters. Indeed, generally, following the pioneering approach of Kuramoto [1], one derives self-consistency conditions for the order parameters, which are formulated as integral equations and constitute an infinite hierarchy of coupled nonlinear relations. Such self-consistency equations have also been derived for oscillators with coupling heterogeneity [2], or for identical oscillators subject to Gaussian white noise and coupling in the first or higher harmonics [3,4]. Self-consistency means that given a stationary prior phase distribution, one calculates the force acting on oscillators of frequency ω, the phase distribution of these oscillators, and, by averaging over a frequency distribution $g(\omega)$, the phase distribution of the whole oscillator ensemble. The latter must be equal to the prior distribution. The self-consistency condition is often formulated in terms of integrals which have to be evaluated numerically.

Only in exceptional cases like one of coupling in the first harmonics, Lorentzian frequency distribution, and zero noise strength can the integrals be evaluated explicitly. A Lorentzian frequency distribution is also central to the low-dimensional dynamics of oscillator ensembles on the Ott-Antonsen manifold [5]. Indeed, this is the main reason a Lorentzian frequency distribution is used by default in many studies of nonidentical phase oscillators.

Although Gaussian white noise acts qualitatively similar to frequency heterogeneity, for ensembles driven by Gaussian noise there is no exact low-dimensional reduction, except for several approximate approaches based on moment closures of the infinite hierarchy [6–8]. As we demonstrate in this paper, the situation is different when the noise is not Gaussian but Lévy stable with exponent $\alpha = 1$, i.e., Cauchy noise. Using Cauchy white noise instead of Gaussian, one simplifies the analysis in a similar way as using a Lorentzian instead of a Gaussian frequency distribution, while keeping the bifurcation scenario qualitatively similar. In the simplest case of purely harmonic coupling, Cauchy noise allows for the Ott-Antonsen reduction (cf. [9]). Furthermore, we demonstrate that a low-dimensional reduction is also possible for *multimode coupling*, albeit being restricted to stationary distributions which we show to be fully characterized by a finite number of modes.

II. BASIC EQUATIONS

We start with formulating general equations for an ensemble of identical phase oscillators driven by independent noise forces,

$$\dot{\vartheta}_n = F(\vartheta_n) + \eta_n(t). \tag{1}$$

For ensembles of coupled oscillators, the driving force $F(\vartheta)$ depends on the phase distribution $P(\vartheta, t)$, which makes the problem of finding a stationary phase distribution nonlinear. This is, for instance, the case with Kuramoto-Daido mean-field coupling,

$$F(\vartheta) = \int_0^{2\pi} H(\vartheta - \vartheta')P(\vartheta')d\vartheta'. \tag{2}$$

The noise terms $\eta_n(t)$ are assumed to be Poisson processes of δ pulses with rate ν and amplitude distribution $W(\cdot)$. Then the evolution equation for the phase density is given by the integro-differential equation

$$\partial_t P = -\partial_\vartheta(F(\vartheta)P) + \nu \int_0^{2\pi} [W(\phi) - \delta(\phi)]P(\vartheta - \phi)d\phi. \tag{3}$$

As special cases, we will consider a wrapped Gaussian distribution of pulse amplitudes $W^G \sim N(0, 2D\nu^{-1})$ and a wrapped Cauchy distribution $W^C \sim C(0, \sigma\nu^{-1})$. In the limit $\nu \to \infty$, the shot noise $\eta(t)$ becomes, respectively, Gaussian $\int_0^\tau \eta^G(t)dt \sim \sqrt{2D\tau}N(0, 1)$ or Cauchy white noise $\int_0^\tau \eta^C(t)dt \sim \sigma\tau C(0, 1)$. Another interesting case is that of random phase resetting with a uniform distribution $W^U = (2\pi)^{-1}$. Equation (3) can be rewritten in terms of the Fourier components of F, with $F(\vartheta) = \sum_{k=-\infty}^{\infty} f_k \exp[-ik\vartheta]$ and order parameters (circular moments) $z_k = \langle e^{ik\vartheta}\rangle_P$ and

$w_k = \langle e^{ik\phi}\rangle_W$ as an infinite system of coupled ordinary differential equations,

$$\dot{z}_k = ik \sum_{l=-\infty}^{\infty} f_{-l} z_{k+l} + \nu(w_k - 1) z_k. \tag{4}$$

Because for globally coupled oscillators the driving force F depends on the phase distribution $P(\vartheta, t)$, in Eq. (4) the Fourier components f_l generally depend on the order parameters z_k, so the system (4) is a nonlinear set of equations for the z_k. The circular moments w_k^G of the wrapped Gaussian distribution, w_k^C of the wrapped Cauchy distribution, and w_k^U of the uniform distribution are

$$w_k^G = \exp\left(-\frac{D}{\nu}k^2\right), \quad w_k^C = \exp\left(-\frac{\sigma}{\nu}|k|\right), \quad w_k^U = \delta_{k0}. \tag{5}$$

In the limit $\nu \to \infty$, we recover the Fourier representation of the second derivative $\nu(w_k^G - 1) \to -Dk^2$ for Gaussian white noise, and of the fractional derivative $\nu(w_k^C - 1) \to -\sigma|k|$ for Cauchy white noise [10,11].

III. INSTABILITY OF DISORDERED STATE FOR DIFFERENT TYPES OF NOISE

As a first result, we show how the type of noise affects the stability of the asynchronous system state with a uniform phase distribution for Kuramoto-Daido coupling (2), where the Fourier modes of the force F depend on the moments of the phase distribution as $f_k = h_k z_k$. In this case, the incoherent state $P(\vartheta) = (2\pi)^{-1}$ or $z_k = \delta_{k0}$ is always a stationary solution of (4). Linearization yields decoupled equations for different order parameters,

$$\dot{z}_k = [ik(h_k + f_0) + \nu(w_k - 1)]z_k. \tag{6}$$

The condition for stability of a mode $k > 0$ is that the real part of the factor on the right-hand side is smaller than zero, i.e.,

$$\mathrm{Re}[ikh_k + \nu(w_k - 1)] < 0. \tag{7}$$

This gives three different stability criteria: $\mathrm{Im}[\bar{h}_k/k] < D$ for Gaussian white noise, $\mathrm{Im}[\bar{h}_k] < \sigma$ for Cauchy white noise, and $\mathrm{Im}[k\bar{h}_k] < \nu$ for random phase resetting. Gaussian white noise, which diffuses phases on continuous trajectories, is most efficient in suppressing higher wave number instabilities by a factor $1/k$, whereas such instabilities more likely occur for strongly anharmonic coupling functions and Cauchy noise or random phase resetting. Phase coupling functions of that kind, e.g., with discontinuities or dead zones [12], are sometimes constructed to design synchronization behavior in ensembles of artificial oscillators.

IV. PHASE DISTRIBUTION UNDER CAUCHY NOISE

We now demonstrate the main findings of the paper—that under Cauchy white noise, which corresponds to the limit $\nu \to \infty$ with Cauchy distributed jumps in the phases, a finite-dimensional reduction for a generally infinite system of mode dynamics (4) is possible. We present a low-dimensional parametrization of all stationary rotating wave solutions with arbitrary degree of synchrony.

A. First harmonic coupling: Ott-Antonsen case

The simplest case is that of sinusoidal forcing. The paradigmatic Kuramoto-Sakaguchi model, among others, belongs to this class. In terms of the phase dynamics (1), this means $F(\vartheta) = \omega + \varepsilon\sin(\alpha - \vartheta)$ with generally time-dependent parameters ω, ε, and α. Then, (4) reduces to

$$\dot{z}_k = ik(f_1 z_{k-1} + f_0 z_k + f_{-1} z_{k+1}) - \sigma|k|z_k. \tag{8}$$

One can straightforwardly check that this infinite system admits the Ott-Antonsen (OA) ansatz $z_k = Z^k$ for $k \geqslant 0$. The dynamics of the mean field $Z = z_1$ on this manifold is the OA equation

$$\dot{Z} = i(f_1 + f_0 Z + f_{-1} Z^2) - \sigma Z. \tag{9}$$

We stress here that in our formulation, Eq. (9) appears for *identical* oscillators subject to independent Cauchy noise, while in the original OA formulation, the same equation has been derived for *noiseless, nonidentical* oscillators with a Cauchy (Lorentz) distribution of natural frequencies. Parameter σ in the latter case characterizes the width of the distribution [5]; in our case, this parameter characterizes the noise intensity. The OA manifold corresponding to the family of wrapped Cauchy distributions (WCDs),

$$P_{OA}(\vartheta) = \frac{1}{2\pi}\frac{1 - |Z|^2}{|e^{i\vartheta} - Z|^2}, \tag{10}$$

where the dynamics (9) is exact, has been shown to be globally attractive under (8) for oscillators with Cauchy distributed random frequencies [13]. This is, therefore, also true for identical oscillators subject to independent Cauchy white noise.

However, the equivalence between oscillator ensembles with Cauchy noise and Cauchy distribution of natural frequencies is no longer valid if *higher harmonics* are present in the forcing $F(\vartheta)$. Then the first order parameter z_1 is coupled to $z_{-1} = \bar{z}_1$ through the mode f_2 in (4), and can therefore not be an analytic function of the oscillator frequency—an essential condition in the OA approach. The averaging of the $z_k = z_k(\omega)$ over the frequency distribution $g(\omega)$ via application of the Cauchy integral theorem, which results in Eq. (8) if only the first harmonics are present, is no longer possible. In contradistinction, for Cauchy noise, the terms $-|k|\sigma$ appear in the ordinary differential equations without the necessity of averaging over a frequency distribution, i.e., the equations are indeed valid for any multimode coupling.

B. Multiharmonic coupling

Our next goal is to generalize the OA approach for white Cauchy noise to the situation where the coupling term $F(\vartheta)$ contains up to L harmonics. Then, Eqs. (4) become

$$\dot{z}_k = ik\left(f_0 z_k + \sum_{l=1}^{L}[f_l z_{k-l} + f_{-l} z_{k+l}]\right) - \sigma|k|z_k. \tag{11}$$

The theory below does not provide a low-dimensional reduction of the dynamics (11), but yields a low-dimensional description of all possible stationary rotating wave solutions. For phase densities which are stationary in a rotating reference frame, i.e., $P^{st}(\vartheta,t)=P^{st}(\vartheta-\Omega t)$, we have $\dot{z}_k^{st}=ik\Omega z_k^{st}$ with a yet unknown frequency offset Ω. Through a shift into a co-rotating reference frame, we can absorb f_0 in the frequency offset, $\Omega\to f_0+\Omega$. The recurrence relations (11) for the stationary solution z_k^{st} with $k\geqslant 1$ become independent of k,

$$0=\sum_{l=1}^{L}\left[f_l z_{k-l}^{st}+f_{-l}z_{k+l}^{st}\right]+(i\sigma-\Omega)z_k^{st}. \tag{12}$$

Recurrence relations of that kind are solved via the transfer matrix method. They have the general solution

$$z_k^{st}=\sum_{m=1}^{2L}c_m\lambda_m^k,\quad k\geqslant 1-L, \tag{13}$$

where complex factors λ_m are the $2L$ roots of the characteristic equation

$$\sum_{l=1}^{L}[f_l\lambda^{-l}+f_{-l}\lambda^{l}]=\Omega-i\sigma. \tag{14}$$

Strictly speaking, ansatz (13) is only valid for roots λ_m of multiplicity one, which is almost always true if the f_l are fixed. Since $k\geqslant 1$ in (12), the ansatz (13) extends up to negative Fourier modes $k\geqslant 1-L$. Using the symmetric version of Rouché's theorem, one can show that if $\sigma\neq 0$, the number of solutions within the unit circle does not depend on the right-hand side. Furthermore, if λ is a root of the Laurent polynomial on the left-hand side, then $\bar{\lambda}^{-1}$ is also a root, i.e., there are L solutions of (14) inside and L solutions outside of the unit circle. Since the order parameters z_k are bounded $|z_k|<1$, only the λ_m with $|\lambda_m|<1$ contribute to the sum (13). Given the L roots λ_m with $|\lambda_m|<1$, the coefficients c_m are the unique solutions of the following set of linear equations for $k=1\dots L-1$:

$$\sum_{m=1}^{L}c_m=1,\quad \sum_{m=1}^{L}c_m\lambda_m^k-\bar{c}_m\bar{\lambda}_m^{-k}=0. \tag{15}$$

The first inhomogeneous equation expresses $z_0=1$, and the $L-1$ homogeneous equations are due to the conditions $z_k^{st}=\bar{z}_{-k}^{st}$ expressed in terms of (13) for $k=1\dots L-1$. This means all z_k are fully determined by the set of eigenvalues λ_m. This is the desired low-dimensional reduction: possible stationary distributions in a population of oscillators forced in L harmonics are fully determined by L complex parameters λ_m with $|\lambda_m|<1$. Furthermore, (14) constitutes a linear set of equations for the Fourier modes f_l, the noise intensity σ, and the frequency Ω, which are thus known explicitly as functions of the λ_m.

Next, we demonstrate that this low-dimensional solution for a stationary phase density is, in fact, a product of WCDs, which can be dubbed poly-WCD (cf. [14]),

$$P^{st}(\vartheta)=\frac{1}{M}\frac{1}{2\pi}\prod_{m=1}^{L}\frac{1-|\lambda_m|^2}{|e^{i\vartheta}-\lambda_m|^2}\,. \tag{16}$$

Given the values of λ_m, we explicitly calculate the coefficients c_m,

$$\begin{aligned} z_k^{st}&=\frac{1}{M}\int_0^{2\pi}\frac{e^{ik\vartheta}}{2\pi}\prod_{m=1}^{L}\frac{1-|\lambda_m|^2}{|e^{i\vartheta}-\lambda_m|^2}d\vartheta\\ &=\frac{1}{2\pi i}\oint_{|z|=1}z^k\left[\frac{1}{M}\prod_{m=1}^{L}\frac{z\left(1-|\lambda_m|^2\right)}{(z-\lambda_m)(1-\bar{\lambda}_m z)}\right]\frac{dz}{z}\\ &=\sum_{m=1}^{L}\lambda_m^k\frac{1}{M}\prod_{n\neq m}^{L}\frac{\lambda_m(1-|\lambda_n|^2)}{(\lambda_m-\lambda_n)(1-\bar{\lambda}_n\lambda_m)}\quad (k\geqslant 1-L).\end{aligned} \tag{17}$$

The integrand has exactly the L poles λ_m on the unit disk if $k\geqslant 1-L$. This begets (13) with coefficients

$$c_m=\frac{1}{M}\prod_{n\neq m}^{L}\frac{\lambda_m(1-|\lambda_n|^2)}{(\lambda_m-\lambda_n)(1-\bar{\lambda}_n\lambda_m)} \tag{18}$$

and with normalizing weight

$$M=\sum_{l=1}^{L}\prod_{p\neq l}^{L}\frac{\lambda_l(1-|\lambda_p|^2)}{(\lambda_l-\lambda_p)(1-\bar{\lambda}_p\lambda_l)}. \tag{19}$$

Equations (18) and (19), together with

$$z_k^{st}=\sum_{m=1}^{L}c_m\lambda_m^k,\quad k\geqslant 1-L, \tag{20}$$

and Eq. (14) (where only roots with modulus smaller than 1 are taken) form the self-consistency conditions for the stationary phase density of identical phase oscillators subject to Cauchy white noise and under a forcing $F(\vartheta)$ with L harmonics. Equation (20) can be regarded as a generalization of the Ott-Antonsen ansatz, although it is restricted to stationary solutions. While algebraic self-consistency equations still require numeric root finding, the evaluation of these equations is much faster and numerical errors are much smaller than for integral self-consistency equations. Before proceeding to an example, we mention that a circle distribution having moment $z_k=c\lambda^k$ has been considered by Kato and Jones [15]. Expression (20) means that our stationary distributions are weighted sums of Kato-Jones distributions.

V. EXAMPLE: BIHARMONIC COUPLING

Let us discuss the simplest nontrivial example: an ensemble of phase oscillators with a phase difference coupling (2),

$$H(\Delta\vartheta)=\varepsilon_1\sin(\alpha_1-\Delta\vartheta)+\varepsilon_2\sin(\alpha_2-2\Delta\vartheta), \tag{21}$$

in the first and the second harmonics, i.e., $h_k=\frac{1}{2i}\varepsilon_k e^{i\alpha_k}$, subject to Cauchy white noise. According to (20), stationary rotating wave solutions have the form

$$z_k^{st}=c_1\lambda_1^k+c_2\lambda_2^k, \tag{22}$$

where, as it follows from (18),

$$c_1=1-c_2=\left[1-\frac{\lambda_2(1-|\lambda_1|^2)(1-\bar{\lambda}_2\lambda_1)}{\lambda_1(1-|\lambda_2|^2)(1-\bar{\lambda}_1\lambda_2)}\right]^{-1}, \tag{23}$$

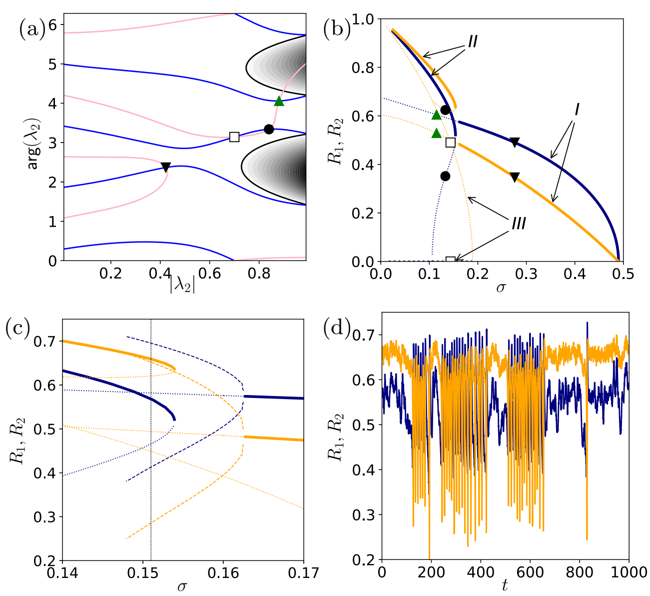

FIG. 1. (a) Zero lines for the imaginary (dark blue) and the real (light red) parts of Δ as a function of λ_2 (amplitude and complex argument) given $\lambda_1 = 0.7$, $\varepsilon_1 = 1.0$, $\varepsilon_2 = 0.7$, $\alpha_1 = -0.2$, and $\alpha_2 = -1.0$. The points $\lambda_2 \neq \lambda_1$ where these lines cross (marker symbols) correspond to stationary solutions. In the shaded region, $\sigma < 0$. (b) Collection of stationary solutions upon variation of λ_1 from zero to one. The dark blue and the light yellow lines mark the order parameters R_1, R_2, respectively. Bold lines are stable solutions and dotted lines are unstable solutions. The markers correspond to the same stationary solutions depicted in (a); they are located on the three different branches (triangle markers are on the same branch). (c) Zoom of the transition region from branch I to branch II. Branch I becomes unstable in a supercritical Hopf bifurcation when $\sigma \approx 0.163$. The dashed lines mark the maxima and the minima of the order parameters of the periodic solution when σ is slowly decreased. For $0.154 < \sigma < 0.163$, no stationary solution exists, whereas for $0.148 < \sigma < 0.154$, the periodic solution coexists with a stable equilibrium. Below $\sigma < 0.148$, the periodic solution has disappeared in a saddle-node on an invariant circle bifurcation. (d) Integration of the Langevin equations (1) for $N = 5000$ phase oscillators with phase difference coupling in the first and second harmonics, and Cauchy noise of strength $\sigma = 0.151$ in a small bistable regime. Stochastic switching between stationary and oscillating order parameters is observed.

and from (14), λ_1 and λ_2 are simultaneous solutions of the algebraic equations

$$\Omega - i\sigma = \bar{h}_2 \bar{z}_2^{st} \lambda_{1,2}^2 + \bar{h}_1 \bar{z}_1^{st} \lambda_{1,2} + h_1 z_1^{st} \lambda_{1,2}^{-1} + h_2 z_2^{st} \lambda_{1,2}^{-2}. \quad (24)$$

One parameter in the problem can be eliminated by a rescaling of time. In this example, we choose $\varepsilon_1 = 1$. Furthermore, because of the rotational invariance $\vartheta \to \vartheta + \text{const}$, we can choose $\lambda_1 \in [0, 1]$ to be real. Given this free parameter, we can calculate the right-hand sides (rhs) of (24) as functions of λ_2. At points $\lambda_2 \neq \lambda_1$ where the difference Δ between the rhs of (24) vanishes, a stationary solution exists in a rotating reference frame with frequency Ω and noise σ, which are thus defined parametrically as the real part and the negative imaginary part of the rhs of (24), respectively. Tuning λ_1 from zero to one, we can quickly pinpoint all λ_2 where the difference Δ is zero [see Fig. 1(a)], and continue these different solution branches. We illustrate the found bifurcation diagram in Fig. 1. Zeros of Δ found according to Fig. 1(a) result in three branches of solutions. Branch I, starting at the point of instability of the first mode $\sigma_{c1} = \cos(0.2)/2 \approx 0.49$, is a state with a bimodal distribution, where both real order parameters $R_{1,2} = |z_{1,2}|$ are nonzero. Another branch III, which starts at the instability point of the second mode $\sigma_{c2} = 0.7\cos(1)/2 \approx 0.19$, is a pure symmetric bimodal solution, where only even modes are present and $R_1 = 0$. This branch in terms of the eigenvalues $\lambda_{1,2}$ corresponds to the case $\lambda_1 = -\lambda_2$, $c_1 = c_2 = \frac{1}{2}$. There is also a third nontrivial mode II that bifurcates at $\sigma_{c3} \approx 0.1$ from the incoherent branch III and folds back at larger values of σ to become the unique stationary solution in the limit of small noise strength. The stability of the stationary states (and periodic solutions) was checked numerically from (11) with truncation at a large number of Fourier modes. Remarkably, there is a stability change from mode I to mode II (mode III is always unstable).

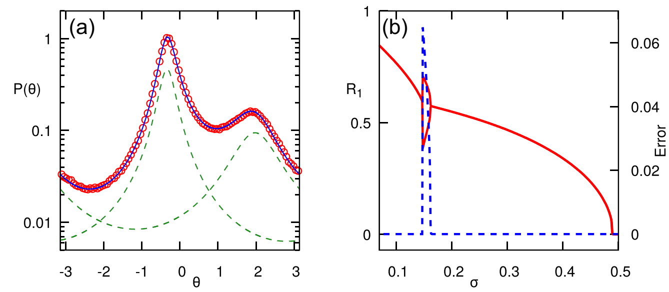

FIG. 2. (a) Comparison of the empirical probability density obtained in simulations of an ensemble of $N = 10^4$ oscillators (red circles) with theoretical prediction of poly-WCD (solid curve under the circles), for branch I at $\sigma = 0.2$. Dashed green lines show the corresponding wrapped Cauchy densities (arbitrarily scaled). (b) Bifurcation diagram (red: span between $R_{1\,\max}$ and $R_{1\,\min}$) obtained from the solutions of system (11) at slowly decreasing parameter σ. The dashed blue line (see scale at the right) shows the deviation from the poly-WCD.

An interesting feature of the bifurcation diagram in Fig. 1 is that there is a range of noise intensities σ where all steady states are unstable, and stable periodic oscillations of the order parameters are observed. Furthermore, there is an even smaller range of bistability where stable periodic oscillations coexist with the stable stationary state of branch II. Only for stationary states do we expect validity of the poly-WCD distribution. To characterize this, it is instructive to consider a Fourier transform of the logarithm of $P^{st}(\vartheta)$ (16). In our example, this is $\ln P^{st}(\theta) = \text{const} + \sum_{k>0}(s_k e^{-ik\vartheta} + \bar{s}_k e^{ik\vartheta})$, with $s_k = k^{-1}(\lambda_1^k + \lambda_2^k)$. Remarkably, this representation does not contain the constants c_m. Because all Fourier modes s_k depend on two complex numbers only, Fourier modes with $k > 2$ can be represented through $s_{1,2}$. In particular, $s_3 = s_1 s_2 - s_1^3/6$. Thus, quantity $\text{Err} = |s_3 - s_1 s_2 + s_1^3/6|$ serves as a measure of the deviation from the poly-WCD. We show this deviation together with the empirical bifurcation diagram in Fig. 2(b). One can see that indeed, while stationary states are given by (16), a clear deviation occurs for periodic solutions.

VI. CONCLUSION

In conclusion, we have demonstrated that an ensemble of identical phase oscillators subject to independent Cauchy white noise admits a finite-dimensional description. In the simplest case of sinusoidal forcing, the resulting reduction is just the Ott-Antonsen ansatz with a wrapped Cauchy distribution of the phases. If forcing contains up to L Fourier modes, stationary states are given by a poly-wrapped Cauchy distribution with L complex roots λ_m on the unit disk as parameters. This finite-dimensional reduction is valid not only for Kuramoto-Daido-type coupling, but also for more general situations, such as Winfree-type models and ensembles with nonlinear coupling.

It is instructive to compare the present analysis of oscillators driven by Cauchy white noise with the perturbative approach developed in [16] for the case of Gaussian white noise. The circular cumulant method of [16] allows one, for a weak noise, to describe states close to the Ott-Antonsen manifold. It is restricted to a harmonic coupling, but yields dynamical equations for generally nonstationary solutions. The approach of this paper is not perturbative, it is valid for any intensity of noise. Furthermore, it is not restricted to a harmonic coupling, but is valid for a coupling term containing any finite number of harmonics. As a result, it allows one to analytically describe states very far from the Ott-Antonsen manifold. The main weakness of the approach is that it is restricted to stationary states only, and does not allow a characterization of a time-dependent solution (as clearly demonstrated in the example of biharmonic coupling above).

We stress here that the special role of the Cauchy distribution is well known in statistical physics, starting from the seminal work by Lloyd on the exact solution [17] for disordered Hamiltonians with Cauchy distributed heterogeneities. In the context of populations of dynamical elements, this property has been explored in studies of homographic maps [18]. Our study shows that considering ensembles of identical oscillators subject to Cauchy noise significantly simplifies the analysis of the collective dynamics, compared to Gaussian noise. While in real experiments it is hard to determine the distribution of acting noise, one way to quantify the validity of the assumption of Cauchy noise could be a comparison of the observed distributions of the phases with the predictions where different model noise terms are employed [7,8].

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