Nonlinear modulation of periodic waves in the cylindrical Gardner equation

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The propagation of dispersive shock waves (DSWs) is investigated in the cylindrical Gardner (cG) equation, which is obtained by employing a similarity reduction to the two-space one-time (2+1) dimensional Gardner-Kadomtsev-Petviashvili (Gardner-KP) equation. We consider the steplike initial condition along a parabolic front. Then, the cG-Whitham modulation system, which is a description of DSW evolution in the cG equation, in terms of appropriate Riemann-type variables is derived. Our study is supported by numerical simulations. The comparison is given between the direct numerical solution of the cG equation and the DSW solution obtained from the numerical solution of the Whitham system. According to this comparison, a good agreement is found between the solutions.

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I. INTRODUCTION

The Gardner equation

$$u_t + 6uu_x \pm 6\alpha u^2 u_x + u_{xxx} = 0 \qquad (1)$$

is a well-known model which describes internal solitary waves in shallow water and it is called the focusing or defocusing Gardner equation depending on the plus or minus sign of the cubic nonlinear term, respectively. This equation is first derived to obtain the infinite set of local conservation laws of the Korteweg–de Vries (KdV) equation [1]. The Gardner equation has importance in modeling large-amplitude inner waves in the ocean (see [2–4]). It also describes a variety of wave phenomena in solid-state and plasma physics [5,6], dynamics of a Bose-Einstein condensate (BEC) [7], and quantum field theory [8].

The Gardner equation is extended to the Gardner-KP equation by using the sense of Kadomtsev and Petviashvili [9], who relaxed the restriction that the waves be absolutely one-dimensional.

In this study, we are interested in a defocusing-type Gardner-KP equation in the following form:

$$(u_t + 6uu_x - 6\alpha u^2 u_x + \epsilon^2 u_{xxx})_x + \lambda u_{yy} = 0, \qquad (2)$$

where $0 < \epsilon \ll 1$, $\alpha > 0$, and $\lambda = \pm 1$ are constants. This equation has two nonlinear terms in the quadratic and cubic forms and the dispersive term is of third order.

Even though the coefficients of $u^2 u_x$ in the Gardner and the Gardner-KP equations cannot be scaled to a fixed value, all derivations and calculations are done by taking $\alpha = 1$ for simplicity through the study. For the general α, similar results can be obtained by applying the procedure which is given in this study.

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The Gardner-KP equation describes strong nonlinear internal waves on an ocean shelf in the two-dimensional case. The propagation of interfacial waves in a two-layer fluid system is considered in [10]. In the study, the fluids in both layers are assumed to be inviscid and incompressible. The evolution equation has been derived for weakly nonlinear and dispersive interfacial waves, propagating near the critical depth level and traveling in a slowly rotating channel with gradually varying topography and sidewalls. Under the assumption that there is no rotation and the variation of topography is weak and behaves like a linear function in the transverse direction, it is shown that weakly nonlinear and dispersive interfacial waves, propagating near the critical depth level, are governed by a Gardner-KP equation which is also called a modified Kadomtsev-Petviashvili (mKP) equation. A similar problem is considered in [11] and two-dimensional interaction of internal solitary waves in a two-layer fluid system is described by a Gardner-KP equation (the authors termed it the extended KP equation), provided the propagation directions of the waves were close to each other. The Gardner-KP equation has been investigated numerically in [11] and [12].

In this study, we consider the formation and the propagation of dispersive shock waves in the cylindrical Gardner (cG) equation [see Eq. (12)], which is obtained by using a similarity reduction to the Gardner-KP equation (2). Dispersive shock waves (DSWs), also termed undular bores in fluid mechanics, are slowly modulated nonstationary wave trains that develop spontaneously in weakly dispersive nonlinear media. In this wave form, the nonlinearity induces front steepening and thus induces the tendency to develop an unphysical hydrodynamic singularity, named the gradient catastrophe. A weak dispersion takes the second place until steep gradients are eventually formed. At this stage dispersion becomes effective. The result is the expanding front characterized by oscillations. These oscillations spread in a characteristic fan in the space-time plane and the borders of this fan represent the leading and the trailing edge of the DSW, where the amplitude of the oscillations is largest and vanishingly small, respectively. These two edges propagate with different speeds.

The investigation of DSWs has a long history that begins with Whitham's pioneering invention of modulation theory [13,14] and continues with the construction of the DSW solution for the KdV equation, which physically describes an undular bore by Gurevich and Pitaevskii [15]. It was then verified numerically by Fornberg and Whitham [16]. The modulation theory for the Gardner equation was developed in [17], where the complete classification of the solutions for the Riemann-type problem was constructed.

In this study, a multiple-scale method [18] is used to investigate DSWs in the cylindrical Gardner equation. By using this method, a system of quasilinear modulation equations describing slow evolution of parameters in the periodic traveling wave solution, such as amplitude, wave number, and mean height, is obtained. These equations are called Whitham modulation equations. For hyperbolic systems, a DSW solution occurs as a stable wave train and for elliptic systems it corresponds to an unstable wave train. When the connection between DSW solutions and hyperbolic modulation equations was realized, the DSW solutions of other integrable nonlinear, dispersive wave equations, such as the nonlinear Schrödinger (NLS) equation [19,20], the modified KdV (mKdV) equation [21], the KdV-Burgers equation [22], the Benjamin-Ono equation [23], and the Gardner equation [17], were found.

The key feature to find the DSW solution from modulation equations is the ability to set them in the appropriate Riemann variable form, which is guaranteed if the underlying equation is integrable. However, most equations governing DSWs in physical applications are not integrable. Then, based on Whitham's and Gurevich and Pitaevskii's research, El proposed the framework of a DSW fitting method which enables the analysis of DSWs governed by nonintegrable equations [24,25]. This method was used to find the leading (solitary wave) and trailing (linear wave) edges. In [26], Kamchatnov also investigated DSWs in nonintegrable equations.

All of these studies were restricted to (1+1)-dimensional PDEs and much less information was known about DSWs in multidimensional PDEs until recently. In the last decade, DSWs in two-space one-time (2+1) dimensional systems have been the subject of few studies. In [27], by using similarity variable, (1+1)-dimensional cylindrical reductions of the Kadomtsev-Petviashvili and two-dimensional Benjamin-Ono equations, their associated DSW solutions were investigated. We note that the method used in [27] works only under the special choice of parabolic front. Later, a generalization of Whitham theory for DSWs in the KP-type equations with a general class of initial conditions was developed [28]. The main result of this work was the derivation of the system of (2+1)-dimensional hydrodynamic-type equations, which describes the slow modulations of the periodic solutions of the corresponding KP-type equation. The method presented in [28] can be applied to DSW investigation in integrable and nonintegrable (2+1)-dimensional PDEs. Actually, the Gardner-KP equation (2) belongs to the equation class studied in [28]. However, quantitative results obtained from our present study about DSWs in the Gardner-KP equation will be very important to verify the theoretical results of the study in [28].

In this study, we employ a parabolic similarity reduction to the integrable (2+1)-dimensional Gardner-KP equation and then this equation reduces to the (1+1)-dimensional equation called the cylindrical Gardner (cG) equation. After that, we apply the Whitham modulation theory to the cG equation. We obtain two secularity (compatibility) conditions associated with the Whitham modulation theory for the cG equation by using a perturbation method. Then, these two secularity conditions with consistency requirement (conservation of waves) are used to derive a system of three quasilinear first-order partial differential equations. This system is called the Whitham modulation system and describes the modulations of the traveling wave solutions of the cG equation. By introducing appropriate Riemann-type variables, the corresponding modulation equations are transformed into simpler form. This simpler form is important to understand the dispersive shock wave phenomena in the cG equation. Next, we obtain the direct numerical solution of the cG equation and compare this solution with the DSW solution. The analytical results from the modulation theory are shown to be in good agreement with direct numerical solution of the cG equation.

The paper is organized as follows. In Sec. II, we consider the Gardner-KP equation. A special ansatz is used for Eq. (2), then the (2+1)-dimensional equation reduces to the cylindrical Gardner equation. In Sec. III, we derive the modulation equations in terms of three Riemann-type variables. The modulation equations have the form of a hyperbolic system of PDEs for the slowly varying parameters of the traveling wave solution. Section IV deals with the direct numerical solution of the cG equation. In Sec. V, the formation of DSW in the cG equation is considered, based on the results obtained in Sec. III. Also, both the direct numerical solution of the cG equation and the DSW solution obtained from the numerical solution of the corresponding Whitham system are compared. In addition, to observe the effect of the cylindrical term, the same examinations were carried out for the Gardner equation, inspired by the study [17]. Finally, all results are summarized in Sec. VI.

II. THE CYLINDRICAL GARDNER EQUATION

In this section, the cG equation is presented by reducing the (2+1)-dimensional Gardner-KP equation (2) with a parabolic similarity reduction [27]. We are interested in a class of initial conditions for the Gardner-KP equation describing almost steplike initial data as follows:

$$u(x,y,0) = \frac{1}{2}\left((f^{-}+f^{+}) + (f^{+}-f^{-})\tanh\left\{A\left[x+\frac{1}{2}\phi(y,0)\right]\right\}\right), \tag{3}$$

where f^{-}, f^{+}, and A are real constants. $\phi(y,t)$ describes the front shape of the solution of Eq. (2). In this study, we choose a parabolic front $\phi(y,0)=\tilde{c}y^2$ where $\tilde{c}$ is a real constant.

Ablowitz *et al.* have used a reduction method for the above initial data type to describe the dispersive shock waves in the Kadomtsev-Petviashvili and two-dimensional Benjamin-Ono equations [27]. The method used in [27] works under the special choice of a parabolic front or a planar front. However, the only way to obtain the cylindrical equation from the reduction of the (2+1) equation is to take the parabolic front.

We use the following ansatz,

$$u = f\left(x + \frac{\phi(y,t)}{2}, t, y\right), \tag{4}$$

for the Gardner-KP equation (2), where the front is then described by $x + \phi(y,t)/2 = \text{constant}$. When we substitute the ansatz (4) into Eq. (2) we obtain

$$\begin{aligned}&\left(\tfrac{1}{2}\phi_t f_\eta + f_t + 6ff_\eta - 6f^2 f_\eta + \epsilon^2 f_{\eta\eta\eta}\right)_\eta \\ +\quad &\lambda\left[\tfrac{1}{4}(\phi_y)^2 f_{\eta\eta} + \tfrac{1}{2}\phi_{yy}f_\eta + \phi_y f_{\eta y} + f_{yy}\right] = 0,\end{aligned} \tag{5}$$

where $\eta = x + \phi(y,t)/2$. Also, we assume that f satisfies the following boundary conditions at the infinities with $f^- > f^+ \geqslant 0$ for nonincreasing-type initial conditions:

$$f \to R(t)f^- \text{ as } \eta \to -\infty \text{ and } f \to R(t)f^+ \text{ as } \eta \to \infty. \tag{6}$$

The function $R(t)$ will be determined at the end of this section with the initial condition $R(0) = 1$.

Assuming that ϕ_{yy} is independent of y, due to the assumption of the parabolic front, and the ansatz is used as $u = f(x + \frac{\phi(y,t)}{2}, t)$, then the system of equations in the following form is obtained:

$$\phi_t + \frac{\lambda}{2}(\phi_y)^2 = 0, \tag{7}$$

$$f_t + 6ff_\eta - 6f^2 f_\eta + \frac{\lambda}{2}\phi_{yy}f + \epsilon^2 f_{\eta\eta\eta} = 0. \tag{8}$$

We call Eq. (7) the front shape equation which describes the evolution of the curvature of the parabolic front. However, Eq. (8) characterizes dispersive shock wave propagation of the wave front. Equation (7) can be transformed to the Hopf equation by using the transformation $v = \phi_y$:

$$v_t + \lambda v v_y = 0. \tag{9}$$

The solution of Eq. (9) with the initial condition $v(y,0) = 2\tilde{c}y$ is

$$v(y,t) = \frac{2\tilde{c}y}{1 + 2\tilde{c}\lambda t}. \tag{10}$$

Thus the front shape function $\phi(y,t)$ is obtained as

$$\phi(y,t) = \frac{\tilde{c}y^2}{1 + 2\tilde{c}\lambda t}. \tag{11}$$

The substitution of (11) into Eq. (8) gives the following cG equation:

$$f_t + 6ff_\eta - 6f^2 f_\eta + \frac{\lambda\tilde{c}}{1 + 2\tilde{c}\lambda t}f + \epsilon^2 f_{\eta\eta\eta} = 0. \tag{12}$$

Denoting $t_0 = 1/\lambda\tilde{c}$, the term $\lambda\tilde{c}/(1 + 2\tilde{c}\lambda t)$ transforms to $1/(2t + t_0)$.

The cG equation (8) with different coefficients of the cylindrical term describes a variety of wave phenomena in plasma physics [29]. It is obtained for the nonlinear propagation of Gardner solitons (GSs) in a nonplanar four-component dusty plasma in [29].

We will consider $\lambda = 1$. The other sign can be obtained by changing the $\tilde{c}$ to $-\tilde{c}$, i.e., changing the direction of the parabolic front.

Note that there is another possibility of the choice for the initial front. When the front is chosen planar as $\phi(y,0) = \tilde{c}y$, then ϕ_{yy} is independent of y and the form of Eq. (8) becomes the classical Gardner equation. DSWs in the Gardner equation were studied in [17].

We construct the DSW solution of the cG equation (12) with the step-type initial condition in the following form:

$$f(\eta,0) = \begin{cases} f^-, & \eta < 0, \\ f^+, & \eta > 0. \end{cases} \tag{13}$$

The structure of the waves modeled by Gardner and cylindrical Gardner equations depends on the values of the initial step parameters f^-, f^+. We require that $f^- > f^+ \geqslant 0$ in order for the generation of a DSW in the cG equation. In this study, we examine the equation (12) by considering the required condition for DSW formation in the Gardner equation [17].

Now to find the function $R(t)$ in the boundary conditions (6), first we neglect η-dependent terms in Eq. (12) and get an ordinary differential equation (ODE). The solution of this ODE with the initial condition $R(0) = 1$ determines the function $R(t)$ in the boundary conditions (6) as

$$R(t) = \frac{1}{\sqrt{1 + 2\tilde{c}t}}. \tag{14}$$

III. DERIVATION OF THE MODULATION EQUATIONS

The key to obtaining DSW solutions is the usage of the Whitham modulation theory [13,14]. We construct a DSW solution of the cG equation by using the multiple-scale method for analyzing slowly varying, nonlinear dispersive waves. The multiple-scale method used in this study was initially introduced by Luke [18]. By using this method, the Whitham modulation equations describing a system of PDEs for the slowly varying parameters of a periodic traveling wave solution, such as amplitude, wave number and mean height, are constructed. These equations are important to understand the dispersive shock wave phenomena in the cG equation.

A DSW consists of two edges, the trailing edge and leading edge, with a modulated dispersive wave train between these edges. These two edges move with different speeds. Also, the trailing edge corresponds to the small-amplitude sinusoidal wave train, while the leading edge corresponds to the large-amplitude solitary waves.

According to the modulation theory, wave parameters change slowly over fast oscillations within the DSW. This requirement can be formalized by introducing a rapidly varying phase variable, where

$$\theta_\eta = \frac{k}{\epsilon}, \quad \theta_t = -\frac{\omega}{\epsilon} = -\frac{kV}{\epsilon}. \tag{15}$$

Here η, t are slow space-time variables and $k(\eta,t)$, $\omega(\eta,t)$, and $V(\eta,t)$ are the wave number, frequency, and the phase velocity, respectively. We assume that $0 < \epsilon \ll 1$.

Since $(\theta_\eta)_t = (\theta_t)_\eta$ we obtain the compatibility condition (conservation of waves) as follows:

$$k_t + (kV)_\eta = 0. \tag{16}$$

By the following relations,

$$\frac{\partial}{\partial \eta} \to \frac{k}{\epsilon}\frac{\partial}{\partial \theta} + \frac{\partial}{\partial \eta}, \quad \frac{\partial}{\partial t} \to -\frac{\omega}{\epsilon}\frac{\partial}{\partial \theta} + \frac{\partial}{\partial t}, \tag{17}$$

Eq. (12) is transformed to

$$\left(-\frac{\omega}{\epsilon}\frac{\partial}{\partial \theta} + \frac{\partial}{\partial t}\right) f + 6f\left(\frac{k}{\epsilon}\frac{\partial}{\partial \theta} + \frac{\partial}{\partial \eta}\right) f + \frac{\lambda\tilde{c}}{1+2\tilde{c}\lambda t} f$$
$$- 6f^2\left(\frac{k}{\epsilon}\frac{\partial}{\partial \theta} + \frac{\partial}{\partial \eta}\right) f + \epsilon^2\left(\frac{k}{\epsilon}\frac{\partial}{\partial \theta} + \frac{\partial}{\partial \eta}\right)^3 f = 0. \tag{18}$$

Grouping the terms in like powers of ϵ, we rewrite Eq. (18) as seen below:

$$\frac{1}{\epsilon}(-\omega f_\theta + 6kff_\theta - 6kf^2 f_\theta + k^3 f_{\theta\theta\theta})$$
$$+\left(f_t + 6ff_\eta - 6f^2 f_\eta + 13k^2 f_{\eta\theta\theta} + 3kk_\eta f_{\theta\theta} + \frac{\lambda\tilde{c}}{1+2\tilde{c}\lambda t} f\right)$$
$$+\epsilon(3kf_{\theta\eta\eta} + 3k_\eta f_{\eta\theta} + k_{\eta\eta} f_\theta) + \epsilon^2 f_{\eta\eta\eta} = 0. \tag{19}$$

When we expand the function f in powers of ϵ as

$$f(\theta,\eta,t) = f_0(\theta,\eta,t) + \epsilon f_1(\theta,\eta,t) + \cdots, \tag{20}$$

the leading and the next order perturbation equations are obtained as

$$O\left(\frac{1}{\epsilon}\right) : -\omega f_{0,\theta} + 6kf_0 f_{0,\theta} - 6kf_0^2 f_{0,\theta} + k^3 f_{0,\theta\theta\theta} = 0, \tag{21}$$

$$O(1) : -\omega f_{1,\theta} + 6k(f_0 f_1)_\theta - 6kf_0^2 f_{1,\theta} - 12kf_0 f_1 f_{0,\theta} + k^3 f_{1,\theta\theta\theta} = U, \tag{22}$$

where

$$U = -\left(f_{0,t} + 6f_0 f_{0,\eta} - 6f_0^2 f_{0,\eta} + 3k^2 f_{0,\eta\theta\theta} + 3kk_\eta f_{0,\theta\theta} + \frac{f_0}{2t+t_0}\right). \tag{23}$$

We can proceed to higher order terms, but doing so is outside the scope of this paper.

In order to solve the leading order problem, the traveling wave solution of the defocusing Gardner equation is examined due to the similarity in the structure of this equation with Eq. (21). We consider the traveling wave ansatz $f = f(\xi)$, $\xi = x - Vt$ in Eq. (1) with the minus sign by taking f instead of u and integrate this equation twice with respect to ξ to obtain

$$f_\xi^2 = f^4 - 2f^3 + Vf^2 + Af + B, \tag{24}$$

where A and B are the constants of integration. The solution of this equation can be expressed in terms of the Jacobian elliptic functions cn and sn. A cnoidal wave solution of the Gardner equation is stable if all roots of the right-hand-side polynomial of Eq. (24) are all real, and unstable if two roots are real, two are complex [17]. Supposing that all real roots of the corresponding right-hand-side polynomial a_1, a_2, a_3, a_4 are ordered as

$$a_1 < a_2 < a_3 < a_4, \tag{25}$$

then a traveling wave solution exists for $a_2 < f < a_3$. In this case, the right-hand side of Eq. (24) is written as

$$f^4 - 2f^3 + Vf^2 + Af + B = (f - a_1)(f - a_2)(a_3 - f)(a_4 - f), \tag{26}$$

where

$$\begin{aligned} a_1 + a_2 + a_3 + a_4 &= 2, \\ a_1a_2 + a_1a_3 + a_1a_4 + a_2a_3 + a_2a_4 + a_3a_4 &= V, \\ -a_1a_2a_3 - a_1a_2a_4 - a_1a_3a_4 - a_2a_3a_4 &= A, \\ a_1a_2a_3a_4 &= B. \end{aligned} \tag{27}$$

Therefore, three of the a_j's are independent. In a modulated wave, which we are interested in, they are slowly varying functions of space coordinate η and time t, $a_i = a_i(\eta, t)$. Their evolution is governed by the Whitham modulation equations, which will describe dispersive shock wave formation of the cG equation.

Provided that $a_2 < f < a_3$, Eq. (24) can be rewritten formally as

$$\frac{df}{\sqrt{(f-a_1)(f-a_2)(a_3-f)(a_4-f)}} = d\xi. \tag{28}$$

If we integrate Eq. (28), the solution of Eq. (24), which is also the solution of the leading order problem (21), in terms of Jacobi elliptic functions is obtained as

$$f_0 = a_2 + \frac{(a_3 - a_2)\mathrm{cn}^2(2(\theta-\theta_0)K, m)}{1 - \frac{a_3-a_2}{a_4-a_2}\mathrm{sn}^2(2(\theta-\theta_0)K, m)}. \tag{29}$$

Here $K = K(m)$ is the complete elliptic integral of the first kind and m is the modulus of the elliptic function cn, where

$$m^2 = \frac{(a_3-a_2)(a_4-a_1)}{(a_3-a_1)(a_4-a_2)}. \tag{30}$$

Note that there is a free constant θ_0 in Eq. (29). It is possible to find θ_0 by constructing Whitham equations to higher order in much the same way as one can develop for higher order KdV or nonlinear Schrödinger type equations in physical applications. Modulation of the phase shift in KdV-type equations was investigated in [30,31] by considering the higher order Whitham theory. Such higher order analysis is outside the scope of this study, because higher order theory is not straightforward. We determine the approximated value of θ_0 by comparison with direct numerical solutions. For more precise results, the higher order Whitham theory must be considered for the Gardner and cylindrical Gardner equations.

As mentioned before our aim is to obtain the three modulation equations for the three independent parameters a_2, a_3, a_4 of the solution (29). k, m, and V will be expressed in terms of these independent variables. One of these modulation equation is Eq. (16), which is the conservation of waves. To obtain the other two equations, the problem $O(1)$ given in Eq. (22) should be examined. If the leading order solution (29) is used in Eq. (22), secular terms, arbitrarily large growing terms with respect to θ, occur. To eliminate these terms, we enforce the periodicity of f_0 in θ and obtain the secularity conditions as

$$\int_0^1 U d\theta = 0 \quad \text{and} \quad \int_0^1 f_0 U d\theta = 0. \tag{31}$$

Replacing U given in (23) into the Eqs. (31), we obtain

$$\frac{\partial}{\partial t}\int_0^1 f_0 d\theta+\frac{\partial}{\partial \eta}\int_0^1 (3f_0^2-2f_0^3)d\theta+\frac{1}{2t+t_0}\int_0^1 f_0 d\theta=0 \tag{32}$$

and

$$\frac{\partial}{\partial t}\int_0^1 f_0^2 d\theta+\frac{\partial}{\partial \eta}\int_0^1 (4f_0^3-3f_0^4-3k^2 f_{0,\theta}^2)d\theta + \frac{2}{2t+t_0}\int_0^1 f_0^2 d\theta=0. \tag{33}$$

Equations (16), (32), and (33) are the required modulation equations. If we calculate the functions $f_0^2, f_0^3, f_0^4, f_{0,\theta}^2$ and their integrals by using the properties of elliptic functions [32], we can obtain modulation equations in terms of a_2, a_3, a_4 variables [a_1 is eliminated with the help of the first equation in (27)].

Note that we need to use the derivative formulas (A5)–(A8) of the elliptic integrals in the first, second, and third types (see Appendix) to obtain the system of modulation equations which is a first-order quasilinear PDE system with the following form,

$$\mathbf{u}_t+A(\mathbf{u})\mathbf{u}_\eta+B(\mathbf{u})\frac{1}{2t+t_0}=0, \tag{34}$$

where $\mathbf{u}(\eta,t)=(a_2,a_3,a_4)$, $A(\mathbf{u})$ is a 3×3 matrix, and $B(\mathbf{u})$ is a 3×1 vector.

The modulation equations can be simplified by replacing the variables a_2, a_3, a_4 with r_1, r_2, r_3 ($r_1\leqslant r_2\leqslant r_3$), where r_i, $i=1,2,3$, are the Riemann variables. For the cG equation we can take Riemann variables as given below [17]:

$$\begin{aligned} r_1&=\frac{1}{4}(a_1+a_2)(a_3+a_4),\\ r_2&=\frac{1}{4}(a_1+a_3)(a_2+a_4),\\ r_3&=\frac{1}{4}(a_2+a_3)(a_1+a_4). \end{aligned} \tag{35}$$

Thus, the Whitham modulation system in Eq. (34) can be transformed to the simpler form:

$$\frac{\partial r_i}{\partial t}+v_i(r_1,r_2,r_3)\frac{\partial r_i}{\partial \eta}+\frac{h_i(r_1,r_2,r_3)}{2t+t_0}=0,\quad i=1,2,3, \tag{36}$$

where

$$\begin{aligned} v_1&=2(r_1+r_2+r_3)+\frac{4(r_2-r_1)K(m)}{E(m)-K(m)},\\ v_2&=2(r_1+r_2+r_3)-\frac{4(r_2-r_1)(1-m^2)K(m)}{E(m)-(1-m^2)K(m)},\\ v_3&=2(r_1+r_2+r_3)+\frac{4(r_3-r_2)K(m)}{E(m)}, \end{aligned} \tag{37}$$

and the cylindrical terms can be expressed in the form below, with the assumption $S_i=1-4r_i,\ i=1,2,3$, for simplicity in processes,

$$\begin{aligned} h_1&=-\frac{S_1E}{E-K}-\frac{\sqrt{S_1}(\sqrt{S_2}+\sqrt{S_3})(\sqrt{S_1}+\sqrt{S_2}\sqrt{S_3})\Pi}{(E-K)(S_1-S_3)}\\ &\quad+\frac{K\sqrt{S_1}[S_1+\sqrt{S_1}+S_2+\sqrt{S_2}+(\sqrt{S_2}-\sqrt{S_1})\sqrt{S_3}]}{2(E-K)(\sqrt{S_1}-\sqrt{S_3})},\\ h_2&=-\frac{\sqrt{S_2}(\sqrt{S_2}+\sqrt{S_3})(\sqrt{S_2}+\sqrt{S_1}\sqrt{S_3})\Pi}{E(S_1-S_3)+K(S_3-S_2)}\\ &\quad+\frac{E(S_3-S_1)S_2}{E(S_1-S_3)+K(S_3-S_2)}+K\sqrt{S_2}(\sqrt{S_2}+\sqrt{S_3})\\ &\quad\times\frac{[S_1+S_2-\sqrt{S_2}(-1+\sqrt{S_3})+\sqrt{S_1}(1+\sqrt{S_3})]}{2[E(S_1-S_3)+K(S_3-S_2)]},\\ h_3&=-S_3+\frac{\sqrt{S_3}(\sqrt{S_2}+\sqrt{S_3})(\sqrt{S_1}\sqrt{S_2}+\sqrt{S_3})\Pi}{E(S_3-S_1)}\\ &\quad+\frac{\sqrt{S_3}(\sqrt{S_2}+\sqrt{S_3})(1+\sqrt{S_1}+\sqrt{S_2}-\sqrt{S_3})K}{2(\sqrt{S_1}-\sqrt{S_3})E}. \end{aligned} \tag{38}$$

In Eq. (37), v_i's are the Whitham characteristic velocities for the defocusing Gardner equation [17]. Also, $K=K(m)$, $E=E(m)$, and $\Pi=\Pi(n,m)$ denote the complete elliptic integrals of the first, second, and the third kinds, respectively. Here n denotes the parameter of the elliptic integral of the third kind. Properties of these complete elliptic integrals are listed in the Appendix.

The expressions of the module, the parameter of the third elliptic integral, the phase speed, and the wave number in terms of Riemann variables are as follows:

$$\begin{gathered} m=\frac{\sqrt{r_2-r_1}}{\sqrt{r_3-r_1}},\qquad n=\frac{\sqrt{1-4r_1}-\sqrt{1-4r_2}}{\sqrt{1-4r_1}+\sqrt{1-4r_3}},\\ V=2(r_1+r_2+r_3),\qquad k=\frac{\sqrt{r_3-r_1}}{2K(m)}. \end{gathered} \tag{39}$$

Note that Eq. (36) reduces to a diagonal system in the absence of cylindrical terms, i.e., $t_0\to\infty$, which agrees with the Whitham system for the defocusing Gardner equation [17].

IV. NUMERICAL SOLUTION OF THE CYLINDRICAL GARDNER EQUATION

In this section, the direct numerical solution associated with the cG equation is examined. Then we will compare this numerical solution with the corresponding Whitham modulation system in the next section.

The defocusing Gardner equation,

$$f_t+6ff_\eta-6f^2f_\eta+\epsilon^2 f_{\eta\eta\eta}=0, \tag{40}$$

with the initial condition (13) depending on the positions of the initial step parameters f^- and f^+, is considered in [17]. According to the analysis, wave structures with different cases such as dispersive shock waves (undular bores), rarefaction waves, solibores, and reversed rarefaction waves have been observed depending on the choice of the initial step parameters f^- and f^+. However, because our study is related to dispersive shock waves, other cases are outside the scope of this paper. For the formation of a dispersive shock wave, f^+

and f^- must satisfy the inequality

$$f^+ < f^- \leqslant 1/2. \quad (41)$$

In [17], the full classification of the solutions of the Gardner equation to the step problem is done. According to this classification, there is a one-to-one correspondence between the dispersionless limits of the KdV and the Gardner equations when f^+ and f^- lie in the region (41), where the function $f(1-f)$ is monotonically increasing and also a quadratic map from a solution of the dispersionless Gardner equation to a solution of the dispersionless KdV equation. These results indicate that in the region (41) there is a qualitative equivalence between DSW solutions of the KdV equation and the Gardner equation at the leading and trailing edges (see the discussion in [17] for details). However, we could not find a similar map from the dispersionless cG equation to the dispersionless cKdV equation. So, we do not have any proof about a qualitative equivalence between DSW solutions of the cG equation and cKdV equations in the region (41). But, this equivalence phenomena must be examined in detail as a classification of the solutions of the cG equation to the step problem.

In our numerical simulations, we use a numerical procedure which is useful for problems with fixed boundary conditions. However, for the cG equation,

$$f_t + 6ff_\eta - 6f^2 f_\eta + \frac{1}{2t+t_0} f + \epsilon^2 f_{\eta\eta\eta} = 0, \quad (42)$$

f satisfies the boundary conditions which are given for nonincreasing-type initial conditions as

$$f \to R(t)f^- \text{ as } \eta \to -\infty \text{ and } f \to R(t)f^+ \text{ as } \eta \to \infty, \quad (43)$$

where $R(t) = \sqrt{\frac{t_0}{2t+t_0}}$ with $t_0 = \frac{1}{\tilde{c}}$. Since these boundary conditions are functions of t, we use the transformation

$$f = R(t)\psi \quad (44)$$

and transform Eq. (42) into the following Gardner equation with variable coefficients:

$$\psi_t + 6R(t)\psi\psi_\eta - 6[R(t)]^2\psi^2\psi_\eta + \epsilon^2\psi_{\eta\eta\eta} = 0. \quad (45)$$

This equation has the constant left boundary condition with $\psi_- = f^-$ and the right boundary condition is $\psi_+ = f^+$. We use a modified version of the exponential time differencing fourth-order Runge-Kutta (ETDRK4) method [33]. For the required spectral accuracy of the method, the initial condition must be smooth and periodic. However, the step initial condition (3) for u or equivalently for f is nonperiodic. To deal with this problem, we differentiate Eq. (45) with respect to η and define $\psi_\eta = z$. We obtain

$$z_t + 6R(t)(\psi z)_\eta - 6[R(t)]^2(\psi^2 z)_\eta + \epsilon^2 z_{\eta\eta\eta} = 0. \quad (46)$$

The initial condition is regularized with the analytic function [34]

$$z(\eta, 0) = -\frac{\hat{C}}{2}\text{sech}^2(\hat{C}\eta), \quad (47)$$

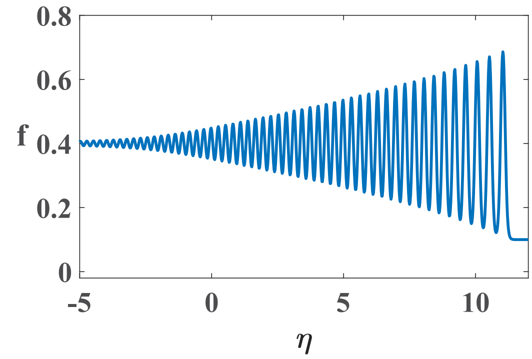

FIG. 1. Numerical solution of Gardner equation at $t = 10$ with the initial condition (13) where $f^- = 0.4$ and $f^+ = 0.1$. Here $t_0 = 10$ and $\epsilon^2 = 0.001$.

where $\hat{C} > 0$ is a large parameter. Therefore, the initial condition is now periodic and smooth, which is convenient for the numerical method.

To work on the Fourier space, we rewrite Eq. (46) as

$$z_t = \boldsymbol{L}\hat{z} + 6R(t)\boldsymbol{N_1}(\hat{z}, t) - 6[R(t)]^2\boldsymbol{N_2}(\hat{z}, t), \quad (48)$$

where $\hat{z} = \mathcal{F}(z)$ is the Fourier transform of z, $\boldsymbol{L}$ is the linear term, and $\boldsymbol{N_1}$, $\boldsymbol{N_2}$ are the nonlinear terms. The expressions of linear and nonlinear terms are given as

$$\begin{aligned} \boldsymbol{L}\hat{z} &= -i\epsilon^2 k^3 \hat{z}, \\ \boldsymbol{N_1}(\hat{z}, t) &= -ik\mathcal{F}\left(\left[\int_{-L}^{\eta} \mathcal{F}^{-1}(\hat{z})d\eta' + \psi_-\right]\mathcal{F}^{-1}(\hat{z})\right), \\ \boldsymbol{N_2}(\hat{z}, t) &= -ik\mathcal{F}\left(\left[\int_{-L}^{\eta} \mathcal{F}^{-1}(\hat{z})d\eta' + \psi_-\right]^2\mathcal{F}^{-1}(\hat{z})\right). \end{aligned} \quad (49)$$

Consequently, equation (46) is solved numerically via Eqs. (49) on a finite spatial domain $[-L, L]$. For the method ETDRK4, we take the number of Fourier modes in space as $N = 2^{12}$, the domain size is $L = 40$, and the time step is $h = 10^{-3}$. Furthermore, the parameters $\tilde{c}^{-1} = t_0 = 10$, $\epsilon^2 = 10^{-3}$, and $\hat{C} = 10$.

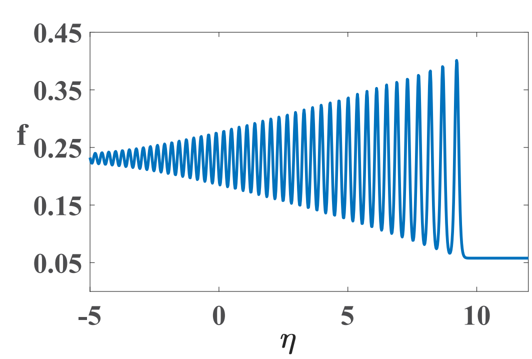

FIG. 2. Numerical solution of cG equation at $t = 10$ with the initial condition (13) where $f^- = 0.4$ and $f^+ = 0.1$. Here $t_0 = 10$ and $\epsilon^2 = 0.001$.

The numerical solutions of the Gardner equation and the cG equation at $t = 10$ are presented in Fig. 1 and Fig. 2, where the step parameters are chosen as $f^- = 0.4$ and $f^+ = 0.1$.

From Figs. 1 and 2, it is observed that the amplitude of the DSW for the Gardner equation locates to a steady state while the amplitude of the DSW for the cG equation decreases in time (see also space-time plots, Fig. 7).

In the next section, DSW solutions obtained by the numerical solutions of the modulation equations and the direct numerical simulations of the Gardner and the cG equation will be compared. This allows us to understand the underlying structure of the DSWs in the cG equation.

V. DISPERSIVE SHOCK WAVES IN THE CYLINDRICAL GARDNER EQUATION AND COMPARISON WITH NUMERICAL RESULTS

In this section we obtain the DSW solution of the cG equation by using the solution of the Whitham modulation system given in (36). Then we compare this DSW solution with direct numerical simulation of the cG equation. Since the Whitham system (36) is not in diagonal form, it is difficult to get its analytical solution. Therefore we solve the corresponding Whitham system by using a numerical method.

It is clear that without the cylindrical term, the system in (36) reduces to the Whitham system for the Gardner equation [17]. To observe the effect of the cylindrical term on the DSW solution, the numerical solution of the corresponding modulation equations for the Gardner equation will be also investigated. For this purpose, we use a first-order hyperbolic PDE solver based on MATLAB by Shampine [35] and choose a two-step variant of the Lax-Wendroff method with a nonlinear filter [36].

In order to solve Whitham modulation equations numerically, we must first obtain boundary conditions for the Whitham system. The boundary conditions for the Gardner equation and associated Whitham system of this equation remain constant at both ends of the domain, since they do not contain time dependency (see Fig. 1). However, the boundary conditions for the cG equation and associated Whitham system are the functions of time (see Fig. 2). The boundary conditions for the cG equation can be seen in Eq. (43). In order to find the boundary conditions for the Whitham system (36), we solve the ODE system obtained from Eq. (36) analytically by neglecting the spatial variable. The reduced ODE system is solved with the initial conditions (50) at both ends separately. The initial values of Riemann variables for the Whitham system are given as follows (see Fig. 3):

$$r_1(\eta, 0) = 0.09, \quad r_3(\eta, 0) = 0.24,$$
$$r_2(\eta, 0) = \begin{cases} 0.09, & \eta \leqslant 0, \\ 0.24, & \eta > 0. \end{cases} \tag{50}$$

Since the structures of the analytic solutions are so complicated, the exact forms of these solutions of the reduced ODE system are not given here. The numerical solutions of Whitham systems including boundary conditions at $t = 10$ are given in Fig. 4 for Gardner and cG equations.

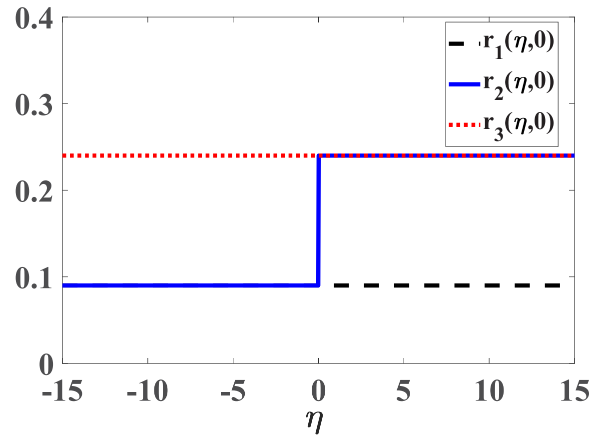

FIG. 3. Initial values of Riemann variables.

In the numerical solutions of the Whitham systems, we use $N = 2^{12}$ points for the spatial domain $[-40, 40]$. Furthermore, the parameter $\tilde{c}^{-1} = t_0 = 10$ in the cG equation.

From Fig. 4, it should be noted that a difference between both of the positions of the intersection points of the Riemann variables r_1 and r_2 at the trailing edges is observed when the behavior of Riemann variables of Gardner and cG equations are compared. A similar difference is observed at the positions of the intersection points of the Riemann variables r_1 and r_2 at the leading edge, too. These differences imply some

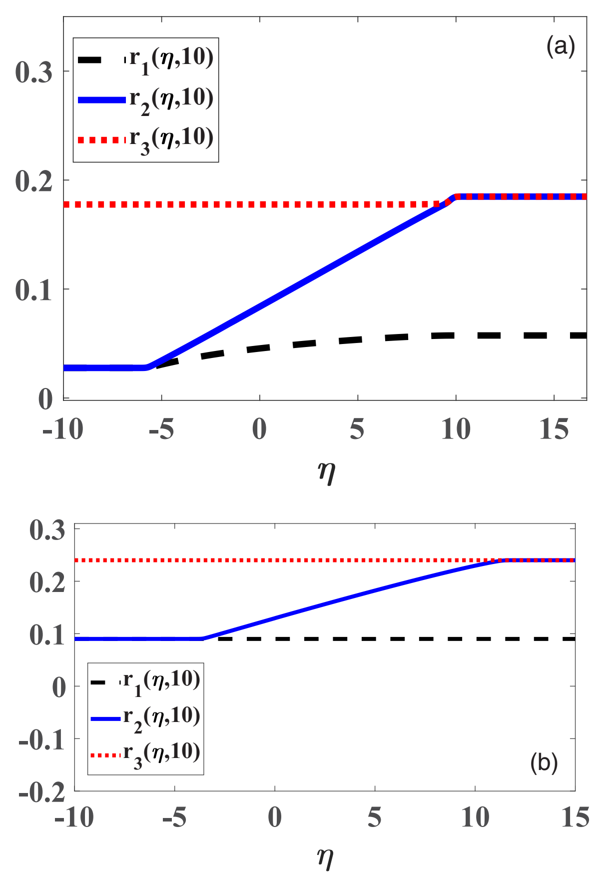

FIG. 4. Riemann variables at $t = 10$ which are found by numerical solutions of (36) (a) for cG equation and (b) for Gardner equation. Here we take $t_0 = 10$.

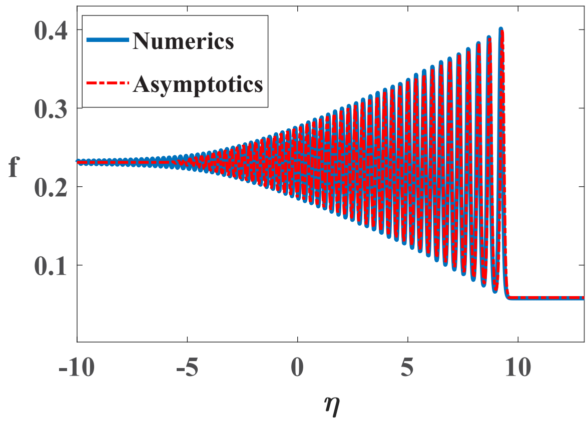

FIG. 5. Numerical and asymptotic solutions of cG equation at $t = 10$ with the initial condition (13) where $f^- = 0.4$ and $f^+ = 0.1$. Here $t_0 = 10$ and $\epsilon^2 = 0.001$.

results about the edge dynamics of DSWs in Gardner and cG equations. Both the leading edge and trailing edge of the DSW in the cG equation move slower than both the leading and trailing edges of the DSW in the Gardner equation. All these results agree with Figs. 5, 6, and 7 and an animation [37]. In the animation [37], the propagation of DSWs in both Gardner and cG equations is between $t = 0$ and $t = 10$.

Now, in order to compare the solution of the cG equation with the direct numerical solution, we first write the periodic wave solution, f_0, in terms of Riemann variables r_i,

$$f_0(\theta, \eta, t) = \frac{1}{2}(1 - \sqrt{S_1} + \sqrt{S_2} - \sqrt{S_3}) + \frac{(\sqrt{S_1} - \sqrt{S_2})(\sqrt{S_1} + \sqrt{S_3})\mathrm{cn}^2(2(\theta - \theta_0)K, m)}{(\sqrt{S_1} + \sqrt{S_3}) + (\sqrt{S_2} - \sqrt{S_1})\mathrm{sn}^2(2(\theta - \theta_0)K, m)}, \tag{51}$$

where $S_i = 1 - 4r_i,\ i = 1, 2, 3$. By integrating (15), the rapid phase θ is obtained as

$$\theta(\eta, t) = \int_{-L}^{\eta} \frac{k(\eta', t)}{\epsilon} d\eta' - \int_0^t \frac{k(\eta, t')V(\eta, t')}{\epsilon} dt'. \tag{52}$$

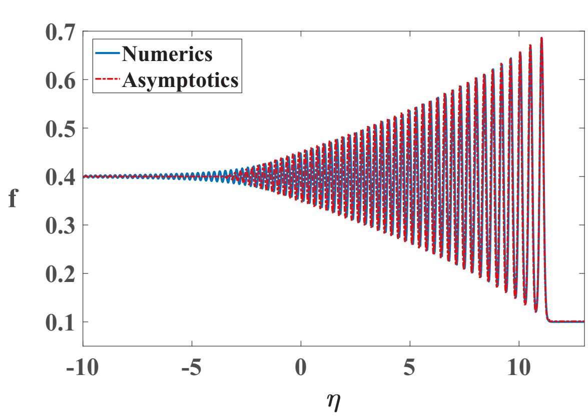

FIG. 6. Numerical and asymptotic solutions of Gardner equation at $t = 10$ with the initial condition (13) where $f^- = 0.4$ and $f^+ = 0.1$. Here $t_0 = 10$ and $\epsilon^2 = 0.001$.

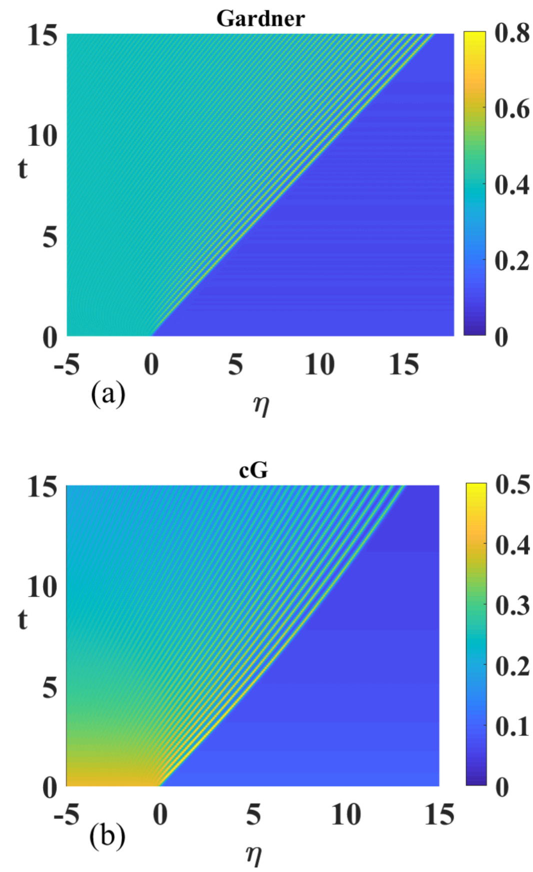

FIG. 7. Space-time plot of the direct numerical solution between $t = 0$ and $t = 15$ (a) for the Gardner equation and (b) for the cylindrical Gardner equation. Here, we take $t_0 = 10$ and $\epsilon^2 = 0.001$.

The asymptotic solution of the cG equation, f_0, is obtained by using (51) and the formula (52) of θ.

Thus, the DSW solutions can be generated at any time for both Gardner and cG equations from the Riemann variables r_i using Eqs. (51) and (52). The direct numerical solution of the cG equation is plotted and compared with the asymptotic solution in Fig. 5. Accordingly, it can be observed from Fig. 5 that the leading edge amplitude and wavelength of oscillations are compatible in both the asymptotic and numerical solutions. Thus, Whitham modulation theory enables us to obtain correct and appropriate approaches for dispersive shock waves in the cG equation.

In the numerical approach, the phase shift θ_0 has been arbitrarily chosen as it is compatible with direct numerical simulations. For this adjustment, we find the mean value of the DSW; that is, we compute the average of the leading hump (the largest amplitude soliton) and the trailing edge. Then θ_0 is selected as the center of the nearest wave determined from the asymptotic solution which is identical to the corresponding hump in the direct numerical simulations. For the cG equation, the average is approximately $(0.4001 + 0.2309)/2 = 0.3155$ and the asymptotic solution has a hump in the middle (center) region with a value of amplitude 0.2958.

A similar analysis can be done for the Whitham system of the Gardner equation to make a comparison. By using the same procedure for this equation, the asymptotic solution can be obtained by taking the cylindrical term $\tilde{c} = 0$. The initial values of the Riemann variables are the same as the cG equation (Fig. 3). Then, the direct numerical simulation of the Gardner equation and the asymptotic solution at $t = 10$ are compared in Fig. 6. For the Gardner equation, the average of the leading hump and the trailing edge is approximately $(0.6878 + 0.4)/2 = 0.5439$ and a hump in the middle region has an amplitude value 0.5199. As we can see, the direct numerical simulations and asymptotic solutions are compatible for both Gardner and cG equations. However, one needs to proceed to higher order terms in the asymptotic expansion (20) to achieve better results for the phase term θ_0. But, this is outside the scope of this paper.

The space-time plots of the direct numerical solutions of the Gardner and cG equations are given in Fig. 7, to emphasize the difference between DSW solutions of these equations. We observed that the DSW humps in the cG equation move slower than in the Gardner equation. The spreading behavior of DSW humps of the Gardner and cG equations are also observed in an animation [37] between $t = 0$ and $t = 10$. In Fig. 7(b), it is seen that the wave fronts in the cylindrical equation are curved. This is the result of geometrical spreading. This phenomenon is reported also for dispersive shock waves in colloidal media [38]. Since the wave front is spreading in the cG equation, the amplitude of the wave decreases to conserve mass and energy.

VI. CONCLUSIONS

We investigate the DSW solution of the cG equation by using Whitham modulation theory. The cG equation is derived from the reduction of the (2+1)-dimensional Gardner-KP equation with steplike initial condition along a parabolic front. The formation of the DSW depends on the selection of the initial step parameters f^-, f^+. We require that $0 \leqslant f^+ < f^- \leqslant 1/2$ in order to generate the DSW solution in the cG equation. Then by using the multiple-scale method we obtain a system of quasilinear modulation equations describing slow evolution of parameters in the periodic solution of a dispersive nonlinear partial differential equation. We solve the corresponding Whitham modulation equations numerically and compare these results with direct numerical solutions of the cG equation. A good agreement is found between these numerics except the negligible phase term. The effect of this phase term can be analyzed by considering higher order terms, but this is outside the scope of this study. In order to observe the contribution of the cylindrical term in the cG equation, a DSW formation of the Gardner equation is also considered in this study. The conclusion we get from the simulations is that the amplitude decreases over time for the cylindrical equation in the trailing and leading edge because of the geometric spreading effect. Observations can be made similarly for other t values.

In [17], other wave-type solutions of the Gardner equation such as rarefaction waves, solibores, and reversed rarefaction waves were studied depending on the choice of the initial step parameters f^- and f^+. A similar analysis can be performed for the cG equation. We address this investigation for near-future studies.

The method introduced in this study for the reduction of the Gardner-KP works only for a special choice of initial front, e.g., a parabolic front. However, Ablowitz *et al.* generalized the Whitham theory to find DSW solutions of the KP-type equations with a general class of initial conditions [28]. In this study, a (2+1)-dimensional Whitham modulation system was derived, which describes the slow modulations of the periodic solutions of the corresponding KP-type equations. To our knowledge any quantitative results for the solutions of derived (2+1)-dimensional Whitham systems have not been reported yet. The Gardner- KP equation belongs to the KP class which was investigated in [28]. Our results about the cG equation can be used as a test subject to verify the DSW solutions obtained from the solution of the (2+1)-dimensional Gardner-KP Whitham system.

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APPENDIX: COMPLETE ELLIPTIC INTEGRALS

In this Appendix, some properties of complete elliptic integrals used in the study will be listed.

The complete elliptic integral of the first kind has the expansion

$$K(m) = \frac{\pi}{2}\left[1 + \frac{m}{4} + \frac{9}{64}m^2 + \cdots + \left(\frac{1 \times 3 \times \cdots \times (2n-1)}{2 \times 4 \times \cdots \times 2n}\right)^2 m^n + \cdots\right]. \tag{A1}$$

The series expansion of the second kind of elliptic integral is

$$E(m) = \frac{\pi}{2}\left[1 - \frac{m}{4} - \frac{3}{64}m^2 - \cdots - \frac{1}{2n-1}\left(\frac{1 \times 3 \times \cdots \times (2n-1)}{2 \times 4 \times \cdots \times 2n}\right)^2 m^n - \cdots\right], \tag{A2}$$

for $|m| < 1$.

The compete elliptic integral of the third kind has the following behavior:

$$\Pi(n, m) = \frac{\pi}{2} \quad \text{when} \quad n = 0, \quad m = 0, \tag{A3}$$

$$\frac{\Pi(n, m)}{K(m)} \approx \frac{1}{1-n} \quad \text{when} \quad m \quad \text{is close to 1.} \tag{A4}$$

The following are the derivative formulas:

$$\frac{dK(m)}{dm} = \frac{E(m) - (1-m)K(m)}{2(1-m)m}, \tag{A5}$$

$$\frac{dE(m)}{dm} = \frac{E(m) - K(m)}{2m}, \tag{A6}$$

$$\frac{d\Pi(n,m)}{dm} = \frac{E(m) - (1-m)\Pi(n,m)}{2(1-m)(m-n)}, \tag{A7}$$

$$\frac{d\Pi(n,m)}{dn} = \frac{nE(m) + (m-n)K(m)}{2n(1-n)(n-m)} + \frac{(n^2-m)\Pi(n,m)}{2n(1-n)(n-m)}. \tag{A8}$$

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