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Quantum entanglement and reflection coefficient for coupled harmonic oscillators

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Quantum entanglement of a system of two coupled quantum harmonic oscillators with a Hamiltonian $\hat{H} = \frac{1}{2}(\frac{1}{m_1}\hat{p}_1^2 + \frac{1}{m_2}\hat{p}_2^2 + Ax_1^2 + Bx_2^2 + Cx_1x_2)$ can be found in many applications of quantum and nonlinear physics, molecular chemistry, and biophysics. Despite this, the quantum entanglement of such a system is still a problem under study. This is primarily due to the fact that the system is multiparametric and the quantum entanglement of such a system is not defined in a simple analytical form. This paper solves this problem and shows that quantum entanglement depends on only one parameter that has a simple physical meaning: the reflection coefficient $R \in (0, 1)$. The reflection coefficient R has a simple analytical form and includes all the parameters of the system under consideration. It is shown that for certain values of the coefficient R, the quantum entanglement can be large. The developed theory can be used not only for calculating quantum entanglement, but also for many other applications in physics, chemistry, and biophysics, where coupled harmonic oscillators are considered.

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I. INTRODUCTION

The study of the properties of coupled harmonic oscillators is an important area of modern physics. This interest is primarily because models of such systems are found in many applications of quantum and nonlinear physics [1–10], molecular chemistry [11–13], and biophysics [14–17]. In quantum physics, this interest is mainly due to quantum entanglement for such a system. In particular, quantum communication protocols, such as quantum cryptography [18], quantum dense coding [19], quantum computing algorithms [20], and teleportation of the quantum state [21,22], can be explained using entangled states. On the other hand, physical models of coupled harmonic oscillators have been used in many physics works, such as the Lee model in quantum field theory [2–4] and others. A similar Hamiltonian is also used in biophysics to explain the problem of photosynthesis [14–17]. The study of the properties of coupled harmonic oscillators, mainly quantum entanglement, is a separate direction in quantum physics. This is primarily because coupled harmonic oscillators are a good model of real physical objects. Such objects include thermal vibrations of bound atoms, photons in cavities, ions in traps, and much more. The study of coupled harmonic oscillators is one of the main methods for studying quantum decoherence (see, e.g., [23,24]).

In particular, of great interest is the system with the Hamiltonian in the form

$$\hat{H} = \frac{1}{2}\left(\frac{1}{m_1}\hat{p}_1^2 + \frac{1}{m_2}\hat{p}_2^2 + Ax_1^2 + Bx_2^2 + Cx_1x_2\right), \qquad (1)$$

where $\hat{p}_k = -i\hbar\frac{\partial}{\partial x_k}$ (where $k = 1, 2$) is the momentum operator. The solution of the stationary (e.g., [6,25]) and nonstationary [25] Schrodinger equation with the Hamiltonian (1) is well known. Despite this, there is a problem of the complexity of calculating and analyzing quantum entanglement for such a system. This is primarily due to the fact that the system is multiparameter and the quantum entanglement of such a system in a simple analytical form is not defined. It should be added that at present, the study of quantum entanglement for a system with the Hamiltonian (1) mainly takes into account interactions with the environment. In this case, the master equation for the reduced density matrix of the two oscillators ρ (e.g., [9,10,26]) is solved. Most of the results obtained for such systems do not have an analytical form, but were obtained by numerical simulation. Therefore, the study of the quantum entanglement of coupled harmonic oscillators based on simple analytical expressions is an actual direction. It should be added that this work is studying quantum entanglement for a dynamical system, thus the nonstationary Schrodinger equation is solved. Such systems can be found in many applications of physics, etc. In this case, it is assumed that at the initial moment of time $t = 0$ the system of oscillators was not coupled, and at $t > 0$ a connection appears in the system. Physically, this means that at $t = 0$ the oscillators did not interact, and at $t > 0$, as a result of some process, a connection appears, as a result of which quantum entanglement of the system arises. As will be shown below, there are many examples of such systems. The quantum entanglement of stationary systems is well studied and is not considered here.

In this paper, an expression is obtained for the quantum entanglement of a coupled harmonic oscillator. It is shown that quantum entanglement depends on only one parameter, which has a simple physical meaning: the reflection coefficient $R \in (0, 1)$. Quantum entanglement was investigated on the basis of the Schmidt [27,28] modes, and the von Neumann entropy and Schmidt parameter were used as a measure of quantum entanglement. In the expressions obtained, the reflection coefficient

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R has a simple analytical form and includes all the parameters of the system under consideration. It is shown that for certain values of the coefficient R quantum entanglement can be large. The results obtained have the simplest and most convenient analytical form for the analysis of the quantum entanglement of the system under study. The developed theory can be used not only for calculating quantum entanglement, but also for many other applications in physics, chemistry, and biophysics, where coupled harmonic oscillators are considered.

II. REFLECTION COEFFICIENT *R* FOR COUPLED HARMONIC OSCILLATORS

Usually, to study the quantum entanglement of coupled harmonic oscillators, the nonstationary Schrodinger equation $\hat{H}\Psi = i\hbar\frac{\partial\Psi}{\partial t}$ is solved, with the Hamiltonian (1), e.g., [25]. We will approach this problem in a different way, namely, we will look for evolution for the operator of the creation or annihilation of quantum states. We assume that the system at the initial time $t = 0$ was in the state $|s_1, s_2\rangle$ [the system was not connected, i.e., at $t = 0$ in Eq. (1) the constant $C = 0$], which are eigenfunctions of unrelated oscillators whose quantum numbers are s_1, s_2. The Hamiltonian (1) for such a system can be represented as

$$\hat{H} = \sum_{k=1}^{2} \hbar\omega_k\left(\hat{a}_k^\dagger \hat{a}_k + 1/2\right) + \hbar\Omega x_1 x_2,$$
$$\omega_1 = \sqrt{A/m_1};\ \omega_2 = \sqrt{B/m_2};\ \Omega = \frac{C}{2(m_1 m_2 A B)^{1/4}}, \tag{2}$$

where $\hat{a}_k = \frac{1}{\sqrt{2}}(x_k + \frac{\partial}{\partial x_k})$, $\hat{a}_k^\dagger = \frac{1}{\sqrt{2}}(x_k - \frac{\partial}{\partial x_k})$, which are, respectively, the operators of annihilation and creation of quantum states $|s_k\rangle$ [29,30]. These operators are related to quantum states by the relations $\hat{a}_k|s_k\rangle = \sqrt{s_k}|s_k - 1\rangle$ and $\hat{a}_k^\dagger|s_k\rangle = \sqrt{s_k + 1}|s_k + 1\rangle$.

Next, we will make variable replacements, similar to [25], we get

$$\hat{H} = \frac{\hbar\sqrt{\omega_1\omega_2}}{2}\sum_{k=1}^{2}\left(A'_k y_k^2 - \frac{\partial^2}{\partial y_k^2}\right),$$
$$A'_1 = \frac{\omega_1}{\omega_2} - \frac{\Omega}{\sqrt{\omega_1\omega_2}}\tan\alpha;\ A'_2 = \frac{\omega_2}{\omega_1} + \frac{\Omega}{\sqrt{\omega_1\omega_2}}\tan\alpha, \tag{3}$$

where $y_1 = (\omega_2/\omega_1)^{1/4}x_1\cos\alpha - (\omega_1/\omega_2)^{1/4}x_2\sin\alpha$, $y_2 = (\omega_2/\omega_1)^{1/4}x_1\sin\alpha + (\omega_1/\omega_2)^{1/4}x_2\cos\alpha$; $\tan\alpha \in (-1, 1)$, because $\tan\alpha = \frac{\varepsilon}{|\varepsilon|}\sqrt{\varepsilon^2 + 1} - \varepsilon$, where $\varepsilon = \frac{\omega_2^2 - \omega_1^2}{2\sqrt{\omega_1\omega_2}\Omega}$, it means $\alpha \in (-\pi/4, \pi/4)$.

The solution of the nonstationary Schrodinger equation with the Hamiltonian (2) will be sought in the form

$$|\Psi(t)\rangle = e^{-i\hat{H}t}|s_1, s_2\rangle = \frac{1}{\sqrt{s_1!s_2!}}e^{-i\hat{H}t}\hat{a}_1^{\dagger s_1}\hat{a}_2^{\dagger s_2}|0, 0\rangle. \tag{4}$$

Equation (4) up to an inessential phase (see, e.g., [31]) can be represented as

$$|\Psi(t)\rangle = \frac{1}{\sqrt{s_1!s_2!}}\hat{b}_1^{\dagger s_1}\hat{b}_2^{\dagger s_2}|0, 0\rangle, \tag{5}$$

where

$$\hat{b}_i^\dagger = e^{-i\hat{H}t}\hat{a}_i^\dagger e^{i\hat{H}t};\ \hat{b}_i = e^{i\hat{H}t}\hat{a}_i e^{-i\hat{H}t}. \tag{6}$$

Further, we consider the case of a rather weak coupling C in oscillators, i.e., $C \ll A, B$. In the case of such a connection, the values of $\omega_1 \approx \omega_2$ should be taken into account, then $\varepsilon = (\omega_2 - \omega_1)/\Omega$. In this case, using Eqs. (3) and (6), one can find in an analytical form $\hat{b}_i, \hat{b}_i^\dagger$ (further, for convenience will consider $\hat{b}_i$). As a result, we get

$$\begin{pmatrix}\hat{b}_1\\ \hat{b}_2\end{pmatrix} = U_{BS}\begin{pmatrix}\hat{a}_1\\ \hat{a}_2\end{pmatrix};\quad U_{BS} = \begin{pmatrix}\sqrt{T} & e^{i\phi}\sqrt{R}\\ -e^{-i\phi}\sqrt{R} & \sqrt{T}\end{pmatrix}, \tag{7}$$

where

$$R = \frac{\sin^2\left(\Omega t/2\sqrt{1+\varepsilon^2}\right)}{(1+\varepsilon^2)};\ T = 1 - R;\ \cos\phi = -\varepsilon\sqrt{\frac{R}{T}}. \tag{8}$$

Equation (7) are well-known transformations for a linear beam splitter, where R is the reflection coefficient, T is the transmission coefficient, and the condition $R + T = 1$ is met, and ϕ is the phase difference. Similarly to the beam splitter (BS) in our case,we introduce the same coefficients R, T, ϕ defined by Eq. (8). Using Eq. (7), you can find $|\Psi(t)\rangle$ by substituting it in Eq. (5). Such calculations for a linear beam splitter are well known (see, e.g., [31,32]). Although it is possible to write a general expression for $|\Psi(t)\rangle$ using this approach, it is not suitable for calculating quantum entanglement, since we need to decompose $|\Psi(t)\rangle$ into Schmidt modes. It is known that by the Schmidt theorem [27,28] the wave function $|\Psi(t)\rangle$ of interacting 1 and 2 of the system can be decomposed in the form $|\Psi(t)\rangle = \sum_k \sqrt{\lambda_k(t)}u_k(x_1, t)v_k(x_2, t)$, where $u_k(x_1, t)$ is wave function of a pure state of system 1 and $v_k(x_2, t)$ is wave function of a pure state of system 2. Where λ_k is the Schmidt mode, which is the eigenvalue of the reduced density matrix, i.e., $\rho_1(x_1, x'_1, t) = \sum_k \lambda_k(t)u_k(x_1, t)u_k^*(x'_1, t)$ or $\rho_2(x_2, x'_2, t) = \sum_K \lambda_k(t)v_K(x_2, t)v_K^*(x'_2, t)$. If we find the Schmidt mode λ_k, we can calculate the quantum entanglement of the system. To do this, various measures of entanglement can be used, for example, the Schmidt parameter [27,28] $K = \left(\sum_k \lambda_k^2\right)^{-1}$ or von Neumann entropy [33,34] $S_N = -\sum_k \lambda_k \ln(\lambda_k)$. The main difficulty in calculating quantum entanglement is the search λ_k of the system in question, so let us move on to the search for this parameter.

You can see that λ_k does not depend on the phase difference ϕ. Indeed, $\lambda_k = |\langle k, k|\Psi(t)\rangle|^2$, and the phase difference is included in $|\Psi(t)\rangle$ as $e^{i\phi}$. As a result, we can say that λ_k depends on only one parameter: R or $T = 1 - R$, which means that quantum entanglement also depends on one parameter. In [25], the Schmidt mode λ_k was found, but in a general form, i.e., depending on many parameters of the system under consideration. Using Eq. (8) and λ_k from [25], you can get λ_k depending on only one parameter R in the form $\lambda_k(R) =$

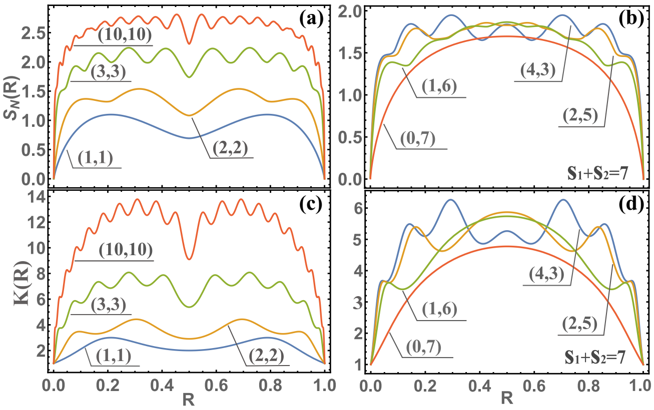

FIG. 1. The dependence of the von Neumann entropy S_N in (a) and (b), as well as the Schmidt parameter in (c) and (d) as a function of R is presented. In the figures, the dependencies are presented for different initial values of quantum numbers (s_1, s_2). For example, when $s_1 = 1$ and $s_2 = 6$, the notation (1,6) is entered.

$|c_{k,s_1+s_2-k}|^2$, where

$$c_{k,p} = \sum_{n=0}^{s_1+s_2} A^{s_1,s_2}_{n,s_1+s_2-n} A^{*k,p}_{n,s_1+s_2-n} e^{-2in\arccos\left(\sqrt{1-R}\sin\phi\right)},$$
$$A^{k,p}_{n,m} = \frac{\mu^{k+n}\sqrt{m!n!}}{(1+\mu^2)^{\frac{n+m}{2}}\sqrt{k!p!}} P_n^{(-(1+m+n),m-k)}\left(-\frac{2+\mu^2}{\mu^2}\right),$$
$$\mu = \sqrt{1+\frac{1-R}{R}\cos^2\phi} - \cos\phi\sqrt{\frac{1-R}{R}}, \quad (9)$$

where $P^{\alpha,\beta}_{\gamma}(x)$ are Jacobi polynomials, k and p are quantum numbers of the oscillator in the state $|k, p\rangle$ with $k + p = s_1 + s_2$. In Eq. (9), no matter what value $\phi \in (0, \pi/2)$ we choose, the value $\lambda_k(R)$ will not depend on ϕ. This amazing property of Eq. (9) can be used depending on the tasks under consideration. For example, if we choose $\phi = 0$, then when summing by n in Eq. (9), the exponent is replaced by $(-1)^n$. If we choose $\phi = \pi/2$, then all the dependence on the reflection coefficient R will be concentrated in the exponent. In this way, the relationship of quantum entanglement with the reflection coefficient R was found. Figure 1 shows the dependence of quantum entanglement for the von Neumann entropy S_N and the Schmidt parameter K, depending on the reflection coefficient R.

For example, for $s_1 = 1$ and $s_2 = 1$, quantum entanglement will be in the form

$$S_N = -(1-2R)^2\ln(1-2R)^2 - 4R(1-R)\ln\left(2R(1-R)\right),$$
$$K = \frac{1}{1-8R(1-R)(1-3R(1-R))}. \quad (10)$$

For Eq. (10) it is interesting to find the value R at which there will be the maximum quantum entanglement and this value $R = 1/2(1 \pm 1/\sqrt{3})$. At this value R, the quantum entanglement will be $S_N = \ln 3$ and $K = 3$. This is quite an interesting result, since at first glance it seems that the maximum entanglement should be at $R = 1/2$. Figure 1 shows that quantum entanglement strongly depends on the reflection coefficient R, but is always zero at $R = 0, 1$. The larger the quantum numbers s_1, s_2, the greater the quantum entanglement. You can see that there are two maxima quantum entanglement for different pairs s_1, s_2, except for the case when one of the quantum numbers is zero. No matter what measure of quantum entanglement we use, all dependencies are similar.

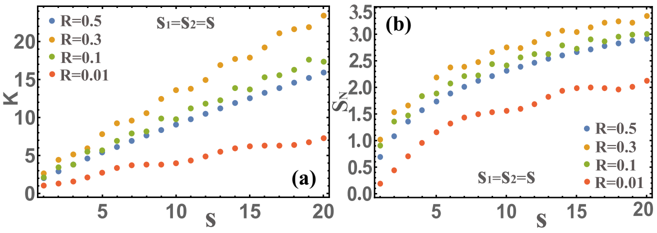

FIG. 2. The dependence of the von Neumann S_N entropy is shown in (a), as well as the Schmidt parameter in (b) for various values of R and s. From bottom to top, respectively $R = 0.01$, $R = 0.5$, $R = 0.1$, $R = 0.3$.

We also present quantum entanglement for various values of $s_1 = s_2 = s$ in Fig. 2. The calculations are given for $R \in (0, 1/2)$, since for symmetric values of $R \in (1/2, 1)$, the results will be the same. Figure 2 shows that the quantum entanglement at $s_1 = s_2 = s$ increases significantly with increasing s. For the Schmidt parameter K, this dependence is close to linear.

In addition to being able to obtain simple expressions for quantum entanglement, other physical characteristics can be calculated using Eq. (9). For example, if one of the quantum numbers is equal to zero, we choose $s_2 = 0$, we can find in a very simple form the average value of the quantum numbers k and p [see Eq. (9)] for the time t of the oscillators interaction (where $k + p = s_1 + s_2$). As a result, we get $\overline{k} = \sum_k k\lambda_k(R) = s_1(1 - R)$ and $\overline{p} = \sum_p p\lambda_p(R) = s_1 R$.

III. GENERALIZATION TO ANOTHER TYPE OF COUPLING IN A QUANTUM OSCILLATOR

Consider the case where the type of connection will be different in the Hamiltonian (1) or (2), namely,

$$\hat{H} = \sum_{k=1}^{2} \hbar\omega_k\left(\hat{a}_k^\dagger\hat{a}_k + 1/2\right) - i\hbar\alpha x_1\frac{\partial}{\partial x_2}. \quad (11)$$

This type of connection can occur when we consider the interaction of a charged oscillator with a quantized mode of the electromagnetic field, see, e.g., [35,36]. The constant α determines the value of the connection and can be different, depending on the problems under consideration. Next, we will use the same initial states as above, i.e., $|s_1, s_2\rangle$ [the system was not connected, i.e., for $t = 0$ in Eq. (11), the constant $\alpha = 0$]. In [35], a solution and Schmidt mode of the nonstationary Schrodinger equation with a Hamiltonian (11) was found. Similarly to what was done above, i.e., using Eq. (8) and the results of [35] we get the Schmidt mode as $\lambda_k(R) = |C_{k,s_1 s_2 - k}|^2$, where

$$C_{k,p} = \sum_{n=0}^{s_1+s_2} B^{s_1,s_2}_{n,s_1+s_2-n} B^{*k,p}_{n,s_1+s_2-n} e^{-2in \arccos\left(\sqrt{1-R}\sin\phi\right)},$$

$$B^{k,p}_{n,m} = i^{k-n}(-1)^{p+m} \frac{\mu^{k+n}\sqrt{m!n!}}{(1+\mu^2)^{\frac{n+m}{2}}\sqrt{k!p!}} P_n^{(-(1+m+n),m-k)}\left(-\frac{2+\mu^2}{\mu^2}\right),$$

$$\mu = \sqrt{1+\frac{1-R}{R}\cos^2\phi} - \cos\phi\sqrt{\frac{1-R}{R}}, \tag{12}$$

where the reflection coefficient R and the phase shift ϕ is determined by Eq. (8) for $\Omega \to \alpha$. You can see that in Eqs. (12) and (9) the amplitudes $C_{k,p}$ and $c_{k,p}$ are different, but the Schmidt modes $\lambda_k(R)$ are the same. From this it follows that the quantum entanglement for such systems will be the same with the same values of R. This is a rather interesting conclusion, which speaks of the identity of the quantum entanglement of these systems. Figures 1 and 2 above, for quantum entanglement, can also be applied to the case of this type of coupling in an oscillator.

IV. PHYSICAL ANALYSIS OF THE REFLECTION COEFFICIENT *R*

In this section of the article, we will conduct a physical analysis of the reflection coefficient R and show examples where the results obtained can be used directly.

Consider the reflection coefficient R, see Eq. (8) for $\omega_2 \approx \omega_1$ so that $|\omega_2 - \omega_1| \ll \Omega$. In this case, the parameter $\varepsilon \ll 1$, then the coefficient $R = \sin^2(\Omega t/2)$. It should be added that such a obtained coefficient R in the theory of a beam splitter in the form of coupled waveguides is well known [37] if we take $t = L/v$, where L is the length of the coupled waveguide, v is the propagation velocity light in the waveguide. In other words, one can directly use the more general expression for the coefficient R, see Eq. (8) in coupled waveguide theory, assuming ω_1 and ω_2 are the frequencies photons for 1 and 2 waveguides, respectively, and Ω is some parameter that determines the degree of coupling of two waveguides.

Next, consider the case $|\omega_2 - \omega_1| \gg \Omega$. In this case, the parameter $\varepsilon \gg 1$ and the coefficient $R \ll 1$, i.e., the system practically does not change its initial state.

It should be added that Eq. (9) can be used in quantum optics to calculate the quantum entanglement of photons in a beam splitter (with two input and output ports) with a reflection coefficient R. In this case, ω_1 and ω_2 are the frequencies of one and two photons, respectively, and Ω is some parameter characterizing the beam splitter. For example, selecting one incoming photon at each port of the beam splitter, i.e., $s_1 = s_2 = 1$ and reflection coefficient $R = 1/2$ we get the well-known Hong-Ou-Mandel effect [38]. This means that for $s_1 = s_2 = 1$ the wave function in the form (5) for $R = 1/2$ will have only two states on the output ports $k = 0$, $p = 2$ and $p = 2$, $k = 0$. It is interesting to note that the HOM effect does not realize maximally entangled photons at the output ports (see Fig. 1 for $s_1 = 1, s_2 = 1$). Such states, as was shown above, are realized at $R = 1/2(1 \pm 1/\sqrt{3})$, see Eq. (10). Combining the parameters $\omega_1; \omega_2; \Omega; t$, you can always choose the value of R with maximum quantum entanglement, see Eq. (8).

One of the interesting applications of the results obtained is the possibility of obtaining the wave functions $\Psi(t)$ Eq. (5) that have practical applications by specifying the coefficient R. By specifying the coefficient R, you can always pick up the parameters included in it, and thus get the wave function $\Psi(t)$ of interest. For example, we need a wave function $\Psi(t)$ that defines the states of Holland-Burnett (HB) [39]. It is well known that this wave function is of great interest in various fields of physics, for example, in quantum metrology [40,41]. To do this, we need to select $R = 1/2$ and $s_1 = s_2 = s$ (for even values s). As a result, we get the wave function (5) in the form

$$\Psi = \sum_{n=0}^{s} e^{2in\phi} \frac{\sqrt{(2n)!(2s-2n)!}}{2^s n!(s-n)!} |2n, 2s-2n\rangle. \tag{13}$$

It should be added that using Eq. (13) one can obtain an expression for quantum entanglement using the Schmidt parameter K in the form

$$K = \frac{\pi(s!)^2}{\Gamma(s+1/2)^2 {}_4F_3(1/2, 1/2, -s, -s; 1, 1/2-s, 1/2-s; 1)}, \tag{14}$$

where $\Gamma(x)$ is the gamma function, ${}_4F_3(x_1, x_2, x_3, x_4; y_1, y_2, y_3; 1)$ is the generalized hypergeometric function. It should be added that Eq. (14) has a fairly simple approximation $K = s^{0.897}$. This clearly shows that quantum entanglement is unbounded from above, which was well known earlier (the more s, the more the quantum entanglement).

Quite an interesting and simple example of where the results obtained can be used is the cooling of mechanical resonators. The most effective method of cooling micro- and nanodevices is the sideband method [42,43]. This method shows great promise for using quantum effects (including coherent properties) in hybrid systems [44] but first it is necessary to prepare the mechanical component in the ground state, i.e., cool it as much as possible. Mechanical resonators are described by Eq. (2), and the connection between them is realized through radiation pressure. Also, cooling can be not only through radiation pressure, but also through the interaction of radiation with a charged resonator, in this case the system is described by Eq. (11), see [36]. We will assume that in the initial state the resonator was in the s_1 state and the interaction with radiation is described by Eqs. (2) or (11). The initial quantum number for radiation is usually chosen as $s_2 = 0$, which we will do. As a result, it is easy to get the average quantum number of the cooled resonator $\overline{k}(R)$ depending on the reflection coefficient R,

$$\overline{k}(R) = \sum_{k=0}^{s_1} k\lambda_k(R) = s_1(1-R). \tag{15}$$

You can see that Eq. (15) is very simple and is a linear function of the coefficient R. Choosing $R = 1$ one can cool the resonator to its ground state. Of course, this is a superficial analysis, without taking into account the thermal noise of the environment, but it shows the simplicity and clarity of the approach presented here. A similar expression was obtained in [36] for cooling a charged resonator, but without taking into account the representation in terms of the reflection coefficient R. Also for this case one can obtain expressions for quantum entanglement in the form of the Schmidt parameter

$$K = \frac{1}{(1-R)^{2s_1}\,{}_2F_1\left(-s_1, -s_1; 1; \left(\frac{R}{1-R}\right)^2\right)}, \tag{16}$$

where ${}_2F_1(a, b; c; x)$ is Gaussian hypergeometric function. The general dependence of Eq. (16) can be seen from Fig. 1(b, d). Also, analyzing Eq. (16), you can get that the maximum of this function at $R = 1/2$. With this value of $R = 1/2$, one can obtain a simpler expression for quantum entanglement

$$K_{\max} = 2^{2s_1}\frac{(s_1!)^2}{(2s_1)!}. \tag{17}$$

You can also find from Eq. (17) the parameter K for large values of the quantum number s_1, we get $K_{\max}(s_1 \gg 1) \rightarrow \sqrt{\pi s_1}$. It can be seen in this case that the quantum entanglement is unbounded from above (the more s_1, the more the quantum entanglement).

It should be added that the cases presented here of using the obtained expressions in quantum optics are only some cases of using the theory developed here, since the Hamiltonian (1) has a large number of applications.

V. CONCLUSION

Thus, in this paper, it was shown that the quantum entanglement of two coupled harmonic oscillators is expressed as a single parameter, this is the reflection coefficient R. Convenient analytical expressions for the study of quantum entanglement are obtained. It is shown that for certain values of R and a given pair of quantum numbers s_1, s_2 quantum entanglement has a maximum and can be large. In general, quantum entanglement is unbounded from above, since it can take an arbitrarily large value for certain parameters R, s_1, s_2. It is quite easy to see from the obtained Eqs. (9) and (12) that for $R = 0$ and $R = 1$ there is no quantum entanglement, which is clearly seen from Fig. 1. Since quantum entanglement is expressed in the form of only one parameter, Eqs. (9) and (12) presented in this work have the most simple analytical form. It should be added that the quantum entanglement, expressed as a single parameter (the reflection coefficient R), was obtained without taking into account the environment. In this case, the environment can significantly affect the quantum entanglement [45]. Is it possible to represent quantum entanglement, taking into account the external environment, depending on the reflection coefficient R and other parameters of the environment? This is an open problem and needs to be solved in the future.

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