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Mechanical analog of quantum bradyons and tachyons

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We present a mechanical analog of a quantum wave-particle duality: a vibrating string threaded through a freely moving bead or "masslet." For small string amplitudes, the particle movement is governed by a set of nonlinear dynamical equations that couple the wave field to the masslet dynamics. Under specific conditions, the particle achieves a regime of *transparency* in which the field and the particle's dynamics appear decoupled. In that special case, the particle conserves its momentum and a guiding wave obeying a Klein-Gordon equation, with real or imaginary mass, emerges. Similar to the double-solution theory of de Broglie, this guiding wave is locked in phase with a modulating group wave comoving with the particle. Interestingly, both subsonic and supersonic particles can fall into a quantum regime as is the case with the slower-than-light bradyons and hypothetical, faster-than-light tachyons of particle physics.

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I. INTRODUCTION

The foundations of quantum mechanics (QM) mainly rely on the pioneering work of de Broglie [1] for which he received the Nobel prize. The key assumption of de Broglie's intuitive approach is that any mass m_p of matter acts like a clock of pulsation ω_p such that its mass energy $m_\mathrm{p}c^2$ balances its vibrational energy $\hbar\omega_\mathrm{p}$ where c is the light velocity and $\hbar$ the Planck constant. Then, relying on special relativity to estimate how this moving clock would appear for an immobile observer, de Broglie showed that the clock must be in phase with a superluminal phase wave, the guiding wave, giving birth to the celebrated but still mysterious wave-particle duality. After de Broglie discovered this phase wave, he proposed a mechanical analog [2–4]: "the double-solution theory." This work was subsequently followed by some hydrodynamical and very interesting analogs, e.g., the Madelung approach [5]. In the same series of works [3,6], de Broglie also introduced the pilot-wave interpretation nowadays known as de Broglie–Bohm, or "Bohmian" mechanics, after its rediscovery by Bohm [7,8]. While the quest for a classical analog of QM is legitimate, it is far from being an easy task and de Broglie failed in extending his earlier results. Quite recently, the interest on this subject was renewed by the pioneering work initiated by Couder and Fort on bouncing droplets [9–11] in which one or several droplets hit a vertically shaken bath and generate a surface wave. Among other works (see [12] for a review), these droplets, sometimes referred to as walkers, were shown to not only mimic a wave-particle duality at macroscale, but also to reproduce most, if not all, of QM features. This is not that surprising because these models share some features with the double-solution and pilot-wave theories [3,7], known to be possible alternative interpretations, however deterministic, of quantum mechanics.

In this paper, we focus on a type of mechanical analog closer in spirit to the original double solution of de Broglie, i.e., transverse waves on which a small bead of mass m_p and of stiffness k_p is submitted to the string impulse and moves without friction. This "masslet" acts as a particle somewhat similar to a free-to-move defect or impedance jump; the density and elasticity are locally altered at the particle location. Its dynamical behavior is governed by usual momentum transfer at the impedance jump, giving rise to reflection, transmission, or absorption of the incoming wave. The masslet being free to move, these mechanisms are accompanied with the radiation force phenomenon common to acoustic and electromagnetic fields. This drives the particle in the whole field, incident and scattered, and eventually leads to memory effects. In the spirit of de Broglie's assumptions, the sliding masslet can be viewed as a a moving clock of pulsation $\omega_\mathrm{p} = \sqrt{k_\mathrm{p}/m_\mathrm{p}}$ in its rest frame.

In the past, quite similar models were first proposed by Rayleigh and Helmholtz [13–15] to study the vibrations of a loaded string. More recently, Boudaoud *et al.* studied the self-adaptation of free-to-move beads on a string submitted to acoustic noise [16] in the context of soap films and pattern formation. A few years ago, Borghesi showed, relying on a relativistic framework, that such a system yields a wave-particle duality governed by the also relativistic Klein-Gordon equation [17]. Here, we restrict ourselves to the nonrelativistic Newtonian framework and unravel the emergence of a *transparency* regime in which the particle-string interaction, and thus the radiation force, vanish. As we show, this regime is reminiscent of de Broglie's double solution [3] (for recent works and reviews see [18–20]) and furthermore leads to a Schrödinger equation for the phase wave associated to the particle. Very interestingly, two classes of transparent particles are possible candidates: (i) the class of subsonic particles that

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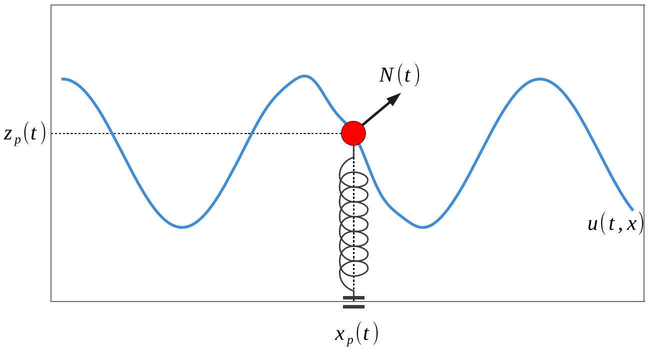

FIG. 1. Sketch of the string-mass mechanical analog. A bead of coordinates (x_{p}, z_{p}) slides without friction on a string with local transverse amplitude $u(t, x)$. A vertical spring acts on the bead as as a vertical restoring force and features an internal clock in the de Broglie double solution of quantum mechanics.

travel uniformly along the string with a velocity smaller than the speed of sound on the string and (ii) the class of supersonic (i.e., faster-than-sound) particles. These two families of dynamical motions are hereafter referred to as bradyons and tachyons, respectively, in reference to their quantum counterparts in particle physics (where the speed of sound must be replaced by the speed of light).

The paper is structured as follows: In Sec. II, we describe the system and derive the general dynamical equations. We discuss analytical solutions and present the numerical approach employed for studying the dynamics. In Sec. III, we discuss in detail the transparency regimes by focusing on the quantum analogs of bradyons. A similar analysis is done for tachyons in Sec. IV. Concluding remarks are addressed in Sec. V.

II. STRING-MASSLET SYSTEM

A. General description

Our system consists of an elastic string of linear density λ stretched along x by tension T. The string oscillates in the transverse z direction and the vibration is characterized by the field $u(t, x)$. A ring-shaped masslet m_{p}, or bead, is threaded on this string at $x = x_{\mathrm{p}}$ and can slide without friction. Importantly, in order to make this particle behave as a clock, the masslet is submitted to an additional elastic force field corresponding to a spring of stiffness k_{p}. Thus, it exerts an elastic force toward the nondeformed axis of the string ($z = 0$) (see illustration in Fig. 1). The effect of gravity is neglected throughout the paper.

Since the mass cannot escape the string, its vertical position z_{p} must satisfy the holonomic condition $z_{\mathrm{p}}(t) = u(t, x_{\mathrm{p}}(t))$ at any time t. Furthermore, the absence of any dissipation (and in particular of friction between the particle and the string) imposes that the reaction force $N(t)$ from the string to the particle is locally normal to the string at the particle's location (in the limit of small vibration amplitudes, i.e., the reaction force is mainly vertical). When the string vibrates, the mass is accelerated vertically and horizontally by the local string acceleration and subsequently moves along it. Since the particle has locally a density or elasticity which are different from the string, its inertia will in turn stretch the string and act as a source for the field $u(t, x)$, generating scattered waves. Thereby, the particle acts as a clock moved by and within the wave field that it has formerly produced.

In this work, we only study nonrelativistic movements, i.e., we assume that the velocities of the particle and wave are much smaller than the celerity of light. Moreover, despite this limitation our model involves a physical constraint defined by the velocity c of the elastic waves along the string. This allows us to define two distinct regimes associated to the Mach number $\mathrm{Ma} = |v_{\mathrm{p},x}/c|$ where $v_{\mathrm{p},x}$ is the horizontal velocity of the particle. $\mathrm{Ma} < 1$ is the subsonic regime, while $\mathrm{Ma} > 1$ is the supersonic one. As we show in the following, the subsonic regime is physically equivalent to the bradyonic regime as defined in particle physics for slower-than-light quanta, while the supersonic regime is similar to the tachyonic one with faster-than-light quantum particles. The mechanical analogy with special relativity is remarkable and we demonstrate that the application of an equivalent Lorentz transformation (where the sound velocity replaces the velocity of light) is crucial for a quantum description of mechanical analogs.

B. Deriving the equations

Let us now derive the general set of equations that governs the dynamics of the masslet-string system. Since our system possesses several degrees of freedom coupled by a holonomic condition, the Lagrangian formalism is well suited to derive the equations of motion. Obviously, the total action I of the system can be split in three parts as

$$I = \int dt\, L_{\mathrm{p}} + \iint dt\, dx\, \mathcal{L}_{\mathrm{s}} + \int dt\, L_{\mathrm{int}}, \tag{1}$$

where L_{p} and $\mathcal{L}_{\mathrm{s}}$ are, respectively, the Lagrangian of the particle and the Lagrangian density of the string (indicated by the cursive letter) in the absence of the holonomic coupling constraint L_{int}. We focus first on the expression of the Lagrangian L_{p} of the particle in our system. Since we impose an attraction toward the string baseline (via the spring of stiffness k_{p}), a potential

$$V(z_{\mathrm{p}}) = \tfrac{1}{2} m_{\mathrm{p}} \omega_{\mathrm{p}}^2 z_{\mathrm{p}}^2 \tag{2}$$

is introduced, with ω_{p} a pulsation or equivalently $k_{\mathrm{p}} = m_{\mathrm{p}}\omega_{\mathrm{p}}^2$ a stiffness applied to the particle solely. The particle's Lagrangian then takes the form

$$L_{\mathrm{p}}(t, z_{\mathrm{p}}, v_{\mathrm{p}}) = \tfrac{1}{2} m_{\mathrm{p}} v_{\mathrm{p}}^2 - V(z_{\mathrm{p}}), \tag{3}$$

where we have introduced $\vec{v}_p = v_{p,x}\vec{u}_x + v_{p,z}\vec{u}_z$ with $v_{\mathrm{p},x}(t) = \frac{dx_{\mathrm{p}}(t)}{dt}$ and $v_{\mathrm{p},z}(t) = \frac{dz_{\mathrm{p}}(t)}{dt}$ the longitudinal and transverse particle velocity, respectively. Without the holonomic constraint, the variational Lagrange principle $\delta[\int dt\, L_{\mathrm{p}}] = 0$ leads to the usual Newtonian equations

$$m_{\mathrm{p}}\ddot{x}_{\mathrm{p}}(t) = 0,\; m_{\mathrm{p}}\big[\ddot{z}_{\mathrm{p}}(t) + \omega_{\mathrm{p}}^2 z_{\mathrm{p}}(t)\big] = 0. \tag{4}$$

Similarly, $\mathcal{L}_{\mathrm{s}}$ is obtained by taking the continuum limit of a chain of springs with constant stiffness and considering small amplitude vibrations (i.e., neglecting nonlinearities):

$$\mathcal{L}_{\mathrm{s}}(u, \partial_t u, \partial_x u) = \tfrac{1}{2}\lambda(\partial_t u(t, x))^2 - \tfrac{1}{2}T(\partial_x u(t, x))^2. \tag{5}$$

In this equation, λ and T are, respectively, the linear density and the tension of the string. We also define our local speed of sound $c = \sqrt{T/\lambda}$ of the transverse waves propagating along the string. Once again, for $L_{\text{int}} = 0$, one can obtain from the variational principle $\delta[\int dt \int dx\, \mathcal{L}_{\text{s}}] = 0$ the wave equation

$$\frac{\lambda}{T}\partial_t^2 u(t,x) - \partial_x^2 u(t,x) = \Box u(t,x) = 0, \tag{6}$$

where $\Box$ is the usual linear d'Alembert operator.

The specific nature of the holonomic constraint is ideally grasped by the coupling Lagrangian

$$L_{\text{int}}(t, x_{\text{p}}, z_{\text{p}}, N) = N[z_{\text{p}} - u(t, x_{\text{p}})] \tag{7}$$

or, equivalently, by a Lagrangian density $\mathcal{L}_{\text{int}}$ (such that $L_{\text{int}} = \int dx\, \mathcal{L}_{\text{int}}$):

$$\mathcal{L}_{\text{int}}(u, x, z_{\text{p}}, N) = N[z_{\text{p}} - u(t,x)]\delta(x - x_{\text{p}}), \tag{8}$$

where $N(t)$ is a Lagrange multiplier defining an additional variable in the variational problem. Variations with respect to N lead automatically to the holonomic constraint

$$z_{\text{p}} = u(t, x_{\text{p}}). \tag{9}$$

Taking into account the action of the mass on the string, the wave dynamics follows a new equation obtained from the variational principle $\delta[\int dt \int dx(\mathcal{L}_{\text{s}} + \mathcal{L}_{\text{int}})] = 0$, whereas symmetrically the modified dynamics of the sliding mass is obtained from $\delta[\int dt(L_{\text{p}} + L_{\text{int}})] = 0$. Altogether, this leads to the following set of coupled equations:

$$m_{\text{p}}\ddot{x}_{\text{p}}(t) = -\partial_x u(t,x)|_{x=x_{\text{p}}(t)}\, N(t), \tag{10a}$$

$$N(t) = m_{\text{p}}\big[\ddot{z}_{\text{p}}(t) + \omega_{\text{p}}^2 z_{\text{p}}(t)\big], \tag{10b}$$

$$\Box u(t,x) = -\frac{N(t)}{T}\delta(x - x_{\text{p}}(t)) \tag{10c}$$

with the holonomic condition (9). Interestingly, after elimination of $N(t)$, Eqs. (10a) and (10b) can equivalently be rewritten as

$$\ddot{x}_{\text{p}}(t) = -\partial_x u(t,x)|_{x=x_{\text{p}}(t)}\big[\ddot{z}_{\text{p}}(t) + \omega_{\text{p}}^2 z_{\text{p}}(t)\big]. \tag{11}$$

Remarkably, this equation is independent of the mass m_{p}, a fact which is reminiscent of acoustic analogs of gravitational forces in hydrodynamical systems [21]. We also point out that Eqs. (10a)–(10c) can easily be interpreted using a Newtonian language. The Lagrange multiplier $N(t)$ is the vertical reaction force acting on the sliding mass while $-\partial_x u(t,x)|_{x=x_{\text{p}}(t)} N(t)$ is the horizontal component of this reaction force [i.e., in the linear limit of small wave amplitude we have $\partial_x u(t,x)|_{x=x_{\text{p}}(t)} = \tan\theta \simeq \sin\theta$ where θ is the angle between the reaction force N and the z vertical direction].

We emphasize that Eq. (10c) can be rewritten as $\tilde{\lambda}(t,x)\partial_t^2 u(t,x) = T\partial_x^2 u(t,x) = 0$ with $\tilde{\lambda}(t,x) = \lambda + m\delta(x - x_{\text{p}}(t))$. In other words, the mass m can also be interpreted as a local translatable defect in the linear density λ. This could have an impact for physical interpretations and future experimental implementations.[1]

C. Energy and momentum conservation

Moreover, the previous dynamics is completed by an analysis of energy and momentum conservation in the coupled system (i.e., of prime integrals of motion). From the full action and Lagrangian (1) [or alternatively from Eqs. (10a)–(10c)] we deduce after lengthy but straightforward calculations the local energy-momentum conservation laws for the field coupled to the mass:

$$\begin{aligned} \partial_t \varepsilon(t,x) + \partial_x S_x(t,x) &= -N\delta(x - x_{\text{p}}(t))\partial_t u(t,x)|_{x=x_{\text{p}}(t)}, \\ \partial_t g_x(t,x) + \partial_x T_{xx}(t,x) &= N\delta(x - x_{\text{p}}(t))\partial_x u(t,x)|_{x=x_{\text{p}}(t)}, \end{aligned} \tag{12}$$

where $\varepsilon = \frac{1}{2}T[\frac{1}{c^2}(\partial_t u)^2 + (\partial_x u)^2]$ and $g_x = -\frac{T}{c^2}\partial_t u \partial_x u$ are, respectively, the u-field energy and linear (pseudo)momentum density of the acoustic field along the string. Similarly, $S_x = c^2 g_x$ and $T_{xx} = \varepsilon$ are, respectively, the energy and (pseudo)momentum density flow along the x direction (T_{xx} is the constraint tensor of the field which has only one component here). Moreover, combining Eq. (12) with Eqs. (10a)–(10c) leads to

$$\begin{aligned} \frac{d}{dt}\bigg\{\int dx\, \varepsilon(t,x) + \frac{1}{2}m_{\text{p}}(\dot{x}_{\text{p}})^2 \\ + \frac{1}{2}m_{\text{p}}\big[(\dot{z}_{\text{p}})^2 + \omega_{\text{p}}^2(z_{\text{p}})^2\big]\bigg\} = 0, \end{aligned} \tag{13a}$$

$$\frac{d}{dt}\bigg\{\int dx\, g_x(t,x) + m_{\text{p}}\dot{x}_{\text{p}}\bigg\} = 0, \tag{13b}$$

which shows that the total energy and linear momentum of the system are conserved.

We stress that despite its nonrelativistic nature our model differs from the one proposed in [17] which considered a Klein-Gordon wave equation from the start, i.e., $[\Box + \frac{\Omega_m^2}{c^2}]u(t,x) = -\frac{N(t)}{T}\delta(x - x_{\text{p}}(t))$. In [17] it was shown that such a Klein-Gordon equation with a source term can be used to generate a mechanical analog of a wave-particle duality. Here we show that it is not necessary to introduce such a complication. An inhomogeneous d'Alembert equation, i.e., as in Eq. (10c), is already sufficient to reproduce a wave-particle duality. As we show in the following, this is in complete agreement with original ideas developed by de Broglie [22].

III. BRADYONIC OR SUBSONIC REGIME: WAVE-PARTICLE DUALITY

A. Transparency regime

Let us now focus on a particular class of motion for which the coupling force $N(t)$ between the field and the particle vanishes: This regime, hereafter referred to as transparency and which corresponds to a uniform movement of the particle

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[1] The effect of gravity can be estimated for a vibrating string at the pulsation ω by comparing the string vertical acceleration with that of gravity g. Typically, as soon as the nondimensional number $\frac{u\omega^2}{g}$ is much larger than unity, gravity is small in comparison with the string acceleration so that gravity can be neglected in the problem. Experimentally, for a string vibrating with an amplitude of about 1 cm at a frequency around 1 kHz, the vertical acceleration is roughly $4\pi^2 \times 10^{-2}$–10^6 that is 410^5 m$/s^2$, much larger than $g = 10$ m$/s^2$.

along the string, is very rich despite its apparent simplicity. In this regime, the vertical oscillation of the masslet is no longer forced and we have

$$m_{\rm p}\big[\ddot{z}_{\rm p}(t)+\omega_{\rm p}^2 z_{\rm p}(t)\big]=N(t)=0 \quad \text{(transparency)} \tag{14}$$

yielding a purely harmonic vertical motion

$$z_{\rm p}(t)=A\cos(\omega_{\rm p}t+\varphi) \tag{15}$$

with A and φ real constants. Equations (10a) and (10c) yield

$$\ddot{x}_{\rm p}(t)=0,\ \ \Box u(t,x)=0 \tag{16}$$

which results in an inertial movement for the particle: In other words, the masslet trajectory is a uniform translation along x combined to a purely harmonic vertical oscillation along z:

$$x_{\rm p}(t)=v_{\rm p}t+x_{\rm p,i} \tag{17}$$

with $x_{\rm p}(0)=x_{\rm p,i}$ and $v_{\rm p}(t)=v_{\rm p}$ two real constants. In this configuration, the different degrees of freedom are decoupled and in particular $\int dx\,\varepsilon(t,x)$, $\frac{1}{2}m_{\rm p}(\dot{x}_{\rm p})^2$, $\frac{1}{2}m_{\rm p}[(\dot{z}_{\rm p})^2+\omega_{\rm p}^2(z_{\rm p})^2]$, $\int dx\,g_x(t,x)$, and $m_{\rm p}\dot{x}_{\rm p}$ are constants of motions. We emphasize that the oscillatory motion $z_{\rm p}(t)$ is reminiscent of de Broglie's original idea [1,22] of associating a local clock to any particle. Here, we have a more detailed mechanical model for which we see that the uniform motion given by Eq. (17) is dynamically linked to the harmonic oscillation defined by Eq. (15).

Concerning the string field $u(t,x)$, several solutions are *a priori* available. Indeed, the general solution of $\Box u(t,x)=0$ reads as $u(t,x)=f(t-x/c)+g(t+x/c)$ corresponding to two arbitrary pulses $f(t)$ and $g(t)$ propagating, respectively, along the $+x$ and $-x$ directions. Considering, for example, the case $g=0$ and using the holonomic condition (9) one gets $f(t(1-\frac{v_{\rm p}}{c})-\frac{x_{\rm p,i}}{c})=A\cos(\omega_{\rm p}t+\varphi)$, i.e.,

$$f(t)=A\cos\left[\frac{\omega_{\rm p}}{1-\frac{v_{\rm p}}{c}}\left(t+\frac{x_{\rm p,i}}{c}\right)+\varphi\right]. \tag{18}$$

This motion would correspond to a mass "surfing" on a monochromatic propagative wave with pulsation $\frac{\omega_{\rm p}}{1-\frac{v_{\rm p}}{c}}$. However, while Eq. (18) is interesting in itself, we are not going here to develop this approach further. Instead, we now follow the physical intuitions of de Broglie and seek solutions for the homogeneous equation $\Box u=0$ such that in a comoving inertial frame $\mathcal{R}'$ translating at velocity $v_{\rm p}$ with the particle, the field u would appear as stationary.

B. Searching for standing fields *u* in the transparency regime: A relativistic perspective

Following the above-mentioned intuition, we search for a coordinate transform yielding a stationary solution for the field. Interestingly, a Galilean transformation of the coordinates ($t'=t$, $x'=x-v_{\rm p}t$) which lets the time flow unchanged fails to bring a standing solution. This coordinate change, although legitimate here for a Newtonian approach, is however too crude to allow for clock synchronization at a distance. A coordinate transformation that does not let the time unaltered is required to bring a stationary solution.

Actually, a "first-order" Poincaré-Lorentz transformation of the form $x'=x-v_{\rm p}t$, $t'=t-\frac{v_{\rm p}}{c^2}x$ is already sufficient to ensure the invariance of the free wave equation. This is because the term $-\frac{v_{\rm p}}{c^2}x$ results from a time synchronization procedure for clocks located at different points of inertial frames $\mathcal{R}$ and $\mathcal{R}'$ as already proposed by Poincaré [23]. Moreover, the d'Alembert operator $\Box$ being invariant under the usual Lorentz transformation $\Box=\Box'$, we make use of the following coordinate change:

$$x'=\gamma_{\rm p}(x-v_{\rm p}t),\quad t'=\gamma_{\rm p}\left(t-\frac{v_{\rm p}}{c^2}x\right), \tag{19}$$

with $\gamma_{\rm p}^{-1}=\sqrt{1-\frac{v_{\rm p}^2}{c^2}}$ so that in the Lorentz-Poincaré group in dimension $1+1$, the field $u(t,x)$ appears as a scalar invariant field, i.e., $u(t,x)=u'(t',x')$. We point out that the variables t' and x' have no direct physical meaning here since we are working in the context of Newtonian dynamics where the time is absolute. However, Eq. (19) is used as a mathematical tool for finding the solutions of the d'Alembert equation under the restriction Ma <1. In this context we emphasize that Voigt [24] already used the Lorentz transformation as a mathematical tool in optics but he did not give a clear interpretation of this transformation. Here, instead the Lorentz transformation is not completely formal but allows us to enlighten the role of the group symmetry associated with the wave equation. This analysis is strongly related to de Broglie's own work where the use of relativistic concepts were key for understanding wave-particle duality.

Searching for a standing solution in the Lorentzian comoving frame $\mathcal{R}'$ that ensures a "time-space" separation $u'(t',x')=F(t')G(x')$, one gets

$$\frac{1}{c^2}\frac{d^2F}{dt'^2}G-F\frac{d^2G}{dx'^2}=0 \tag{20}$$

which turns into a set of equations

$$\frac{d^2F}{dt'^2}+\omega'^2F=0,\quad \frac{d^2G}{dx'^2}+\frac{\omega'^2}{c^2}G=0 \tag{21}$$

with ω' a complex constant to be determined. For the case of interest when ω' is real, F and G are harmonic so that we obtain an "amplitude modulated" field $u'(t',x')$:

$$\begin{aligned}u'(t',x')&=B\cos(\omega't'+\eta)\cos\left(\frac{\omega'}{c}x'+\xi\right)=u(t,x)\\&=B\cos\left[\omega'\gamma_{\rm p}\left(t-\frac{v_{\rm p}}{c^2}x\right)+\eta\right]\\&\quad\times\cos\left[\frac{\omega'}{c}\gamma_{\rm p}(x-v_{\rm p}t)+\xi\right],\end{aligned} \tag{22}$$

B, η, and ξ being three real constants. They can be determined by using the holonomic condition expressed for the case of the uniform motion $z_{\rm p}(t)=u(t,x=v_{\rm p}t+x_{\rm p,i})$. It follows that

$$\begin{aligned}u(t,x=v_pt+x_{\rm p,i})&=B\cos\left(\frac{\omega'}{\gamma_{\rm p}}t+\eta-\frac{\omega'\gamma_{\rm p}x_{\rm p,i}v_{\rm p}}{c^2}\right)\\&\quad\times\cos\left(\frac{\omega'}{c}\gamma_{\rm p}x_{\rm p,i}+\xi\right)\end{aligned}$$

which, by identification with $z_0(t)$ given by Eq. (15), yields

$$\varphi = \eta - \frac{\omega' \gamma_{\rm p} x_{\rm p,i} v_{\rm p}}{c^2} + 2\pi n \ (\text{with} n \in \mathbb{Z}), \tag{23}$$

$$A = B \cos\left(\frac{\omega'}{c}\gamma_{\rm p} x_{\rm p,i} + \xi\right), \tag{24}$$

and above all the relation

$$\omega_{\rm p} = \frac{\omega'}{\gamma_{\rm p}}. \tag{25}$$

Equation (25) differs from the original de Broglie model which imposes the relativistic phase harmony condition $\omega_{\rm p} = \omega'$. We will go back to this issue in Sec. III E.

C. Physical interpretation and the de Broglie–Bohm pilot-wave analogy

Let us now examine this transparent regime and its underlying quantum interpretation. To give a meaningful description of the above-cited condition, we can write $u' = u$ in Eq. (22) in the form $u = B \cos S_{\rm brad} \cos \Phi_{\rm brad}$ with

$$S_{\rm brad} = \omega' t' + \eta = \omega t - kx + \eta,$$
$$\Phi_{\rm brad} = \frac{\omega'}{c}x' + \xi = \frac{\omega}{c}(x - v_{\rm p} t) + \xi \tag{26}$$

with $\omega = \omega'\gamma_{\rm p}$, $k = \omega v_{\rm p}/c^2$. Besides, the following dispersion relation applies:

$$\frac{\omega^2}{c^2} - k^2 = \frac{\omega'^2}{c^2} \tag{27}$$

between the pulsation ω and the wave vector k. The quantity $S_{\rm brad}$ plays the role of a phase for a plane (carrying) wave $\psi(t,x) = e^{iS_{\rm brad}}$, solution of the Klein-Gordon equation

$$\left(\frac{1}{c^2}\partial_t^2 - \partial_x^2\right)\psi(t,x) = -\frac{\omega'^2}{c^2}\psi(t,x). \tag{28}$$

The phase velocity $v_{\rm ph}$ is defined by the condition $dS_{\rm brad} = 0$, which implies

$$v_{\rm ph} = \frac{\omega}{k} = \frac{c^2}{v_{\rm p}} > c \tag{29}$$

in accordance with the formulas obtained by de Broglie [2]. Besides, $\Phi_{\rm brad}$ in Eq. (26) defines an envelope (i.e., a group) velocity which identifies with the particle's velocity $v_{\rm p}$ (setting $d\Phi_{\rm brad} = 0$ we get $dx/dt = v_{\rm p}$). Furthermore, we deduce the Rayleigh formula $v_{\rm gr} = \frac{d\omega}{dk} = v_{\rm p}$ which was also obtained by de Broglie. This is clearly reminiscent of Hamilton's formula $\frac{dH_{\rm p}}{dP_{\rm p}} = v_{\rm p}$ if we identify the Hamiltonian or energy function $H_{\rm p} := m_{\rm p} c^2 \gamma_{\rm p}$ and the linear momentum $P_{\rm p} := m_{\rm p}\gamma_{\rm p} v_{\rm p}$ with, respectively, $Q\omega$ and Qk where Q is a constant having the dimension of an action. The similarity between Q and $\hbar$ is clear and is reenforced if we write Eq. (28) in the limit $M_a \ll 1$ as

$$iQ\partial_t \Psi \simeq -\frac{Q^2}{2m_{\rm p}}\partial_x^2 \psi(t,x) + m_{\rm p} c^2 \psi(t,x) \tag{30}$$

which is identical to Schrödinger's equation after the substitution $Q \to \hbar$ [similarly in the Klein-Gordon equation (28) we can replace $\frac{\omega'^2}{c^2}$ by $\frac{m_{\rm p}^2 c^2}{Q^2}$in agreement with standard textbooks]. Moreover, we can also write $H_{\rm p} = -Q\partial_t S_{\rm brad}$ and $P_{\rm p} = Q\partial_x S_{\rm brad}$ which are reminiscent of Hamilton-Jacobi equations with QS playing the role of an action and, therefore, we have

$$v_{\rm p} = -c^2 \frac{\partial_x S_{\rm brad}}{\partial_t S_{\rm brad}} \tag{31}$$

which is the guidance formula introduced by de Broglie in his pilot-wave interpretation [3,6] and which leads to $v_{\rm p} = Q\frac{\partial_x S_{\rm brad}}{m_{\rm p}}$ in the limit Ma $\ll 1$ in agreement with Bohmian mechanics [6,7]. Therefore, altogether we recover de Broglie's assumptions casting the double-solution theory [3] with a ψ wave also called the guiding wave (solution of the Klein-Gordon equation) together with the u wave (solution of the homogeneous d'Alembert equation) associated with the particle's movement. Importantly, in our approach (and in contradistinction to [17]) we start from the d'Alembert equation and not from the Klein-Gordon equation. Still, we are able to obtain a guiding wave ψ which is solution of Eq. (28), i.e., the Klein-Gordon equation. In other words, the mass term of the Klein-Gordon equation has been generated from the u field itself. This agrees with a model already presented by de Broglie in [22], i.e., two years before the model named traditionally the double solution [3] and based on the Klein-Gordon equation.

The condition (25) on the frequency can also be written in terms of phase and becomes a phase-locking condition, since for $x = x_{\rm p}(t)$ one gets

$$S_{\rm brad} = \omega' t' + \eta = \frac{\omega'}{\gamma_{\rm p}}t + \varphi = \omega_{\rm p} t + \varphi \tag{32}$$

which expresses the phase locking of the particle's clock (with pulsation $\omega_{\rm p}$) to that of the wave (with pulsation ω or ω').

D. Generating the transparent field $u(x,t)$

Following de Broglie [22,25], we can rewrite the total wave u as a sum of waves [by means of the trigonometric identity $2 \cos F \cos G = \cos(F+G) + \cos(F-G)$] to give a physically meaningful interpretation

$$u'(t',x') = \frac{B}{2}\cos\left[\omega'\left(t' + \frac{x'}{c}\right) + \eta + \xi\right] + \frac{B}{2}\cos\left[\omega'\left(t' - \frac{x'}{c}\right) + \eta - \xi\right] \tag{33}$$

or, equivalently,

$$u(t,x) = \frac{B}{2}\cos\left[\omega'\gamma_{\rm p}\left(1 - \frac{v_{\rm p}}{c}\right)\left(t + \frac{x}{c}\right) + \eta + \xi\right] + \frac{B}{2}\cos\left[\omega'\gamma_{\rm p}\left(1 + \frac{v_{\rm p}}{c}\right)\left(t - \frac{x}{c}\right) + \eta - \xi\right]. \tag{34}$$

It is another way to express u as the sum of two counterpropagating waves $u(t,x) = u_+(t,x) + u_-(t,x)$ with (i) $u_- = \frac{B}{2}\cos[\omega_- t + \omega_- x/c + \eta + \xi]$ a wave with a low-frequency Doppler shift $\omega_- = \omega(1 - \frac{v_{\rm p}}{c})$ propagating along the $-x$ direction, and (ii) $u_+ = \frac{B}{2}\cos[\omega_+ t - \omega_+ x/c + \eta - \xi]$ a wave with a high-frequency Doppler shift $\omega_+ = \omega(1 + \frac{v_{\rm p}}{c})$ propagating along the $+x$ direction. Experimentally, this decomposition would allow us to generate the resulting modulated transparent u field appearing in Eq. (22) as the sum of two plane waves.

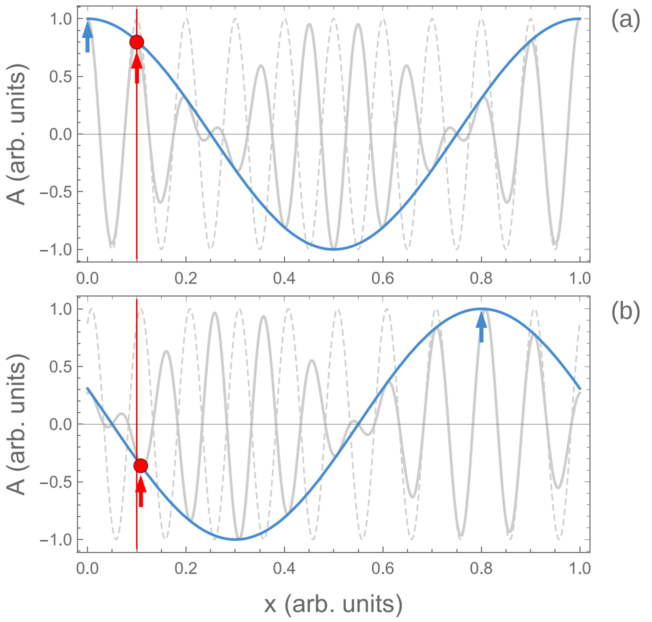

FIG. 2. The transparent regime for a bradyonic (i.e., subsonic) particle plotted from the analytical solutions described in the paper. An equivalent animation obtained from the numerical code of the full set of equations is available in the Supplemental Material [26]. For this example, we have $v_p/c = 0.1$, $x_{p,i} = 0.1$ for a string length of $L = 1$ and $c = 1$ (in natural units). We have chosen a ψ-wave temporal period $T_{\text{phase}} = \frac{2\pi}{\omega} = 10$ and $\eta = \xi = 0$. The continuous thick gray line is the total field $u(x, t)$ which can be decomposed into the phase wave ψ (continuous blue line) and the group wave (dashed gray line). The whole field is shown at two different instants (a) $t = 0$ and (b) $t = 0.08$ clearly demonstrating the subsonic (or bradyonic) regime of the particle. The particle's initial condition is set so that it is at a maximum of the group phase Φ_{brad} [see Eq. (26)] and is clearly locked to it along time. The red arrows in (a) and (b) compare the position of the particle at time $t = 0$ and 0.08, whereas the blue arrow shows the position and displacement of a phase maximum at the two instants $t = 0$ and 0.08.

The analytical solution is plotted in Fig. 2 for this transparency regime (the normalized parameters are indicated in the caption). Going back to Eq. (26) we have for the phase wave a temporal period $T_{\text{phase}} = \frac{2\pi}{\omega}$ and a spatial period $\lambda_{\text{phase}} = \frac{2\pi}{\omega}\frac{c^2}{v_p}$ which have to be compared with the temporal and spatial periods of the group wave $T_{\text{group}} = \frac{2\pi}{\omega}\frac{c}{v_p} \geqslant T_{\text{phase}}$, $\lambda_{\text{group}} = \frac{2\pi}{\omega}c \leqslant \lambda_{\text{phase}}$ (as illustrated in Fig. 2 for $\lambda_{\text{phase}}/\lambda_{\text{group}} = 10$). Importantly, we have here the well-known de Broglie formula [1]

$$P_p = \frac{2\pi Q}{\lambda_{\text{phase}}}. \tag{35}$$

It is interesting to note that in de Broglie's approach the wave field was a solution of a nonhomogeneous equation including a source term for the particle $\Box u(t, x) \propto \delta(x - x_p(t))$ [i.e., like in Eq. (10c)]. Here, instead the transparency condition $N(t) = 0$ imposes the field equation $\Box u(t, x) = 0$ and this simplifies the interpretation of Eq. (34) as the sum of two plane waves whereas de Broglie had to involve a complicated sum of advanced and retarded waves emitted by the particle itself [22,25].

More generally, it is clear that the equations of motion presented here are strongly nonlinear, and except for a few specific regimes like the transparent one discussed in this paper, no analytical solutions are usually reachable. In order to further investigate the physics of our model, we have employed a numerical scheme based on standard finite differences through a Runge-Kutta of order 4 (see, for example, [27]). Since the mutual interaction between the mass and the string only enters as an external source term for the each other, we have implemented separately their differential equations and solve the full system self-consistently. We first start by fixing initial compatible configurations for both the string and the mass. During the time resolution process, an update of the string is performed, accounting for the presence of the mass. Once this space-time update is achieved, we proceed to the update of the mass now accounting back to the new state of the string. We continue so on and so forth until desired time periods. The convergence is ensured by respecting the von Neumann stability criterion. Despite the above-mentioned nonlinearities in the complete set of equations, the algorithm appears to be well stable and reproduces very precisely the analytical solutions discussed here (transparency). This validates our approach and indicates that the coupling between the mass and the string does not seem to be that crucial in the convergence. For this first work, we have employed our algorithm in order to complement with the dynamics of the discussed analytical solutions. In particular, we have recorded a movie [26] showing the behavior of the mass in the transparency regime. The question of other possible emerging exotic regimes in our coupled system is very interesting, though, and deserves a proper study. We dedicate a more systematic numerical analysis to a future work.

E. Discussions

The calculations above deserve some comments in relation with de Broglie's picture. Indeed, it is noteworthy that the pulsation ω_p remains unchanged regardless of the particle's speed and is in particular different from ω' (which is velocity dependent). It is slightly different from de Broglie's results and is a consequence of our mixed approach, combining a Lorentz transformation (by means of c the speed of sound) in a Newtonian framework for which relativistic dynamics do not apply. More precisely, in de Broglie's approach which is fully relativistic (i.e., with the sound celerity replaced everywhere by the light velocity) the particle's internal clock with pulsation ω_p is defined in the rest frame $\mathcal{R}'$, not in the laboratory frame $\mathcal{R}$. The phase-locking condition reads as now

$$S_{\text{brad}} = \omega' t' + \eta = \frac{\omega'}{\gamma_p} t + \eta - \frac{\omega' \gamma_p x_{p,i} v_p}{c^2} = \omega_p t' + \varphi \tag{36}$$

and we have thus $\omega' = \omega_p = \frac{\omega}{\gamma_p}$ which differs from Eq. (25) by a prefactor γ_p associated with a relativistic time dilation (we have also $\eta = \varphi$ since φ is now defined in the rest frame [17,22]). In the nonrelativistic limit where $\frac{v_p}{c} \ll 1$ and $\gamma_p \simeq 1$ de Broglie's theory reduces to $\omega \simeq \omega' = \omega_p$. This is identical in our model to the regime $\text{Ma} \ll 1$ (where c is now the sound velocity) so that the difference between the two approaches vanishes for sufficiently slow particle motions. It is interesting to remark that in the $\text{Ma} \ll 1$ regime the Hamiltonian $H_\mathrm{p} := m_\mathrm{p}c^2\gamma_\mathrm{p}$ which was formally introduced reduces to $H_\mathrm{p} \simeq m_\mathrm{p}c^2 + \frac{1}{2}m_\mathrm{p}v_\mathrm{p}^2$ which, up to an additive constant, is the translational kinetic energy associated with the particle motion along x (similarly $P_\mathrm{p} \simeq m_\mathrm{p}v_\mathrm{p}$ which identifies with the translational linear momentum of the particle). Since c is here the sound velocity, $m_\mathrm{p}c^2$ can not physically be identified with a rest energy which is a relativistic Einsteinian concept. This once again stresses the similarities and differences between our mechanical analog and de Broglie's own approach. For similar reasons, we have no right to identify the integration factor Q discussed previously with the Planck constant $\hbar$. This could only hold in de Broglie's model for a genuine quantum particle. Still, the mechanical analogy works fine for Ma $\ll 1$ and could be actually extended to the relativistic regime by implementing a covariant, i.e., Einsteinian, mechanical model. Here, we nevertheless stick to the Newtonian framework which is closer to the experimental realization and is already sufficient to grasp the essential features of de Broglie's mechanical model.

IV. TACHYONIC OR SUPERSONIC REGIME

So far, we have discussed the dynamics of a particle moving on the string with a velocity v_p smaller than that of the waves on this string. Since this motion is completely governed by the interaction with the field u, we do not expect the particle to spontaneously cross the sound barrier without the use of an external force. However, one could choose initial conditions such that $v_\mathrm{p} > c$. We would then find a whole new supersonic regime, which we can investigate. As explained in the Introduction we choose to refer to such a particle as a "tachyon" by analogy with the eponymous hypothetical particles introduced in special relativity, which are a class of solution for the dynamics associated with faster-than-light motions [28–31] (symmetrically the case studied previously with $v_\mathrm{p} < c$ is referred to as "bradyonic" motion [28–31]). Indeed, this velocity c appears in the Lorentz transformation that we use for the field, as in the Lorentz transformation of special relativity, and appears as an asymptotic limit for the velocity v_p in both cases. However, one should once again not confuse those two velocities: it is a fundamental limit within the special relativity framework, while it is not in the context of a particle sliding on a string. Supersonic particles (analogous to supersonic aircrafts in a fluid) are indeed possible solutions. Thus, it is completely reasonable to study these "acoustic" tachyons, which are physically viable solutions of the dynamical equations. For this purpose, we see from Eq. (19) that if we set $x' = 0$, we get $x = v_\mathrm{p}t$ and therefore the ct' axis corresponds to the trajectory of a particle with velocity $v_\mathrm{p} < c$ (this is of course the definition of a comoving rest frame for such a bradyonic motion). If we now set $t' = 0$, we get $x = \frac{c^2}{v_\mathrm{p}}t = w_\mathrm{p}t$ which corresponds to a case where the x' axis identifies with the trajectory of a tachyonic particle with velocity $w_\mathrm{p} = \frac{c^2}{v_\mathrm{p}} > c$. This is the first hint that the dynamics of tachyons is completely symmetrical with that of "normal" bradyonic particles. Indeed, we can rewrite Eq. (19) as

$$t' = \frac{1}{c}\Gamma_\mathrm{p}(w_\mathrm{p}t - x), \quad x' = c\Gamma_\mathrm{p}\left(\frac{w_\mathrm{p}}{c^2}x - t\right), \tag{37}$$

with $\Gamma_\mathrm{p} = (\frac{w_\mathrm{p}^2}{c^2} - 1)^{-1/2} = \gamma_\mathrm{p}\frac{v_\mathrm{p}}{c}$ and where we see, by comparing with the original Lorentz transformation, that the roles of space and time have been reversed in the tachyonic case, compared to the bradyonic one. Furthermore, like for Eq. (19) the physical meaning of Eq. (37) is not immediate and here we use it mainly as a mathematical tool for guessing at an interesting solution of the wave equation. More precisely, let us consider once again the stationary field

$$u'(t', x') = B\cos(\omega't' + \eta)\cos\left(\frac{\omega'}{c}x' + \xi\right) \tag{38}$$

defined with the variables t' and x'. From the analysis made before we can use this field to match the motion of a tachyonic particle with velocity $w_\mathrm{p} = \frac{c^2}{v_\mathrm{p}}$. For this we use in Eq. (38) a wave with the same pulsation ω' as in the badryonic case. However, as we will see below it implies that we use a different spring with pulsation $\Omega_\mathrm{p} \neq \omega_\mathrm{p}$. Hence, using Eq. (37) we get in the laboratory frame

$$\begin{aligned} u(t, x) = B\cos&\left[-\frac{\omega'}{c}\Gamma_\mathrm{p}(x - w_\mathrm{p}t) + \eta\right] \\ &\times \cos\left[-\omega'\Gamma_\mathrm{p}\left(t - \frac{w_\mathrm{p}}{c^2}x\right) + \xi\right]. \end{aligned} \tag{39}$$

We find that, once again, the roles of two quantities have been swapped. We have now

$$\begin{aligned} \Phi_\mathrm{tach} &= \omega't' + \eta = -\frac{\omega'}{c}\Gamma_\mathrm{p}(x - w_\mathrm{p}t) + \eta, \\ S_\mathrm{tach} &= \frac{\omega'}{c}x' + \xi = -\omega'\Gamma_\mathrm{p}\left(t - \frac{w_\mathrm{p}}{c^2}x\right) + \xi. \end{aligned} \tag{40}$$

Here, the carrying phase wave and the envelope have been swapped: the supersonic phase wave that we had in the bradyonic case is now traveling alongside the particle; the group wave is now slower than the particle and has become the phase wave (i.e., $S_\mathrm{tach} = \Phi_\mathrm{brad}$ and $S_\mathrm{brad} = \Phi_\mathrm{tach}$). The field itself remains unchanged, and only the roles of its two components with respect to the particle have been changed. Thus, all the results that were obtained in the case of a particle with velocity v_p can be used for a tachyonic particle of velocity $w_\mathrm{p} = c^2/v_\mathrm{p}$ if we keep in mind the symmetrical nature of this supersonic regime. More precisely, considering a uniform motion $x_\mathrm{p}(t) = w_\mathrm{p}t + X_\mathrm{p,i}$ with $X_\mathrm{p,i}$ a constant and using the holonomic condition (9) [i.e., $u(t, x_\mathrm{p}(t)) = z_\mathrm{p}(t)$] together with the oscillatory $z_\mathrm{p}(t)$ motion of Eq. (15) we get by identification with Eq. (39)

$$\begin{aligned} \varphi &= -\xi - \frac{\omega'\Gamma_\mathrm{p}X_\mathrm{p,i}w_\mathrm{p}}{c^2} + 2\pi p(\text{with} p \in \mathbb{Z}), \\ A &= B\cos\left(\frac{\omega'}{c}\Gamma_\mathrm{p}X_\mathrm{p,i} - \eta\right), \\ \Omega_\mathrm{p} &= \frac{\omega'}{\Gamma_\mathrm{p}} = \frac{\omega'\frac{c}{v_\mathrm{p}}}{\gamma_\mathrm{p}}. \end{aligned} \tag{41}$$

Comparing the frequency-locking condition with the result obtained in Eq. (25) we deduce the constraint

$$\omega_\mathrm{p}\frac{c}{v_\mathrm{p}} = \Omega_\mathrm{p} \tag{42}$$

and, consequently, $\omega_\mathrm{p} < \Omega_\mathrm{p}$.

There are, however, a few noteworthy differences in some of the equations, mainly the dispersion relation between the wave pulsation $\Omega = \Omega'\Gamma_p$ (here we use the notation $\Omega' = \omega'$ to emphasize the role of the phase wave) and wave vector $K = \Omega\frac{w_p}{c^2}$ which becomes

$$\frac{\Omega^2}{c^2} - K^2 = -\frac{\Omega'^2}{c^2} \tag{43}$$

and the Klein-Gordon equation for the wave $\Psi = e^{iS_{\text{tach}}}$ (with $S_{\text{tach}} = Kx - \Omega t + \xi$)

$$\Box\Psi = +\frac{\Omega'^2}{c^2}\Psi, \tag{44}$$

where the signs in front of $\frac{\Omega'^2}{c^2}$ have been reversed compared with $\frac{\omega'^2}{c^2}$ in Eqs. (27) and (28). This is specific of tachyonic motions where the pulsation or mass can be envisioned as being purely imaginary $\tilde{\Omega} = i\Omega'$. We have

$$\Omega = \frac{\tilde{\Omega}}{\sqrt{1 - \frac{w_p^2}{c^2}}} \tag{45}$$

and for the tachyon the pure imaginary number i in the numerator is exactly compensated by the same pure imaginary number in the denominator [28,29,31]. Equation (44) thus equivalently reads as $\Box\Psi = -\frac{\tilde{\Omega}^2}{c^2}\Psi$ which is the usual form for the Klein-Gordon equation but now with a purely imaginary mass.

Once more, we point out that our supersonic particle is only superficially looking as a tachyon. Indeed, genuine relativistic tachyons would induce reluctant causality and thermodynamic problems [28,29,31,32] which are often considered as fatal objections to their mere existence. Here, the analogy with tachyon is not complete. For instance, observe that we can in analogy with the subsonic regime define the Hamiltonian and linear momentum as $H_p := m_p c^2 \Gamma_p = Q\Omega$ and $P_p := m_p w_p \Gamma_p = QK$. Physically, this corresponds to a tachyonic particle of imaginary mass $im_p = iQ\Omega'/c^2 = iQ\omega'/c^2$ while the physical "Newtonian" mass is of course m_p. These expressions are in general clearly different from the usual kinetic energy and momentum of a Newtonian particle.

To illustrate the tachyon dynamics we show in Fig. 3 an example of this transparency regime (the normalized parameters are indicated in the caption). Here, we have for the phase wave the temporal period $T_{\text{phase}} = \frac{2\pi}{\Omega} = \frac{2\pi}{\omega}\frac{c}{v_p} = 1$ and a spatial period $\lambda_{\text{phase}} = \frac{2\pi}{\Omega}\frac{c^2}{w_p} = \frac{2\pi}{\omega}c = 0.1$ which have to be compared with the temporal and spatial periods of the group wave $T_{\text{group}} = \frac{2\pi}{\Omega}\frac{c}{w_p} = \frac{2\pi}{\omega} = 0.1 \leqslant T_{\text{phase}}$, $\lambda_{\text{group}} = \frac{2\pi}{\Omega}c = \frac{2\pi}{\omega}\frac{c}{v_p} = 1 \geqslant \lambda_{\text{phase}}$ (as illustrated in Fig. 3 for $\lambda_{\text{phase}}/\lambda_{\text{group}} = 0.1$). Clearly, the role of the phase and group waves has been swapped compared to the bradyonic case.

V. CONCLUSION

A. Speculations about Bose-Einstein statistics

We would like to start this conclusion by pointing out a possible interpretation in terms of Bose-Einstein statistics of the *collective* transparent movement of multiple particles. Indeed, the last section stressed the strong symmetry existing between bradyonic and tachyonic particle.

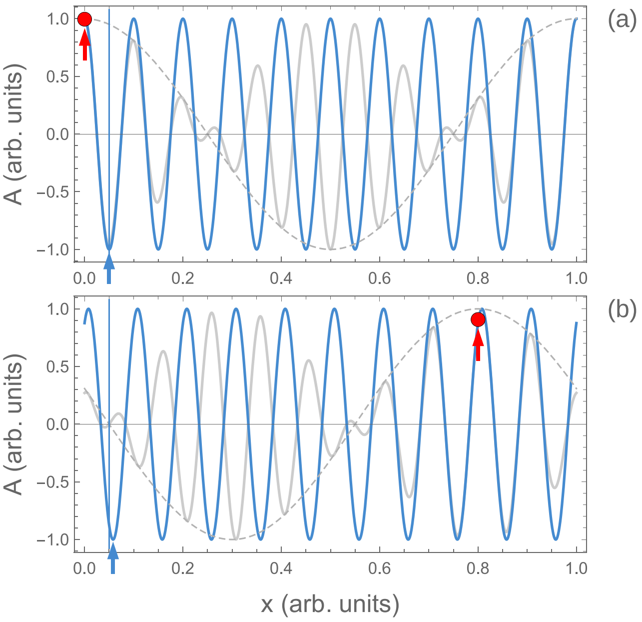

FIG. 3. The transparent regime for a tachyonic (i.e., supersonic) particle of velocity $w_p/c = 10$, and initial coordinate $X_{p,i} = 0$ (in natural units). The two panels correspond to observation times (a) $t = 0$ and (b) $t = 0.08$. The other parameters of the wave and string are unchanged with respect to Fig. 2. In particular, we have $T_{\text{phase}} = \frac{2\pi}{\Omega} = \frac{2\pi}{\omega}\frac{c}{v_p} = 1$ (see text). Like the subsonic case, the particle is locked to the group wave $\cos\Phi_{\text{tach}}$ (dashed gray line) and is clearly faster than the velocity of the phase wave $\cos S_{\text{tach}}$ on the string (blue continuous line). The red and blue arrows compare the displacement of the particle and phase wave between $t = 0$ and 0.08.

Remarkably, it means that a same u wave can carry several bradyons of velocity v_p associated with a spring of pulsation ω_p and tachyons of velocity $w_p = \frac{c^2}{v_p}$ but with a spring of pulsation Ω_p (compare Figs. 2 and 3). The different particles could actually move together on the same string since the transparency condition $N(t) = 0$ ensures that the particles do not interact with the wave and completely ignore each other. Therefore, we speculate that we have here the possibility to generate collective excitations actually reminiscent of bosons surfing on a given wave.

Consider, for instance, the case of n particles (bradyons or tachyons) coherently driven by a carrying u wave. The situation shows some similarity with a Fock state $|n\rangle$ in quantum mechanics for a collective excitation (e.g., photons or phonons) with n energy quanta. To make sense the analogy should be discussed in the context of the pilot-wave interpretation advocated by de Broglie and Bohm. More precisely, in the pilot-wave interpretation a bosonic collective excitation must be analyzed in the configuration space. Therefore, to a Fock state $|n\rangle$ we associate the many-body wave function $\Psi_n(x_1, x_2, \ldots, x_n, t) = \prod_{i=1}^{i=n}\Psi(x_i, t)$ with $\Psi(x_i, t)$ a single-particle wave function [in this work we have a plane wave $\Psi(x, t) = e^{i(kx-\omega t)}$] and $x_1, x_2, \ldots, x_n$ are the coordinates of the different particles [7,8]. Moreover, the single-particle wave-function $\Psi(x, t)$ is the same for all the particles from

$i = 1$ to n which means that all these particles are guided by the same wave. Indeed, in the pilot-wave interpretation the velocities of the different particles are given by

$$\frac{dx_i(t)}{dt} = \frac{\hbar}{m}\frac{\partial}{\partial x_i} S_n(x_1, x_2, \dots, x_n, t) = \frac{\hbar k}{m}, \tag{46}$$

where $S_n(x_1, x_2, \dots, x_n, t)$ is the phase of the n-particle wave function $\Psi_n(x_1, x_2, \dots, x_n, t)$. Clearly, all the particles have the same constant velocity. This clearly motivates our analogy between a pure Fock state in quantum mechanics and the collective motion of beads guided by a base wave.

Following this analogy, we could subsequently imagine a statistical ensemble of such strings carrying a various number of quanta $n = 0, 1, \dots, +\infty$ and subsequently develop a particle bosonic statistics with partition function $Z = \sum_{n=0}^{n=+\infty} e^{-\frac{n\varepsilon_{\omega'}}{K_B T}}$ (K_B is the Boltzmann constant and T the temperature of the ensemble). The energy $\varepsilon_{\omega'}$ is the Hamiltonian H_{p} associated with the bradyons or tachyons characterized by the frequency ω'. We naturally obtain Planck's law or more generally and realistically a bosonic statistics if the number of particles is finite and fixed. Clearly, this issue is very interesting for developing mechanical analogs of quantum statistics and will deserve further studies in connection with equilibrium and nonequilibrium thermodynamics.

B. Discussion and perspectives

We have developed a simple nonrelativistic mechanical analog of de Broglie's wave-particle duality. In the model considered, an oscillating bead or particle mimics an internal quantum clock in phase with a transverse acoustic wave field $u(t, x)$. Despite its nonrelativistic nature the model is based on a Lorentz transformation where the sound celerity c of waves on the string replaces the light velocity. Therefore, the model offers strong similarities with de Broglie's approach which was based on special relativity. De Broglie named this phase-locking feature the "phase harmony" [3] since it emphasizes the fundamental role of the quantum relation $m_{\mathrm{p}}c^2 = \hbar\omega_{\mathrm{p}}$. Like with de Broglie's theory the acoustic u field here generates a phase wave Ψ acting as a guiding field for the particle in pure analogy with Bohmian interpretation of QM. In other words, in the presented model the u field unifies the particle and the wave in an inseparable structure as summarized by the holonomic condition $z_{\mathrm{p}} = u(t, x_{\mathrm{p}})$. Thus, one can speak about "wave monism" and in that limited sense our analogy deciphers the meaning of wave-particle duality. Therefore, our work enables to figure out and visualize clearly a possible mechanical interpretation of the wave-particle duality and especially of the phase harmony. It gives a deeper understanding of the possible cause for the clock synchronization on which de Broglie's theory leans.

Interestingly, our approach which was developed for both the "bradyonic" (i.e., subsonic) and "tachyonic" (i.e., supersonic) regime is *a priori* not limited to uniform motions. Indeed, the set of equations (10a), (10b), and (10c) can easily be extended in order to include the effects of a more complex potential $V(t, x_{\mathrm{p}}, z_{\mathrm{p}})$. For example, by adding a longitudinal potential $V(t, x_{\mathrm{p}})$ Eq. (10a) becomes

$$m_{\mathrm{p}}\ddot{x}_{\mathrm{p}}(t) = -\partial_x u(t, x)|_{x=x_{\mathrm{p}}(t)} N(t) - \partial_x V(t, x)|_{x=x_{\mathrm{p}}(t)}, \tag{47}$$

whereas Eqs. (10b) and (10c) are left unaffected. The model could thus in principle be applied to regimes involving confining potentials which are of particular interest for further studies on mechanical analogs of QM.

Now, the similarity between our acoustic model and the bouncing drop model would deserve a dedicated discussion. Although the two approaches do not seem very different at first glance, they do not in fact rely on the same fundamentals. First of all, unlike bouncing drops, the particle in our acoustic model does not "surf" on the scattered wave it previously produced. In our system, the uniform movement corresponds to a perfect decoupling between the bath and the oscillator. Second, no Faraday instability is required in the acoustic system, while it is essential to bouncing drop dynamics. Even more crucially, there is no superluminal phase wave in the bouncing drop system. Because both systems yield a quantum analog to a certain extent, a careful examination of these differences is probably key to understanding the quantum world.

Finally, while the basis of our mechanical model is rather simple, the complexity of the various dynamics (with $N = 0$ and $N \neq 0$) may reveal interesting and fundamental questions when extending to higher dimensions (2D or 3D) and to many interacting particles. In this context, the continued development of the numerical method employed to generate the video (see [26]) will bring further insight into the complex dynamics of the system [especially when the "transparency" condition $N(t) = 0$ is no longer valid] and for several interacting particles.

From an experimental point of view, this work naturally appeals for an experimental demonstration of the wave-particle dynamics on a string. Here, we would like to suggest at least two possible routes for observing the transparent regime with subsonic and supersonic particles moving along a string.

The first setup could rely on a simple vibrating string (like a piano string), excited with a modulated (group × phase) signal from both ends of a fork. For a given length of the string (and therefore of the fork), it is possible to obtain periodic conditions at both ends so that both vibrate in phase with exactly

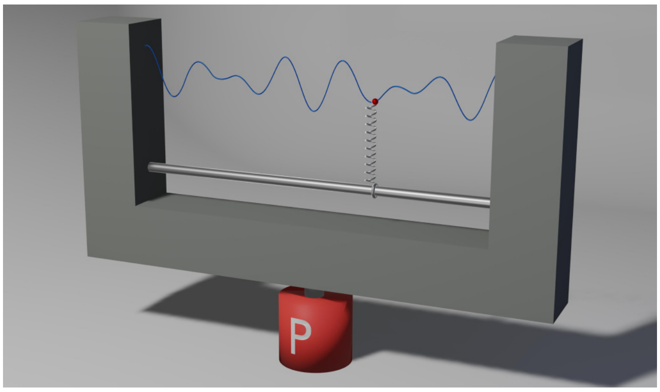

FIG. 4. Principle of a typical experiment for testing a mechanical analog of wave-particle duality. A rigid fork transmits mechanical vibrations from an excitation source (vibrating pot P) to an oscillating string attached to the fork at both ends. A sliding (red) bead constrained to move along the string is connected to a spring attached to a moving mass (ring on a rail). The bead is acting like a quantum particle excited by the wave propagating along the string (see text).

the same signal and generate the amplitude-modulated signal required to produce transparent movement of the bead. As for the particle, the elastic force (responsible for the internal particle frequency) can be achieved either by attaching a vertical spring to a mass, moving without friction along a horizontal guiding rod, or by creating magnetic elastic potential such that any departure from the neutral line of the string would yield an opposite vertical restoring force on the mass, for instance. A second and more complex setup would take advantage of a levitation configuration: by using a superconductive masslet flowing above a linear magnetic rail, one could obtain a stable elastic trap and ensure minimal friction along the string. An example of realization is illustrated on Fig. 4. By tracking and imaging the particle moving along the string with a high-speed camera, one should be able to visualize the standing field associated with the particle regardless of its nature (bradyonic or tachyonic).

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[1] L. de Broglie, Ann. Phys. **10**, 22 (1925).
[2] L. de Broglie, Nature (London) **118**, 441 (1926).
[3] L. de Broglie, J. Phys. Radium **8**, 225 (1927), translated in L. de Broglie and L. Brillouin, *Selected Papers on Wave Mechanics* (Blackie and Son, Glasgow, 1928).
[4] L. de Broglie, Ann. Fond. Broglie **12**, 1 (1987).
[5] E. Madelung, Z. Phys. **40**, 322 (1927).
[6] G. B. A. Valentini, *Quantum Theory at the Crossroads: Reconsidering the 1927 Solvay Conference* (Cambridge University Press, Cambridge, 2009).
[7] D. Bohm, Phys. Rev. **85**, 166 (1952).
[8] D. Bohm and B. J. Hiley, *The Undivided Universe* (Routledge, London, 1993).
[9] Y. Couder, S. Protière, and E. Fort, Nature (London) **437**, 208 (2005).
[10] Y. Couder and E. Fort, Phys. Rev. Lett. **97**, 154101 (2006).
[11] D. M. Harris, J. Moukhtar, E. Fort, Y. Couder, and J. W. M. Bush, Phys. Rev. E **88**, 011001(R) (2013).
[12] J. W. M. Bush, Phys. Today **68**(8), 47 (2015).
[13] L. Rayleigh, *The Theory of Sounds* (University of Michigan library, 1894) [reprint (Dover, New York, 1945)].
[14] H. V. Helmholtz, *Vorlesungen Über Theoretische Physik, Vol. 3* (Ambrosuis Barth, Leipzig, 1908).
[15] P. M. Morse and K. U. Ingard, *Theoretical Acoustics* (Princeton University Press, Princeton, NJ, 1992).
[16] A. Boudaoud, Y. Couder. and M. Ben Amar, Eur. J. Phys. B **9**, 159 (1999).
[17] C. Borghesi, Ann. Fond. Broglie **42**, 161 (2017).
[18] D. Fargue, Ann. Fond. Broglie **42**, 9 (2017).
[19] S. Colin, T. Durt, and R. Willow, Ann. Fond. Broglie **42**, 19 (2017).
[20] A. Drezet, Ann. Fond. Broglie **45**, 1 (2020).
[21] P. Y. Gires, J. Duplat, A. Drezet, and C. Poulain, Europhys. Lett. **127**, 34002 (2019).
[22] L. de Broglie, C. R. Acad. Sci. **180**, 498 (1925).
[23] H. Poincaré, in *Recueil Des Travaux Offerts Par Les Auteurs À H. A. Lorentz* (Nijhoff, The Hague, 1900), pp. 252–278.
[24] W. Voigt Voigt, Göttinger Nachrichten **8**, 177 (1887).
[25] L. de Broglie, *Ondes Et Mouvements* (Gauthier Villars, Paris, 1926).
[26] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevE.102.052206 for completes Fig. 2 and includes a movie showing the uniform motion of a masslet guided by a wave in the transparency regime.
[27] J. Loustau, *Numerical Differential Equations* (World Scientific, Singapore, 2016).
[28] J. P. Terletsky, J. Phys. Radium **21**, 681 (1960).
[29] O. M. P. Bilaniuk, V. K. Deshpande, and E. C. G. Sudarshan, Am. J. Phys. **30**, 718 (1962).
[30] G. Feinberg, Phys. Rev. **159**, 1089 (1967).
[31] J. P. Terletsky, J. Phys. Radium **23**, 910 (1962).
[32] J. S. Bell, in *"La Nouvelle Cuisine" in Between Science and Technology*, edited by A. Sarlemijn and P. Kroes, North-Holland Delta Series (Elsevier, Amsterdam, 1990), pp. 97–115.