Entanglement, coherence, and charging process of quantum batteries

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Quantum devices are systems that can explore quantum phenomena, such as entanglement or coherence, for example, to provide some enhancement performance concerning their classical counterparts. In particular, quantum batteries are devices that use entanglement as the main element in their high performance in powerful charging. In this paper, we explore quantum battery performance and its relationship with the amount of entanglement that arises during the charging process. By using a general approach to a two- and three-cell battery, our results suggest that entanglement is not the main resource in quantum batteries, where there is a nontrivial correlation-coherence tradeoff as a resource for the high efficiency of such quantum devices.

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I. INTRODUCTION

Recently, the idea of quantum batteries (QBs) was proposed to exploit quantum effects in order to gain the charging time and charging power compared to their classical counterparts. The concept of quantum batteries was first introduced as two-level systems for energy storage and transmission to consumer centers [1]. Therefore, the issue of efficient and operational quantum batteries is always an essential subject. In most scenarios, quantum batteries are considered as N independent systems that are charged by a temporary field. However, so far there have been many efforts to model protocols to extract more work from a quantum battery, in particular by employing quantum entanglement [2–5].

As an alternative approach, the concept of quantum batteries has been developed as many-body systems, where N cells of a QB are charged locally [6,7], different from previous processes in which the cells are jointly charged by using global operations. In this model, the quantum battery is presented as a one-dimensional Heisenberg spin chain composed of N spins, which provides the intrinsic interactions between the spins and the possibility of entanglement. In a spin chain, we can consider a coupling given by the XXZ Heisenberg model, where an anisotropic parameter Δ develops a role in the dynamics of such a system. It is known that the XXZ Heisenberg chain has been applied to quantum batteries [6], but the role of the quantum correlations, e.g., entanglement and coherence, is still an open question. Moreover, since it has been shown that entanglement is not necessary for optimal work extraction [8], this leads us to ask whether the quantum supremacy of QBs is due to the entanglement.

To address this question, one needs to consider a suitable approach in which the collective charging process can be done without entanglement generation. In this paper, we consider a two-qubit QB (a two-qubit cell), where we display the battery charge dynamics for both collective and noncollective (parallel) charging processes. Our results suggest that entanglement is not always the best resource to charge QBs, where in this scenario the coherence generation is the quantum resource for optimal charging of QBs. Finally, we investigate the relation between entanglement and coherence with the performance of three-qubit QBs.

II. ERGOTROPY AND CHARGING PROCESS OF QUANTUM BATTERIES

The work extraction from quantum batteries is well defined by the *ergotropy* [9], where we can define the notion of *passive states*, which are states in which no amount of work can be extracted from them by unitary transformations. It is important to highlight the nonuniqueness of the passive states, in general [10]. However, for pure states, the passive state can be well defined as the ground state of the system because it is the lowest-energy state of the system [11]. Here, we focus on processes in which the system is thermally isolated so that no heat is exchanged at any point during the process. We also consider cyclic processes, in the sense that the driving Hamiltonian is the same at the beginning and at the end of the dynamics. Since the system is thermally isolated, the evolution of state ρ can be described by a unitary operator. Therefore, the extracted work is given by

$$\mathcal{E} = W_{\max} = \mathrm{Tr}(\rho H_0) - \max_{U\in\mathcal{U}} \mathrm{Tr}(U\rho U^{\dagger} H_0), \qquad (1)$$

where $\mathcal{U}$ is the set of all accessible unitary evolutions, and the internal (time-independent) Hamiltonian H_0 of the system can be decomposed as $H_0 = \sum_i \varepsilon_i |\varepsilon_i\rangle\langle\varepsilon_i|$, with $\varepsilon_{i+1} \geqslant \varepsilon_i$. It is possible to show that the work can be extracted from a system if and only if the system is nonpassive, where a passive system

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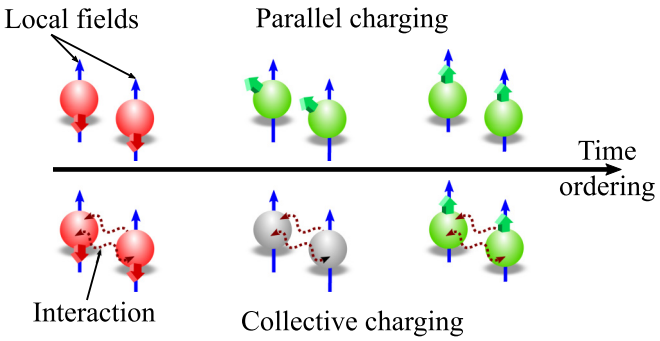

FIG. 1. Schematic diagram of a two-cell QB, e.g., a two-spin system, being charged through the parallel and collective charging, respectively. Local fields act on the cells, and the cells interact with each other along a collective charging. In collective charging, the system can evolve through an entangled state (gray balls).

has the form $\sigma_\rho = \sum_i p_i|\varepsilon_i\rangle\langle\varepsilon_i|$, where $p_{i+1} \leqslant p_i$ [9,12]. That is, passive states are diagonal in the energy basis and do not have population inversions. Then, any unitary acting on ρ can only increase its energy, hence no work can be extracted from it. It easily follows that given a pure state, the passive state reads $\sigma_\rho^{\text{pure}} = \rho_g = |\varepsilon_f\rangle\langle\varepsilon_f|$, with $|\varepsilon_f\rangle$ being the fundamental state of H_0 [5,11]. Therefore, the available energy of a QB that is unitarily charged reads

$$\mathcal{E} = W_{\max} = \text{Tr}(\rho H_0) - \text{Tr}(\rho_g H_0). \tag{2}$$

Throughout the analysis presented here, we are dealing with unitary processes. Therefore, the above equation corresponds to the internal energy variation of the system concerning the energy scale defined by H.

III. TWO-CELL QB

First, we start by introducing our physical model (as illustrated in Fig. 1), i.e., the two-qubit cell QB consisting of two coupled two-level systems. At the same time, in order to charge the QB, we need to consider that each cell couples individually with local fields. Without loss of generality [5], we consider the driving Hamiltonian for our model in the interaction picture as $H = H_{\text{ch}} + H_{\text{int}}$, where $H_{\text{ch}} = \hbar\Omega\sum_{n=1}^{2}\sigma_n^x$, with σ_n^x being the Pauli X-matrix acting on the nth spin. The second Hamiltonian is the interaction one given by the XXZ Heisenberg Hamiltonian,

$$H_{\text{int}} = J\hbar\left(\sigma_1^x\sigma_2^x + \sigma_1^y\sigma_2^y + \Delta\sigma_1^z\sigma_2^z\right), \tag{3}$$

where σ^i $(i = x, y, z)$ are the Pauli matrices, J is the strength of two-body interaction, and Δ is a dimensionless parameter associated with the anisotropy of the chain.

The status of the battery charging depends on the system state concerning the spectrum of the reference Hamiltonian H_0 considered here as $H_0 = \hbar\omega_0\sum_{n=1}^{2}\sigma_n^z$, with identical Larmor frequency ω_0 for both qubits. Here, as $|\uparrow\rangle$ and $|\downarrow\rangle$ are the ground and excited states of a single spin, respectively, we define the fully charged state of the battery as $|\text{full}\rangle = |\uparrow\uparrow\rangle$ with energy $E_{\text{full}} = 2\hbar\omega_0$, and the empty one as $|\text{emp}\rangle = |\downarrow\downarrow\rangle$ with low energy $E_{\text{emp}} = -2\hbar\omega_0$. Therefore, the maximum energy that can be stored in the battery reads $\mathcal{E}_{\max} = 4\hbar\omega_0$.

Now, we investigate the charging process in two different situations. As sketched in Fig. 1, we can drive the system with interaction between the cells (collective) and without interaction (parallel), where different results are expected [2–4]. To study both processes, we will start from the most general cases in which interaction is considered. Since the Hamiltonian is time-independent, the system dynamics is given by

$$|\psi(t)\rangle = \sum_{n=1}^{4} c_n e^{-\frac{i}{\hbar}E_n t}|E_n\rangle, \tag{4}$$

where E_n are the eigenenergies of H associated with the eigenstate $|E_n\rangle$, and c_n are the coefficients of the expansion of the initial state of the system in the basis $\{|E_n\rangle\}$. The eigenenergies of H are given by $E_1 = J\Delta\hbar$, $E_2 = -J(\Delta+2)\hbar$, $E_3 = (J-\beta)\hbar$, $E_4 = (J+\beta)\hbar$ with their respective eigenstates

$$\begin{aligned}
|E_1\rangle &= (|\downarrow\downarrow\rangle - |\uparrow\uparrow\rangle)/\sqrt{2},\ |E_2\rangle = (|\downarrow\uparrow\rangle - |\uparrow\downarrow\rangle)/\sqrt{2},\\
|E_3\rangle &= \gamma_1(|\downarrow\downarrow\rangle + |\uparrow\uparrow\rangle) - \gamma_2(|\downarrow\uparrow\rangle + |\uparrow\downarrow\rangle),\\
|E_4\rangle &= \gamma_2(|\downarrow\downarrow\rangle + |\uparrow\uparrow\rangle) + \gamma_1(|\downarrow\uparrow\rangle + |\uparrow\downarrow\rangle),
\end{aligned} \tag{5}$$

with

$$\gamma_1 = \frac{2\,\Omega}{\sqrt{2(\alpha+\beta)^2 + 8\,\Omega^2}},\quad \gamma_2 = \frac{\alpha+\beta}{\sqrt{2(\alpha+\beta)^2 + 8\,\Omega^2}}, \tag{6}$$

where we defined $\beta = \sqrt{J^2(\Delta-1)^2 + 4\,\Omega^2}$ and $\alpha = J(\Delta-1)$.

As a first analysis, let us consider the parallel charging process of the battery ($J = 0$), where each cell will independently evolve driven by the charging field. Therefore, from the above equations, we find the instantaneous ergotropy given by

$$\mathcal{E}_\parallel(t) = \mathcal{E}_{\max}\sin^2(\Omega t). \tag{7}$$

Immediately from this result, we establish the maximum average power for the parallel charging as $\bar{\mathcal{P}}_{\max}^{\parallel} = 2\mathcal{E}_{\max}\Omega/\pi$, where we used that $t_{\min} = \pi/2\Omega$ is the minimum time interval to get the maximum charge $\mathcal{E}_{\max}$. For the sake of completeness, from Eq. (7) we find the instantaneous power as

$$\mathcal{P}_\parallel(t) = \frac{d\mathcal{E}_\parallel(t)}{dt} = \mathcal{P}_{\max}^{\parallel}\sin(2t\,\Omega), \tag{8}$$

with $\mathcal{P}_{\max}^{\parallel} = \mathcal{E}_{\max}\Omega$ being the maximum instantaneous charging power. As we shall see, the quantities $\mathcal{P}_{\max}^{\parallel}$ and $\mathcal{E}_{\max}$ will be useful to study the role of the quantumness of the battery for a parallel and collective charging process.

On the other side, the instantaneous ergotropy and charging power for the collective charging process ($J \neq 0$) reads, respectively (see Appendix A),

$$\frac{\mathcal{E}_{\text{col}}(t)}{\mathcal{E}_{\max}} = \frac{1}{2} - \gamma_1^2\cos[(\beta + J\alpha)t] - \gamma_2^2\cos[(\beta - J\alpha)t] \tag{9}$$

and

$$\mathcal{P}_{\text{col}}(t) = 2\mathcal{P}_{\max}^{\parallel}\Omega\cos(\alpha J t)\sin(\beta t)/\beta. \tag{10}$$

Now, as a first remark, we explore the role of the anisotropy parameter Δ in the special limit $\Delta \to 1$, where we have $\alpha \to 0$ and $\beta \to 2\Omega$, so Eqs. (9) and (10) give $\mathcal{E}_{\text{col}}(t)|_{\Delta\to 1} = \mathcal{E}_\parallel(t)$ and $\mathcal{P}_{\text{col}}(t)|_{\Delta\to 1} = \mathcal{P}_\parallel(t)$, thus recovering results for the parallel charging process of a two-cell quantum battery. This quick remark allows us to conclude that the choice of Δ is relevant

to the performance of QBs. This leads us to ask the following: What is an effective collective charging process?

Entanglement, coherence, and charging power

The study of the quantumness of the two-cell QB will be addressed here from the standpoint of the amount of entanglement $\mathcal{Q}$ and normalized coherence C_0 of the system state. Given a pure state written in the reference basis as $|\psi\rangle = \alpha_{\uparrow\uparrow}|\uparrow\uparrow\rangle + \alpha_{\downarrow\uparrow}|\downarrow\uparrow\rangle + \alpha_{\uparrow\downarrow}|\uparrow\downarrow\rangle + \alpha_{\downarrow\downarrow}|\downarrow\downarrow\rangle$, we consider the entanglement given by the Wootters' measure of entanglement of a pair of qubits as [13]

$$\mathcal{Q} = 2|\alpha_{\uparrow\uparrow}\alpha_{\downarrow\downarrow} - \alpha_{\downarrow\uparrow}\alpha_{\uparrow\downarrow}|. \tag{11}$$

The energy content of the battery and the change in its energy distribution are of great interest in the context of quantum batteries. From this perspective, it is very useful to compute the coherence in the eigenstates of a bare Hamiltonian H_0 as the energy battery basis. As we shall see, this choice leads to a better understanding of the relationship between coherence as a quantum resource and the efficiency of quantum batteries. In addition, coherence in an energy battery basis has been considered in recent works [14,15]. Therefore, we define the coherence in the empty and charged basis of the battery as

$$C_0(t) = (1/C_{\max}) \sum_{i,j\neq i} |\rho_{ij}(t)|, \tag{12}$$

where the quantity $C_{\max}$ is the maximum coherence of the system. For example, for a two-qubit state one has $C_{\max} = 3$, which corresponds to the case $|\psi_{C_{\max}}\rangle = (1/2)(|\uparrow\rangle + |\downarrow\rangle)(|\uparrow\rangle + |\downarrow\rangle)$. We define the above quantity by normalizing the definition of the l_1 norm of coherence [16–18], so that $0 \leqslant C_0 \leqslant 1$. Then, from Eqs. (11) and (12) one can study how "quantum" the QB is. In addition, we are interested here in analyzing the role of entanglement for the charging process of the battery.

As previously discussed, through a parallel charging of the QB, the maximum charge state is achieved for the minimum time interval $t_{\min}$. Therefore, we will analyze here the dynamics of charging within the interval $t \in \mathcal{T}_{\min} = [0, t_{\min}]$. For our discussion, the time interval $\mathcal{T}_{\min}$ under consideration is appropriate, since we want to investigate both the role of correlations and the internal battery interaction. In this scenario, because $t_{\min}$ is the minimum charging time of a parallel charging process, quantum correlations develop an important role if we can achieve the maximum charge for some time smaller than $t_{\min}$. Otherwise, quantum correlations are not a resource. For completeness, it is worth mentioning that different values of $t_{\min}$ have been considered in the literature. For example, we can consider the minimum time given by the instant where we get maximum *instantaneous* power $t_{\max\mathcal{P}}$ [6]. However, as we want to consider situations in which we fully charge the battery, considering $t_{\max\mathcal{P}}$ as a reference is not appropriate because $t_{\max\mathcal{P}}$ is not associated with the maximum charge instant. Actually, from the definition of instantaneous power for the parallel charging [see Eq. (8)], the instantaneous time $t_{\min}$ of maximum values for $\mathcal{E}(t)$ is associated with instantaneous power zero, because $t_{\min}$ corresponds to a critical (maximum) point of $\mathcal{E}(t)$ in time. Therefore, by considering the collective charging process $J \neq 0$, Fig. 2 shows the instantaneous power of the quantum battery for different choices of the anisotropy parameter Δ. We highlight here the case with $\Delta = 1$, in which no entanglement is present [as we can see in Fig. 2(c)] and the charging power is better than the other cases with $\Delta = 0$ and -1. However, such zero entanglement production does not mean the battery is classical. As we can see from Fig. 2(d), the maximum coherence is obtained when $\Delta = 1$. Different from other works [15], here we stress that the maximum ergotropy is not stored in the system coherence (the full charged state is $|\uparrow\uparrow\rangle$), but coherence works as a resource to speed up the charging process of the QB. It is worth mentioning that when we consider the case in which the effects destroy coherence (the decoherence process), the battery performance for the optimal configuration $\Delta = 1$ becomes negatively affected, and maximum charge is not achieved (see Appendix B). For this reason, we identify coherence as a resource to enhance the QB performance.

The role of the parameter Δ for the charging process can be better understood by defining the average quantities for charge, power, entanglement, and coherence. In other words, one can define $\bar{\mathcal{P}}(\Delta)$, $\bar{\mathcal{Q}}(\Delta)$, and $\bar{C}(\Delta)$ in the interval $t \in [0, t_{\min}]$, given by $\bar{X}(\Delta) = (1/t_{\min})\int_0^{t_{\min}} X(t)dt$. In general, the average power is an important tool to investigate the charging performance of a quantum battery. However, in batteries in which spontaneous discharging is present [5,11], we can get ambiguous results because the average power depends on the entire time window considered in the integration. For this reason, our analysis takes into account averaged values and the instantaneous quantities shown in Fig. 2, so that a robust analysis can be done [19]. It is worth mentioning the physical meaning of $\bar{\mathcal{Q}}(\Delta)$ and $\bar{C}(\Delta)$, which can be understood as the average amount of entanglement and coherence, respectively, generated in the battery along the charging process. For completeness, we compute the value for the ergotropy at the end of the evolution $\mathcal{E}_{\text{fin}}(\Delta) = \mathcal{E}(t = t_{\min})$ for different values of Δ. From these sets of quantities, one can characterize the role of quantumness in the QB. In Fig. 3 we present the results for each quantify $\mathcal{E}_{\text{fin}}(\Delta)$, $\bar{\mathcal{P}}(\Delta)$, $\bar{\mathcal{Q}}(\Delta)$, and $\bar{C}(\Delta)$ as a function of Δ.

By remarking that the collective charging for the case in which $\Delta = 1$ is identical to the parallel charging process, Fig. 3 suggests that entanglement-like quantum correlations in the QB are not beneficial for the performance of the QB considered in our study. It is indeed possible to see that the difference in the QB performance becomes enhanced for the situation in which the amount of entanglement generated along the entire evolution is vanishing. The quantum characteristic of the two-cell QB considered here is maintained due to the system state coherence, as we can see in Figs. 2 and 3.

It is worthwhile to study the effect of parameter J by creating different regimes to identify the optimal charging protocols for the QBs. Then, we compute the relevant quantities introduced in Fig. 3 as a two-variable function for Δ and the relative strength coupling J/Ω, as shown in Fig. 4. By comparing Fig. 4, we can identify situations in which, by decreasing the coupling strength, the average work and power increases that converge to the values given in the $\Delta = 1$ region (where the coherence plays an effective role in the charging process) become optimal in all situations for values of J/Ω. Physically, this means that whether we are increasing

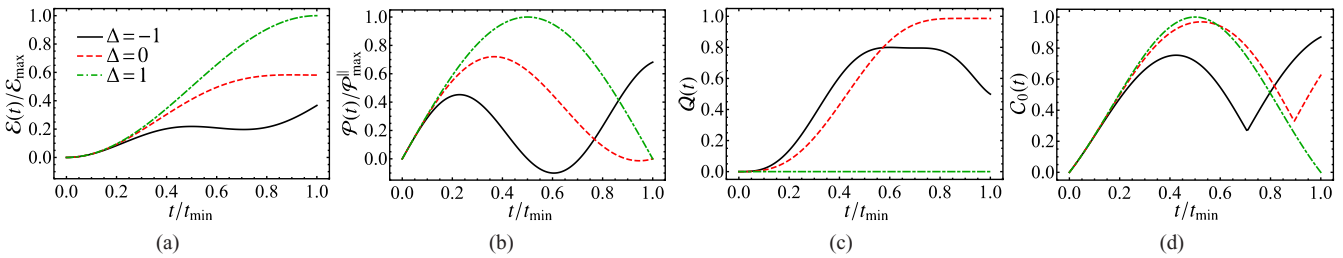

FIG. 2. Time evolution for (a) ergotropy, (b) instantaneous charging power, (c) entanglement, and (d) coherence of the two-cell QB for different values of the anisotropy parameter Δ. The coupling regime between the qubits is $J = \Omega$.

the pumping field intensity (Ω) or we are just turning off the internal battery interactions, we are close to the charging process with $\Delta = 1$, which is independent of the coupling strength. Again, our regime of observation is given by $t \in [0, t_{\min}]$, providing the optimal time window for our study. We remark that the behavior of the coherence in the system, Fig. 4(d), seems to be in agreement with the behavior of the power and charge, Figs. 4(a) and 4(b), for all values of J/Ω, while the entanglement (average) behavior does not explain the increasing battery efficiency in the regimes considered here. It is worth highlighting that our results are consistent with the specific case of the strong-coupling limit $J \gg \Omega$ in Ref. [6]. Moreover, one can observe that there is a significant reduction in the work and power of the battery when the coupling constant J for other values of $\Delta \neq 1$ is increased. In other words, these quantities tend to the maximum value in the limit $J \to 0.1\ \Omega$. Consequently, this implies that nonzero anisotropy has no effect on the charging process of many-body quantum batteries in this regime. In fact, it means that we have an intense charging field, and we expected that no internal interactions in the battery would become relevant for the charging process. In the same way, in the low-intensity regime of the charging field ($J \to 10\ \Omega$), the dynamics is drastically governed by interaction, and we can see the relevant role that anisotropy plays in the battery charging performance.

IV. THREE-CELL QB

As an immediate application, let us now discuss the quantumness of a three-cell QB. It can be achieved by adding a new cell to the battery, and the new interaction Hamiltonian reads $H'_{\rm int} = J\hbar \sum_{n=1}^{2}(\sigma_n^x\sigma_{n+1}^x + \sigma_n^y\sigma_{n+1}^y + \Delta\sigma_n^z\sigma_{n+1}^z)$. Then, for this case we numerically solve the system dynamics $\rho(t)$ and compute the quantities $\mathcal{E}(t)$ and $C_0(t)$ as

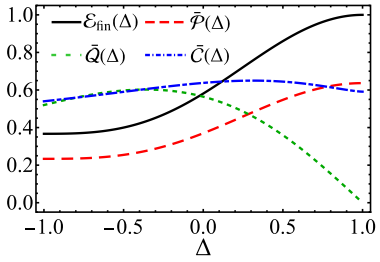

FIG. 3. Graph for the quantities $\mathcal{E}_{\rm fin}(\Delta)$ (in units of $\mathcal{E}_{\max}$), $\bar{\mathcal{P}}(\Delta)$ (as a multiple of $\mathcal{P}^{\parallel}_{\max}$), $\bar{\mathcal{Q}}(\Delta)$, and $\bar{C}(\Delta)$ as a function of Δ. The coupling regime between the qubits is $J = \Omega$.

was done previously, but the quantities $\mathcal{P}(t)$ and $\mathcal{Q}(t)$ need to be computed in a different way. Due to the numerical solution, $\mathcal{P}(t)$ is computed here from the energy current operator $\hat{\mathcal{P}}$ as $\mathcal{P}(t) = {\rm Tr}(\hat{\mathcal{P}}\rho(t))$, where [5]

$$\hat{\mathcal{P}} = (1/i\hbar)[H_0, H'_{\rm int}]. \tag{13}$$

As for the correlation $\mathcal{Q}(t)$, we cannot use Eq. (11) in this case because we have a tripartite system [20–22]. Therefore, one defines a quantity based on the average purity of each subsystem as $\mathcal{Q}_{\rm av} = \sum_{n=1}^{N}\mathcal{Q}_n/N$, where $\mathcal{Q}_n = {\rm Tr}(\rho_n^2)$, with ρ_n being the reduced matrix density of the nth cell. We stress here that the above quantity cannot be taken as a measure of correlations for a general $\rho(t)$, but in the case in which $\rho(t)$ is a pure state it can be used as a measure of nonseparability (correlations) of the system state. In fact, for a separable state of N qubits, ${\rm Tr}(\rho_n^2) = 1$ for all N, we get $\mathcal{Q}_{\rm av} = 1$ for a fully uncorrelated state of N qubits. Otherwise, in the case in which the system is correlated (even for nearest-neighbor qubits) we shall find ${\rm Tr}(\rho_n^2) \neq 1$ for some n, thus revealing a correlated system.

Figure 5 shows the relevant quantities for the three-cell quantum battery. From Figs. 5(a) and 5(c) it is possible to see that an entanglement charging process implies a nonoptimal charging process, since the case without correlation achieves maximum charge at $t = t_{\min}$. Furthermore, we remark that this case corresponds to the situation in which maximum coherence is created in the system during its evolution. A detailed analysis of Fig. 5 shows that it is not evident that we have a trivial tradeoff between correlations and power, but if we take the coherence into account we gain a better understanding of the system.

V. CONCLUSIONS

In this work, we studied the relation between entanglement and coherence with the performance of two- and three-cell quantum batteries. By using a system of coupled two-level systems, we explore the role of an anisotropy parameter of the XXZ Heisenberg linear chain. Through a counterexample, we have shown that the generation of entanglement along the charging process of QBs can negatively contribute to the performance of QBs. Our results suggest that a nontrivial relation between the amount of entanglement and high-performance QBs is not universal and depends on the system with which we are dealing. On the other hand, coherence develops a relevant role as a resource for the efficiency of the system considered in our study. As a general conclusion, we highlight

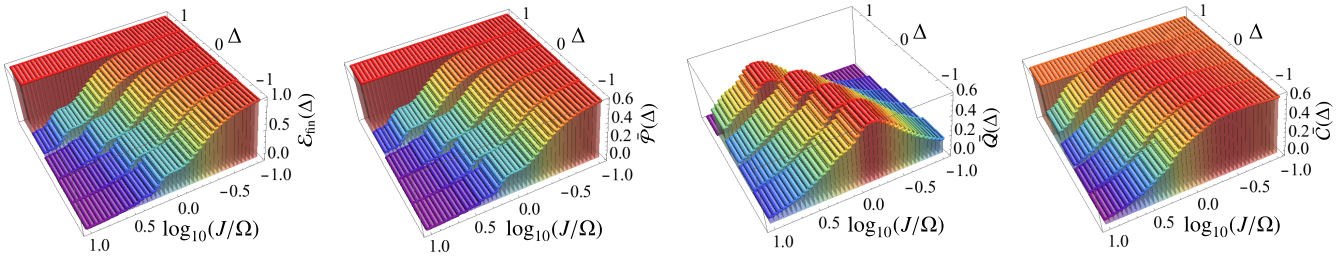

FIG. 4. Graphs for the quantities $\mathcal{E}_{\rm fin}(\Delta)$ (in units of $\mathcal{E}_{\rm max}$), $\bar{\mathcal{P}}(\Delta)$ (as a multiple of $\mathcal{P}^{\parallel}_{\rm max}$), $\bar{\mathcal{Q}}(\Delta)$, and $\bar{\mathcal{C}}(\Delta)$ as a function of Δ and J/Ω (in log scale). The regime of values for J varies from $J = 0.1\ \Omega$ [$\log_{10}(J/\Omega) = -1$] to $J = 10\ \Omega$ [$\log_{10}(J/\Omega) = 1$] and $\Delta \in \{-1, -0.5, 0, 0.5, 1\}$.

a correlation-coherence tradeoff in the optimal performance of QBs, so that the high charging efficiency of the QBs adopted here cannot be explained by correlations only.

We recognize the validity of the large number of works in the literature showing the role of collective charging processes for scalable N-cell QBs. However, we highlight here the requirement of a detailed analysis of the real role of quantum correlations in the collective charging of such devices. By considering the results present in this paper and previous discussion on the work extraction from the coherence of quantum states [14,15,23], we stress that a possible "quantum supremacy" of QBs needs to be investigated in more detail. In addition, the definition of a class of different devices and charging processes would be a consequence of this study. The extension of this work to a scenario of N-cell QBs is content for future research, where a study of which physical quantity can be a good resource for optimal performance of QBs can be appropriately provided.

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APPENDIX A: ANALYTICAL SOLUTION FOR THE SYSTEM DYNAMICS

The most general state of two qubits reads

$$|\Psi(0)\rangle = \mu|\uparrow\uparrow\rangle + \nu|\uparrow\downarrow\rangle + \eta|\downarrow\uparrow\rangle + \delta|\downarrow\downarrow\rangle. \tag{A1}$$

With the help of Eqs. (5) and (6), its time evolution will be

$$|\Psi(t)\rangle = \mu(t)|\uparrow\uparrow\rangle + \nu(t)|\uparrow\downarrow\rangle + \eta(t)|\downarrow\uparrow\rangle + \delta(t)|\downarrow\downarrow\rangle, \tag{A2}$$

where

$$\begin{aligned}
\mu(t) &= -\frac{(\delta-\mu)}{2}e^{-iE_1t} + (\delta+\mu)\left(\gamma_1^2 e^{-iE_3t} + \gamma_2^2 e^{-iE_4t}\right) \\
&\quad + \gamma_1\gamma_2(\nu+\eta)(e^{-iE_4t} - e^{-iE_3t}), \\
\nu(t) &= -\frac{(\eta-\nu)}{2}e^{-iE_2t} + (\eta+\nu)\left(\gamma_2^2 e^{-iE_3t} + \gamma_1^2 e^{-iE_4t}\right) \\
&\quad + \gamma_1\gamma_2(\delta+\mu)(e^{-iE_4t} - e^{-iE_3t}), \\
\delta(t) &= (\delta-\mu)e^{-iE_1t} + \mu(t), \\
\eta(t) &= (\eta-\nu)e^{-iE_2t} + \nu(t).
\end{aligned} \tag{A3}$$

At this point, let us consider the most general state of two nonentangled qubits,

$$\begin{aligned}
\mu &= \sin[\theta_1]\sin[\theta_2]e^{i(\varphi_1+\varphi_2)}, \\
\nu &= \sin[\theta_1]\cos[\theta_2]e^{i\varphi_1}, \\
\eta &= \cos[\theta_1]\sin[\theta_2]e^{i\varphi_2}, \\
\delta &= \cos[\theta_1]\cos[\theta_2],
\end{aligned} \tag{A4}$$

where we can consider $\theta_1, \theta_2 \in [0, \pi]$ and $\varphi_1, \varphi_2 \in [0, 2\pi]$. Also, we find the energy $\mathrm{tr}(\rho(t)H_0)$

$$\begin{aligned}
U(t) = -2\omega_0\big[&\Gamma_1\left(\gamma_1^2\cos[(E_3-E_1)t] + \gamma_2^2\cos[(E_4-E_1)t]\right) \\
&+ \Gamma_2\left(\gamma_1^2\sin[(E_3-E_1)t] + \gamma_2^2\sin[(E_4-E_1)t]\right) \\
&+ \Gamma_3(\sin[(E_4-E_1)t] - \sin[(E_3-E_1)t]) \\
&+ \Gamma_4(\cos[(E_4-E_1)t] - \cos[(E_3-E_1)t])\big],
\end{aligned} \tag{A5}$$

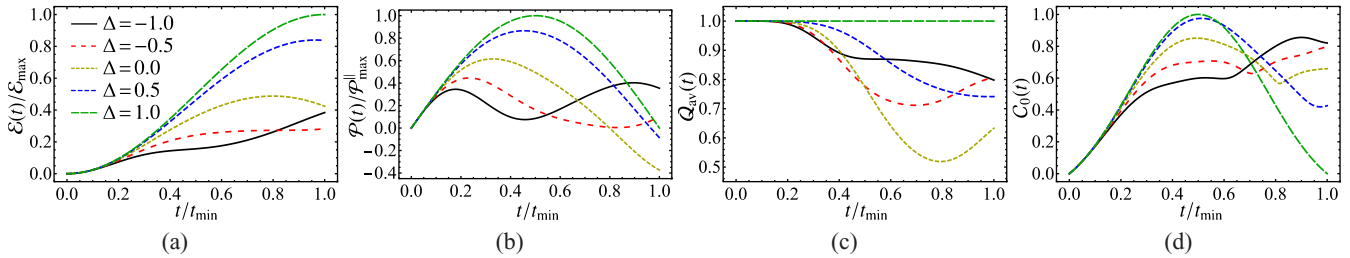


FIG. 5. Time evolution for (a) ergotropy, (b) instantaneous charging power, (c) average entanglement, and (d) coherence of the three-cell QB for different values of the anisotropy parameter Δ. The coupling regime between the qubits is $J = \Omega$, and for the three-cell QB we find $\mathcal{E}_{\rm max} = 6\hbar\omega$ and $\mathcal{P}^{\parallel}_{\rm max} = \mathcal{E}_{\rm max}\Omega$.

with

$$\Gamma_1 = 2(\cos^2[\theta_1]\cos^2[\theta_2] - \sin^2[\theta_1]\sin^2[\theta_2]),$$
$$\Gamma_2 = (\sin[2\theta_1]\sin[2\theta_2]\sin[\varphi_1+\varphi_2]),$$
$$\Gamma_3 = (\sin[2\theta_1]\sin[\varphi_1] + \sin[2\theta_2]\sin[\varphi_2]),$$
$$\Gamma_4 = \gamma_1\gamma_2[(\sin[2\theta_1]\cos[2\theta_2]\cos[\varphi_1] + \sin[2\theta_2]\cos[2\theta_1]\cos[\varphi_2])]. \quad \text{(A6)}$$

In addition, we define the instantaneous charge (ergotropy) as

$$\mathcal{E}(t) = U(t) - E_{\text{emp}}, \quad \text{(A7)}$$

and the instantaneous power

$$\mathcal{P}(t) = \frac{d}{dt}\mathcal{E}(t). \quad \text{(A8)}$$

At the beginning of the charging process, the battery is assumed to be empty, i.e., $\rho(0) = |\text{emp}\rangle\langle\text{emp}|$. This is achieved when we have $\theta_1 = \theta_2 = 0$ in Eq. (A4), which leads to $\Gamma_1 = 2$ and $\Gamma_2 = \Gamma_3 = \Gamma_4 = 0$ in Eq. (A6). Therefore, we have

$$\mathcal{E}(t) = -4\omega_0\left(\gamma_1^2\cos[(E_3-E_1)t] + \gamma_2^2\cos[(E_4-E_1)t] - \tfrac{1}{2}\right) \quad \text{(A9)}$$

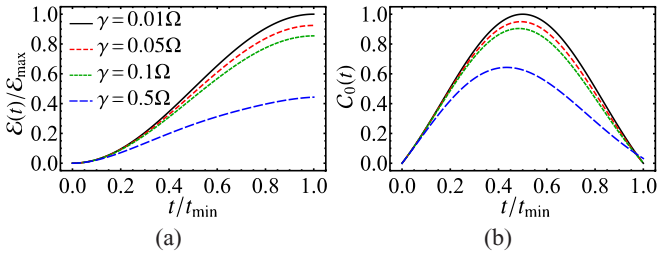
and

$$\mathcal{P}(t) = 4\omega_0\left(\gamma_1^2(E_3-E_1)\sin[(E_3-E_1)t] + \gamma_2^2(E_4-E_1)\sin[(E_4-E_1)t]\right). \quad \text{(A10)}$$

APPENDIX B: DEPHASING EFFECTS ON BATTERY PERFORMANCE

To discuss the performance of the battery with regard to decoherence and to describe how the coherence is a resource of the charging process, in this Appendix we briefly present some results of the performance of a two-qubit QB driven by the Hamiltonian $H = H_{\text{ch}} + H_{\text{int}}$, where $H_{\text{ch}} = \hbar\Omega\sum_{n=1}^{2}\sigma_n^x$, and H_{int} is given by Eq. (3). Because we are interested in understanding the role of quantum coherence for the charging process, it is worthwhile to study battery performance in the presence of dephasing. For this purpose, we investigate our charging protocol driven by the Lindblad master equation for dephasing as

$$\frac{d\rho(t)}{dt} = -\frac{i}{\hbar}[H,\rho(t)] + \gamma\sum_{i=1,2}\left(\sigma_i^z\rho(t)\sigma_i^z - \rho(t)\right), \quad \text{(B1)}$$



(a) (b)

FIG. 6. (a) Instantaneous ergotropy and (b) coherence for the dissipative charging process of the two-cell QB driven by Eq. (B1) for different values of the dissipative charging process. The Hamiltonian parameters are $J = \Omega$, $t_{\min} = \pi/2\Omega$, and $\Delta = 1$.

with the local Lindblad operators σ_i^z $(i = 1, 2)$ acting on each qubit with an identical dephasing rate γ.

Now, because the dynamics is not unitary and leads the system to a nonpure density matrix, the ergotropy cannot be computed from internal energy, as we did in Eq. (2). In fact, for a general density matrix ρ with dimension N, the ergotropy given in Eq. (1) reads [9,15]

$$\mathcal{E} = \sum_{i,n}^{N,N} x_n\epsilon_i(|\langle x_n|\epsilon_i\rangle|^2 - \delta_{ni}), \quad \text{(B2)}$$

where $|x_n\rangle$ and x_n are the eigenvectors and eigenvalues of ρ, so that $x_1 \geqslant x_2 \geqslant \cdots \geqslant x_N$, which is obtained from the spectral decomposition of ρ, and ϵ_i are eigenvalues of the reference Hamiltonian H_0 with eigenstates $|\epsilon_i\rangle$, with $\epsilon_1 \leqslant \epsilon_2 \leqslant \cdots \leqslant \epsilon_N$. Therefore, in Fig. 6 we present the ergotropy and coherence for different dissipative rates γ for the case in which we have the anisotropy parameter $\Delta = 1$ and $t \in [0, t_{\min}]$, which leads to the optimal parallel charging process of the battery. As one can see, concerning the absence of dephasing effects, i.e., $\gamma = 0$, by increasing γ we have a destructive impact on battery performance such that the extractable work becomes smaller. This occurs due to the amplification of dephasing effects in the charging process, as we can see from the graph for ergotropy and coherence in Figs. 6(a) and 6(b), respectively. Consequently, the coherence plays the role of a quantum advantage in the QBs.

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