Kinetic theory for spin-1/2 particles in ultrastrong magnetic fields

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When the Zeeman energy approaches the characteristic kinetic energy of electrons, Landau quantization becomes important. In the vicinity of magnetars, the Zeeman energy can even be relativistic. We start from the Dirac equation and derive a kinetic equation for electrons, focusing on the phenomenon of Landau quantization in such ultrastrong but constant magnetic fields, neglecting short-scale quantum phenomena. It turns out that the usual relativistic γ factor of the Vlasov equation is replaced by an energy operator, depending on the spin state, and also containing momentum derivatives. Furthermore, we show that the energy eigenstates in a magnetic field can be computed as eigenfunctions of this operator. The dispersion relation for electrostatic waves in a plasma is computed, and the significance of our results is discussed.

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I. INTRODUCTION

Quantum kinetic descriptions of plasmas are typically of most interest for high densities and modest temperature [1]. For this purpose, typically the starting point is the Wigner-Moyal equation [1–6], or various generalizations thereof also accounting for physics associated with the electron spin, such as the magnetic dipole force [7,8], spin magnetization [7,9], and spin-orbit interaction [9]. Both weakly [10,11] and strongly [12,13] relativistic treatments have been presented in the recent literature.

Certain quantum phenomena depend strongly on the magnitude of the electromagnetic field, however, rather than on the density and temperature parameters. Phenomena such as radiation reaction (reviewed in, e.g., Ref. [14]) and pair creation fall into this category.[1] Another field-dependent phenomenon is Landau quantization [17], which becomes prominent whenever the Zeeman energy due to the magnetic field is comparable to or larger than the thermal energy, or the Fermi energy, in the case of degenerate electrons.

In the atmospheres of pulsars and magnetars [18], the electron motion may become relativistic, and the magnetic field strength can be ultrastrong, i.e., the Zeeman energy may be comparable to or even larger than the electron rest mass energy [19]. Further information about pulsar properties can be gained through the emission profiles, see, e.g., Refs. [20–22]. Theoretical studies of wave propagation relevant for strongly magnetized objects have been made both with kinetic [23] and hydrodynamic [24,25] models. However, most previous theoretical studies starting from the Dirac equation (see also Refs. [11,12]) have been limited to cases where the magnetic field strength is well below the critical field $B_{\rm cr} = m^2c^2/|q|\hbar$.

Our objective in this work is to derive a fully relativistic kinetic model of spin-1/2 particles, applicable for ultrastrong magnetic fields, i.e., with

$$B \sim B_{\rm cr} = m^2c^2/|q|\hbar = 4.4 \times 10^9\,\mathrm{T}, \tag{1}$$

relaxing the conditions given in previous works. Assuming that the electric field is low enough to avoid pair creation (i.e., below the critical electric field $E_{\rm cr} = m^2c^3/|q|\hbar = 1.3 \times 10^{18}\,\mathrm{V/m}$), we can use the Foldy-Wouthuysen transformation [26,27] to separate particle states from antiparticle states in the Dirac equation. Moreover, we limit ourselves to the case where the characteristic spatial scale length of the fields is much longer than the Compton length, $L_c = \hbar/mc$.

Making a Wigner transformation of the density matrix, our approach results in an evolution equation for a 2×2 Wigner matrix, where the four components encode information regarding the spin states. However, the off-diagonal elements of the matrix is associated with the spin-precession dynamics which is too rapid to be resolved by the theory. Thus a further reduction is made, where only the diagonal components representing the spin-up and the spin-down states relative to the magnetic field remain.

Together with Maxwell's equations, we obtain a closed system describing the plasma dynamics. A unique feature of the model is the energy expression, where the usual γ factor

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[1]While the classical radiation reaction, where the response is a smooth function of the orbit, is a useful approximation as long as the emitted spectrum is soft (dominated by photons with energies well below mc^2), the description based on QED (see, e.g., Refs. [15,16]), containing discrete probabilistic contributions due to the emission of high-energy quanta, is more generally applicable.

from classical relativistic theory is replaced by an operator in phase space, also depending on the magnetic field. Interestingly, the Landau-quantized states in a constant magnetic field turn out to be eigenfunctions of the energy operator of this theory, which is helpful when computing the thermodynamic background state for the Wigner matrix. To demonstrate the usefulness of the theory, we compute the dispersion relation for Langmuir waves propagating parallel to an external magnetic field. Finally, the consequences of the theory and applications to astrophysics are discussed

II. BACKGROUND AND PRELIMINARIES

In this section we will briefly summarize the fundamental approaches we have used in order to derive a kinetic theory for spin-1/2 particles in ultrastrong magnetic field; see the next section for a technical derivation.

In the formulation of quantum mechanics in phase space [28] the system is described by a quasidistribution function $W(\mathbf{r},\mathbf{p})$. This is in contrast to the Hilbert space formulation where the system is described by a density matrix $\rho(\mathbf{r},\mathbf{r}')$. The two formulations are related via the Wigner transformation. From the von Neumann equation for the density matrix,

$$i\hbar\partial_t\hat{\rho} = [\hat{H},\hat{\rho}], \tag{2}$$

one obtains an equivalent equation for W, involving $\partial_x W$ and $\partial_p W$. To lowest order in $\hbar$, this equation will be the Liouville equation for the Hamiltonian at hand. This generalizes to a many-particle system, and because the phase-space formulation is analogous to classical statistical physics, the Bogoliubov-Born-Green-Kirkwood-Yvon (BBGKY) hierarchy [29] applies, and assuming exchange and correlation effects are small, one obtains an analog of the classical Vlasov equation.

While this approach is completely general (see Ref. [28] for numerous applications in a wide range of fields), for the Dirac Hamiltonian in particular, the physical interpretation is complicated due to that the Dirac Hamiltonian is mixing particle and antiparticle states. Through a Foldy-Wouthuysen transformation [26,27], particle and antiparticle states can be decoupled to a given order in some small expansion parameter ε. Using the resulting Hamiltonian $\hat{H}_{\mathrm{FW}}$ and applying a Wigner transformation, one can get a kinetic theory for the quasidistribution W involving particles or antiparticles only. For this to be physically valid, the pair production rate should be low when ε is small.

The obtained kinetic equation is an evolution equation for a 2×2 Wigner matrix, where the four components represent the different spin states. There are several ways of treating the spin degrees of freedom in a phase-space approach [11], e.g., extending phase-space with spin dimensions [7], dividing the Wigner matrix into a scalar and a 3-vector [30], or using other generalizations of the Wigner function for spin [31].

The kinetic equation for the particles or antiparticles is closed with Maxwell's equations with sources as moments of W, e.g., the free charge density is $\rho_f(\mathbf{r}) = \int qW(\mathbf{r},\mathbf{p})\,d^3p = q\,\langle\mathbf{r}|\,\hat{\rho}\,|\mathbf{r}\rangle$.

In the next section, we present the derivation of a kinetic theory for plasmas in ultrastrong magnetic fields, including the spin degrees of freedom. In our derivation, we need to split the magnetic field $\mathbf{B} = \mathbf{B}_0 + \delta\mathbf{B}$ into a strong but constant background $\mathbf{B}_0$ and a weak but freely varying perturbation $\delta\mathbf{B}$. We will also not resolve dynamics on length scales comparable to the de Broglie wavelength and not on timescales comparable to the inverse Compton frequency.

III. DERIVATION OF THE THEORY AND CONSERVATION LAWS

In this section we review the Foldy-Wouthuysen transformation for strong fields (Sec. III A), and then we apply it to obtain a particle-only Hamiltonian that is applicable in ultrastrong magnetic fields (Ref. III B). Next, in Sec. III C, we derive the corresponding kinetic equation in phase space. Finally, in Sec. III D, we present conservation laws for particle number and energy.

A. Foldy-Wouthuysen transformation

To derive our theory for ultrastrong magnetic fields, we will take the Dirac Hamiltonian

$$\hat{H} = \beta m + \hat{\mathcal{E}} + \hat{\mathcal{O}}, \tag{3}$$

as our starting point. Here m is the mass, $\hat{\mathcal{E}} = q\phi(\hat{\mathbf{r}})$, $\hat{\mathcal{O}} = \boldsymbol{\alpha}\cdot\hat{\boldsymbol{\pi}}$, and $\boldsymbol{\alpha}$ and β are the Dirac matrices. Furthermore, $\hat{\mathbf{p}}$ is the canonical momentum, $\hat{\boldsymbol{\pi}} = \hat{\mathbf{p}} - q\mathbf{A}(\hat{\mathbf{r}},t)$, q is the charge, and ϕ and $\mathbf{A}$ are, respectively, the scalar and vector potentials. From now one we use units such that $c = 1$.

In Foldy and Wouthuysen's seminal paper [26] the expansion parameter ε was E/m, meaning that the expansion is valid for sufficiently small energies, i.e., E represents relevant energies (kinetic energy, magnetic dipole energy, etc.). In this paper we will use the results of Refs. [27,32] where a modified Foldy-Wouthuysen transformation was developed for the case where the expansion parameter is instead the scale length of the fields. This transformation hence makes it possible to take into account arbitrarily strong fields as long as we are only concerned with variations on sufficiently long scale lengths.

An operator is called *odd* if it couples the upper and the lower pairs of components of the Dirac four-spinor and *even* if it does not. The goal of a Foldy-Wouthuysen transformation is thus to obtain a Hamiltonian H_{FW} that is even. The odd and even terms of the Hamiltonian (3) are $\hat{\mathcal{O}}$ and $\hat{\mathcal{E}}$, respectively, satisfying $[\beta,\hat{\mathcal{E}}] = 0$ and $\{\beta,\hat{\mathcal{O}}\} = 0$.

In Ref. [32], Silenko found a unitary transformation of the Hamiltonian operator

$$\hat{H}' = \hat{U}(\hat{H} - i\partial_t)\hat{U}^\dagger + i\partial_t, \tag{4}$$

where

$$\hat{U}^{(\dagger)} = \frac{\hat{\epsilon} + m \pm \beta\hat{\mathcal{O}}}{\sqrt{2\hat{\epsilon}(\hat{\epsilon}+m)}}, \tag{5}$$

where $\hat{\epsilon} = \sqrt{m^2 + \hat{\mathcal{O}}^2}$. Using the Dirac Hamiltonian $\hat{H}$ in Eq. (4), one obtains

$$\hat{H}' = \beta\hat{\epsilon} + \hat{\mathcal{E}}' + \hat{\mathcal{O}}', \tag{6}$$

where

$$\hat{\mathcal{E}}' = \hat{\mathcal{E}} + \frac{1}{2\hat{T}}([\hat{T}, [\hat{T}, \beta\hat{\epsilon} + \hat{\mathcal{F}}]] - [\hat{\mathcal{O}}, [\hat{\mathcal{O}}, \hat{\mathcal{F}}]] - [\hat{\epsilon}, [\hat{\epsilon}, \hat{\mathcal{F}}]])\frac{1}{\hat{T}}, \quad (7)$$

$$\hat{\mathcal{O}}' = \frac{1}{2\hat{T}}\beta(\{\hat{\epsilon} + m, [\hat{\mathcal{O}}, \hat{\mathcal{F}}]\} - \{\hat{\mathcal{O}}, [\hat{\epsilon}, \hat{\mathcal{F}}]\})\frac{1}{\hat{T}}, \quad (8)$$

where $\hat{T} = \sqrt{2\hat{\epsilon}(m + \hat{\epsilon})}$ and $\hat{\mathcal{F}} = \hat{\mathcal{E}} - i\hbar\partial_t$. The Hamiltonian H' still has odd terms, the anticommutators in $\hat{\mathcal{O}}'$ are linear in $\hat{\mathcal{O}}$, i.e., linear in $\boldsymbol{\alpha}$. However, the odd term $\hat{\mathcal{O}}'$ in H' should be small compared to $\hat{\epsilon}$, after choosing an expansion parameter. Since we are interested in the effects of an ultrastrong magnetic field, our small expansion parameters will be the electric field strength E/E_{cr} (such that pair production is negligible) and the inverse scale length of the fields L_c/L, but we will make no assumption on the *magnetic* field strength.

Thus a second transformation can be preformed with the following operator:

$$\hat{U}' = e^{i\hat{S}'}, \quad \hat{S}' = -\frac{i}{4}\left\{\hat{\mathcal{O}}', \frac{1}{\hat{\epsilon}}\right\}. \quad (9)$$

Since $\hat{\mathcal{O}}'$ is small, only the major corrections are taken into account. Finally, the transformed Hamiltonian is

$$\hat{H}_{\text{FW}} = \beta\hat{\epsilon} + \hat{\mathcal{E}}' + \frac{1}{4}\left\{\hat{\mathcal{O}}'^2, \frac{1}{\hat{\epsilon}}\right\}. \quad (10)$$

This Hamiltonian has no odd terms, so we are done with the transformation of the Dirac Hamiltonian. Until now, we have presented the Foldy-Wouthuysen transformation derived in Ref. [27].

B. Strong-field Hamiltonian

Our approach now is to calculate the commutators occurring in (6) to all orders in the magnetic field. In doing this we will expand some of them in a series using Ref. [33]; see the Appendix for more details. After the calculation, the Hamiltonian H' in (6) becomes

$$\hat{H}' = \beta\hat{\epsilon} + \hat{\mathcal{E}}' + \hat{\mathcal{O}}', \quad (11)$$

where

$$\hat{\mathcal{E}}' = \hat{\mathcal{E}} + \frac{im\mu_B}{\hat{T}}\left[i\hbar q\left(\frac{2}{\hat{T}} + \frac{m}{\hat{\epsilon}\hat{T}}\right)^2(\mathbf{E}\times\mathbf{B})\cdot\hat{\boldsymbol{\pi}} - \frac{i\hbar q}{\hat{\epsilon}^2}(\mathbf{E}\times\mathbf{B})\cdot\hat{\boldsymbol{\pi}} + i\hat{\boldsymbol{\Sigma}}\cdot(\mathbf{E}\times\hat{\boldsymbol{\pi}} - \hat{\boldsymbol{\pi}}\times\mathbf{E})\right]\frac{1}{\hat{T}}, \quad (12)$$

$$\hat{\mathcal{O}}' = \frac{im\mu_B\beta}{\hat{T}}\left[2\boldsymbol{\alpha}\cdot\mathbf{E}(\hat{\epsilon} + m) - 2\left(\frac{\hat{\boldsymbol{\pi}}}{\hat{\epsilon}}\cdot\mathbf{E}\right)\boldsymbol{\alpha}\cdot\hat{\boldsymbol{\pi}} + \frac{\hbar}{\hat{\epsilon}}(\hat{\boldsymbol{\Sigma}}\cdot\partial_t\mathbf{B})\boldsymbol{\alpha}\cdot\hat{\boldsymbol{\pi}}\right]\frac{1}{\hat{T}}, \quad (13)$$

where $\mu_B = q\hbar/2m$ is the Bohr magneton and

$$\hat{\boldsymbol{\Sigma}} = \begin{pmatrix}\boldsymbol{\sigma} & 0\\ 0 & \boldsymbol{\sigma}\end{pmatrix}.$$

Note that we kept all orders of the magnetic field. However, we only kept up to first order of the combination of $\hbar$ with E and ∇_r. In the expression for $\hat{H}'$ there are still some odd operators, but since $\hat{\mathcal{O}}'$ is linear in $\mu_B E$ and $\mu_B\partial_t B$, they should be smaller than $\hat{\epsilon}$. Thus it is fine to only include the minor correction of the second transformation in (9). Thus the Hamiltonian $\hat{H}_{\text{FW}}$ after the second transformation will have the same structure as in (10). However, the anticommutator in $\hat{H}_{\text{FW}}$ in (10) is proportional to the square of $\hat{\mathcal{O}}'$. Since we only kept up to first order of E and $\partial_t B$, we keep only the first two terms of $\hat{H}_{\text{FW}}$,

$$\hat{H}_{\text{FW}} = \beta\hat{\epsilon} + q\phi(\hat{\mathbf{r}}) + \frac{\mu_B m}{\sqrt{2\hat{\epsilon}(\hat{\epsilon} + m)}} \times \left\{\hat{\boldsymbol{\Sigma}}\cdot(\hat{\boldsymbol{\pi}}\times\mathbf{E} - \mathbf{E}\times\hat{\boldsymbol{\pi}}) - \frac{2\mu_B m}{\hat{\epsilon}^2}\left[1 + \frac{m^2}{2\hat{\epsilon}(m + \hat{\epsilon})}\right] \times(\hat{\boldsymbol{\pi}}\times\mathbf{B})\cdot\mathbf{E}\right\}\frac{1}{\sqrt{2\hat{\epsilon}(\hat{\epsilon} + m)}}, \quad (14)$$

where $\hat{\epsilon} = \sqrt{m^2 + \hat{\boldsymbol{\pi}}^2 - 2\mu_B m\hat{\boldsymbol{\Sigma}}\cdot\mathbf{B}}$. Taking the limit of weak B field, i.e., keeping up to first order in $\mu_B B/m$, we recover the Hamiltonian in Ref. [27]. The Hamiltonian in (14) includes the spin orbit interaction, see Ref. [12] for more details, but since $E \ll B$, it is negligible compared to the magnetic interaction. We will therefore neglect the spin-orbit interaction from now on, as the main idea of this paper is to study the effects of a strong magnetic field on the dynamics of a plasma. The transformed Hamiltonian is now

$$\hat{H}_{\text{FW}} = \beta\sqrt{m^2 + \hat{\boldsymbol{\pi}}^2 - 2\mu_B m\hat{\boldsymbol{\Sigma}}\cdot\mathbf{B}} + q\phi(\hat{x}). \quad (15)$$

This Hamiltonian includes all orders of $\mu_B B/m$ and is fully relativistic and hence is suitable to be used in deriving a kinetic equation for plasma in an environment where the magnetic field is of the order of the critical field B_{cr}. However, in order to obtain a kinetic equation in phase space (see next subsection for more details), we will use the identities from Ref. [34], see (19) and (20) below. These identities can be utilized for a function F that only depends on $\hat{\boldsymbol{\pi}}$. However, our Hamiltonian in Eq. (15) depends on both $\mathbf{B}(\hat{\mathbf{r}})$ and $\hat{\boldsymbol{\pi}}$, and thus we divide the magnetic field as

$$\mathbf{B}(\hat{\mathbf{r}}) = \mathbf{B}_0 + \delta\mathbf{B}(\hat{\mathbf{r}}), \quad (16)$$

where $\mathbf{B}_0$ is a constant strong magnetic field (i.e., $\mu_B B_0/m \sim 1$) and $\delta\mathbf{B}(\hat{\mathbf{r}})$ is a varying magnetic field. Furthermore, a Taylor series around $\mathbf{B} = \mathbf{B}_0$ can be done; see Ref. [35] for more details. However, the Taylor series gets more complicated for the higher-order terms, and thus a restriction on $\delta\mathbf{B}(\hat{\mathbf{r}})$ needs to be done. Considering the case where $\mu_B\delta B/m \ll 1$, similarly to the previous assumption $\mu_B E/m \ll 1$, the Hamiltonian becomes

$$\hat{H}_{\text{FW}} = \beta\sqrt{m^2 + \hat{\boldsymbol{\pi}}^2 - 2\mu_B m\boldsymbol{\sigma}\cdot\mathbf{B}_0} + q\phi(\hat{\mathbf{r}}). \quad (17)$$

This Hamiltonian is the final one that will be used in next section to derive the kinetic equation. Note that all operators in (17) are even, and thus we will let $\beta \to 1$ and $\hat{\boldsymbol{\Sigma}} \to \boldsymbol{\sigma}$ from now on.

Note that even though $\delta\mathbf{B}$ does not appear explicitly in Eq. (17), the perturbed magnetic field will still be contained in the Lorentz force, as will be seen below, because $\hat{\boldsymbol{\pi}}$ contains the full vector potential **A** from which the full magnetic field is obtained. This is an example of how the phase-space formulation can make the physics of a system more explicit.

C. Gauge-invariant Wigner function and kinetic equation

Now we want to derive a kinetic equation on phase space using the Hamiltonian (17). We will use the gauge-invariant Wigner function defined by Stratonovich [34],

$$W_{\alpha\beta}(\mathbf{r},\mathbf{p},t)=\left(\frac{1}{2\pi\hbar}\right)^3\int d^3\lambda\exp\times\left\{\frac{i\boldsymbol{\lambda}}{\hbar}\cdot\left[\mathbf{p}+q\int_{1/2}^{1/2}d\tau\mathbf{A}(\mathbf{r}+\tau\boldsymbol{\lambda})\right]\right\}\times\rho_{\alpha\beta}\left(\mathbf{r}+\frac{\boldsymbol{\lambda}}{2},\mathbf{r}-\frac{\boldsymbol{\lambda}}{2},t\right),\tag{18}$$

for each spinor-space component of the density matrix, indiced by α,β. Thus, we write out Eq. (2) and use the identities [34]

$$F[-i\hbar\nabla_r-q\mathbf{A}(\mathbf{r})]\rho_{\alpha\beta}(\mathbf{r},\mathbf{r}')\to F[\mathbf{p}-i\hbar/2\nabla_r+im\mu_B\mathbf{B}\times\nabla_p]W_{\alpha\beta}(\mathbf{r},\mathbf{p}),\tag{19}$$

$$F[i\hbar\nabla_{r'}-q\mathbf{A}(\mathbf{r})]\rho_{\alpha\beta}(\mathbf{r},\mathbf{r}')\to F[\mathbf{p}+i\hbar/2\nabla_r-im\mu_B\mathbf{B}\times\nabla_p]W_{\alpha\beta}(\mathbf{r},\mathbf{p}).\tag{20}$$

Using the Hamiltonian in (17) and keeping up to first order in ∇_r as we did in the derivation of the Hamiltonian, the kinetic equation is

$$\partial_t W_{\alpha\beta}+\frac{1}{\epsilon'}\mathbf{p}\cdot\nabla_r W_{\alpha\beta}+q\left(\mathbf{E}+\frac{1}{\epsilon'}\mathbf{p}\times\mathbf{B}\right)\cdot\nabla_p W_{\alpha\beta}=0,\tag{21}$$

where **B** is the full magnetic field and

$$\epsilon'=\sqrt{m^2+\mathbf{p}^2-2m\mu_B\boldsymbol{\sigma}\cdot\mathbf{B}_0-m^2\mu_B^2(\mathbf{B}_0\times\nabla_p)^2}.\tag{22}$$

Also note that ϵ' is a function of $\boldsymbol{\sigma}$ and that the momentum derivatives act on everything to the right of the operator. Thus, in the second and fourth terms of (21), $1/\epsilon'$ acts also on **p**. In order to get a scalar theory, we can Taylor-expand ϵ' around $\boldsymbol{\sigma}$

$$\epsilon'=\tfrac{1}{2}(\epsilon'_++\epsilon'_-)I+\tfrac{1}{2}(\epsilon'_+-\epsilon'_-)\sigma_z,\tag{23}$$

where I is the identity matrix and

$$\epsilon'_\pm=\sqrt{m^2+\mathbf{p}^2\mp 2m\mu_B B_0-m^2\mu_B^2(\mathbf{B}_0\times\nabla_p)^2}.\tag{24}$$

Next, we note that if initially $W_{\alpha\beta}$ has no off-diagonal elements, let us say that $W_{11}=W_+$, $W_{22}=W_-$, then the evolution (21) for W_+ and W_- will decouple into separate equations for the spin-up and spin-down populations, as defined relative to $\mathbf{B}_0$. While limiting ourselves to such initial conditions may seem unwarranted, the only thing left out by this restriction is the spin precession dynamics. However, since the timescale for spin precession is the inverse Compton frequency, we note that the present theory, based on the assumption $\partial/\partial t\ll c/L_c$, is not designed to resolve the spin precession dynamics anyway. Hence, from now on, we will be using the above representation for $W_{\alpha\beta}$, in which case (21) decouples into the scalar equations for W_+ and W_- as follows:

$$\partial_t W_\pm+\frac{1}{\epsilon'_\pm}\mathbf{p}\cdot\nabla_r W_\pm+q\left[\mathbf{E}+\frac{1}{\epsilon'_\pm}\mathbf{p}\times\mathbf{B}\right]\cdot\nabla_p W_\pm=0.\tag{25}$$

The kinetic equation in Eq. (25) is our main result in this work. The new effects of this kinetic equation are hiding in $\epsilon'_\pm$. First, we have all orders of the spin magnetic moment, compared to previous models [11,12] where only the first-order correction is included. Note also that the equations for W_+ and W_- are decoupled and can be solved separately. Moreover, we have momentum derivatives in ϵ', which turn to be energy operators, see Sec. IV for more details.

Equation (25) describes the dynamics of an ensemble of spin-1/2 particles in an ultrastrong magnetic field in the mean-field approximation. In this approximation, the electric and magnetic fields are generated by the sources via

$$\nabla\cdot\mathbf{E}=\rho_f,\quad\text{and}\quad\nabla\times\mathbf{B}=\mathbf{j}_f+\partial_t\mathbf{E},\tag{26}$$

where ρ_f and j_f are the free charge and current density, respectively,

$$\rho_f=q\sum_\pm\int d^3p\,W_\pm,\tag{27}$$

$$\mathbf{j}_f=q\sum_\pm\int d^3p\frac{1}{\epsilon'_\pm}\mathbf{p}W_\pm.\tag{28}$$

Under the assumptions we have made, the bound sources arising from the spin are negligible, but these and have been included in other models, e.g., Refs. [7,11]. A more thorough discussion of our model, including comparison with previous works [8–13,36–38] is found in Sec. VI.

D. Conservation laws

To check the validity of the derived model, we derive the conservation law of energy and the mass continuity. Starting with the mass continuity, the number density of spin-up (spin-down) particles $n_\pm$ can be given by $n_\pm=\int d^3p\,W_\pm$. To show that this quantity is conserved we take the time derivative of it and use Eq. (25). Since ϵ' is independent of **r**, it is trivial to show that

$$\partial_t n_\pm+\nabla_r\cdot\int d^3p\frac{1}{\epsilon'_\pm}\mathbf{p}W_\pm=0.\tag{29}$$

The number densities are separately conserved because transitions between spin-up and spin-down states require absorption or emission of quanta with energies on the order of m. This is based on the assumption that $\mu_B B_0/m\sim 1$ holds, since this is the regime where the strong-field quantum effects of our model are most significant. Note that this is closely related to the approximation made when deriving Eq. (25).

Moving to the conservation of energy, the total energy density is

$$E_{\text{tot}}=\frac{1}{2}(E^2+B^2)+\sum_\pm\int d^3p\,\epsilon'_\pm W_\pm.\tag{30}$$

We want now to show that the energy is conserved, taking the time derivative of E_{tot}, and using Maxwell's equations together with the kinetic equation, we get

$$\partial_t E_{\rm tot} + \nabla_r \cdot \mathbf{K} = 0, \tag{31}$$

where $\mathbf{K}$ is the energy flux

$$\mathbf{K} = \mathbf{E}\times\mathbf{B} + \sum_{\pm}\int d^3p\,\mathbf{p}W_{\pm}. \tag{32}$$

This is precisely what one would expect: the Poynting vector for the fields and the kinetic energy flux for the particles as required in a relativistic theory. Since we have neglected polarization and magnetization, there is no Abraham-Minkowski dilemma in this model; see Ref. [13] and references therein for a related discussion.

IV. BACKGROUND WIGNER FUNCTION IN A CONSTANT MAGNETIC FIELD

In principle we can compute the time-independent solutions for W in a constant magnetic field $\mathbf{B} = B_0\hat{\mathbf{z}}$ by solving the Dirac equation for this geometry, making a sum over different particle states and then perform the Foldy-Wouthuysen and Wigner transformations of Sec. III C. Except for the Foldy-Wouthuysen transformation, this was done in a covariant approach in Ref. [36]. However, here we will take a shorter route to arrive at the same results. Noting that for a constant magnetic field, both the Dirac equation and the Pauli equation results in electrons obeying a quantum harmonic oscillator equation, we can make a trivial generalization of the Pauli case [7]. First we need to find the Wigner function corresponding to the Landau quantized energy eigenstates. Both for the Pauli and the Dirac equations, the spatial dependence of the wave function in Cartesian coordinates can be expressed as a Hermite polynomial times a Gaussian function [39] only the energy eigenvalues are different. Specifically, applying the Dirac theory, the energy of the Landau quantized states become

$$E_{n\pm} = m\sqrt{1 + (2n+1\pm 1)\frac{\hbar\omega_{ce}}{m} + \frac{p_z^2}{m^2}}, \tag{33}$$

where $n = 0, 1, 2, \ldots,$ corresponds to the different Landau levels for the perpendicular contribution to the kinetic energy, the index $\pm$ represents the contribution from the two spin states, and the term proportional to p_z^2 gives the continuous dependence on the parallel kinetic energy. Since the Pauli and Dirac equations for individual particle states have the same spatial dependence for the wave function, we can adopt the expression for the Wigner function from Ref. [7] (based on the Pauli equation) with some relatively minor adjustments.

(1) Contrary to Ref. [7], we have made no Q transform to introduce an independent spin variable, and thus the spin-dependence of Ref. [7] reduces to $W_{\pm}$.

(2) The Wigner function of Ref. [7] must be expressed in terms of the momentum, i.e., $m(v_x^2 + v_y^2)/2 \to (p_x^2 + p_y^2)/2m$.

(3) The nonrelativistic energy of Ref. [7] is replaced by the relativistic expression Eq. (33) of the Dirac theory.

(4) The normalization of the Wigner function must be adopted to fit the present case.

With these changes, the Wigner function for an energy eigenstate $W_{n\pm}$ can be written

$$W^0_{n\pm} = g_{n,\pm}(p_z)(-1)^n\phi_n(p_\perp), \tag{34}$$

where

$$\phi_n(p_\perp) = \exp\left(-\frac{p_\perp^2}{m\hbar\omega_{ce}^2}\right)L_n\left(\frac{2p_\perp^2}{m\hbar\omega_{ce}^2}\right), \tag{35}$$

where L_n denotes the Laguerre polynomial of order n and $g_{n\pm}(p_z)$ is a function that is normalizable but otherwise arbitrary. The number of particles in each Landau quantized eigenstate $n_{0n,\pm}$ obeys the condition

$$n_{0n\pm} = \int g_{n\pm}(p_z)(-1)^n\phi_n(p_\perp)\,d^3p$$
$$\Rightarrow$$
$$n_{0n\pm} = \frac{(2\pi\hbar)^3}{2}\int g_{n\pm}(p_z)dp_z. \tag{36}$$

Naturally, a general time-independent solution $W^0_{\pm}$ to Eqs. (25) can be written as a sum over the energy eigenstates (34), according to

$$W^0_{\pm} = \sum_n W^0_{n\pm} = \sum_n g_{n,\pm}(p_z)(-1)^n\phi_n(p_\perp). \tag{37}$$

That the factor $\phi_n(p_\perp)$ gives us the proper Wigner function for the Landau quantized eigenstates can be confirmed by an independent check. Since the expression (40) contains no dependence on the azimuthal angle in momentum space, we can write

$$\epsilon'_{\pm} = m\sqrt{1 + p_\perp^2/m^2 - \mu_B^2B_0^2\left(\frac{\partial}{\partial p_\perp} + \frac{1}{p_\perp}\right)\frac{\partial}{\partial p_\perp} \mp \frac{2\mu_B B_0}{m} + \frac{p_z^2}{m^2}}, \tag{38}$$

when $\epsilon'_{\pm}$ acts on $\phi_n(p_\perp)$. Computing $\epsilon'_{\pm}\phi_n(p_\perp)$ by Taylor-expanding the square root to infinite order, using the properties of the Laguerre polynomials, and then converting the sum back to a square root, it is straightforward to verify the relation

$$\epsilon'_{\pm}\phi_n(p_\perp) = m\left[1 + (2n+1\pm 1)\frac{\hbar\omega_{ce}}{m} + \frac{p_z^2}{m^2}\right]^{1/2}\phi_n(p_\perp), \tag{39}$$

where $\omega_{ce} = \frac{|qB_0|}{m}$ is the electron cyclotron frequency, confirming that $\phi_n(p_\perp)$ generates the proper energy eigenvalues for the perpendicular kinetic energy and the spin degrees of freedom.

Next we turn our attention to the case of a background state in thermodynamic equilibrium. Given the energy eigenstates, we only need to apply Fermi-Dirac statistics, in which case the total thermodynamic Wigner function $W^{\rm TB} = \sum_{\pm} W^{\rm TB}_{\pm}$, including both spin states, is given by

$$W^{\mathrm{TB}} = \frac{n_0}{(2\pi\hbar)^3} \sum_{n,\pm} \frac{2(-1)^n \phi_n(p_\perp)}{\exp\left[(E_{n,\pm} - \mu_c)/k_B T\right] + 1}, \tag{40}$$

where n_0 is the total electron number density of the plasma, μ_c is the chemical potential, and T is the temperature.

Naturally, the expressions presented above are of most significance for relativistically strong magnetic fields, when Landau quantization is pronounced. As a consequence, the formula (40) will reduce to more well-known expressions when the limit $\hbar\omega_{ce}/m \ll 1$ is taken. Specifically, (40) will become a relativistically degenerate Fermi-Dirac distribution in case we let $T = 0$ and $\mu_c = E_F \gg \hbar\omega_{ce}$, where E_F is the Fermi energy. Alternatively, for $k_B T \gg E_F$ and $k_B T \gg \hbar\omega_{ce}$, (40) reduces to a Synge-Juttner distribution.

To give a concrete illustration, in Fig. 1 we have made a bar chart for the normalized number density $n_{0n\pm}/n_0$ in the different energy states for a few values of the temperature and magnetic field, under the assumption that the density is low enough for the system to be nondegenerate, i.e., assuming $T \gg T_F$.

As will be demonstrated in the next section, to a large degree the electrons behave as a multispecies system, where each particle species has its own rest mass, as given by Eq. (33) but with $p_z = 0$. This is because the separation between Landau levels is on the order of the rest mass, and all excitation quanta with energies of that order have been neglected. Noting that the effective number density of each "species" (discrete energy state) is given by

$$n_{0n\pm} \equiv \frac{n_0}{(2\pi\hbar)^3} \int d^3p \frac{2(-1)^n \phi_n(p_\perp)}{\exp\left[(E_{n\pm} - \mu_c)/k_B T\right] + 1}, \tag{41}$$

we see that $n_{0n\pm}$ essentially will be determined by the Boltzmann factors of (40). However, for the cases where our model Eq. (25) is of most interest, the magnetic field B_0 is strong enough to make relativistic Landau quantization prominent. Thus, in the next section and for the remainder of this paper, we will consider background distributions $W_{0\pm}$ where simplifications based on $\hbar\omega_{ce}/m \ll 1$ do not apply.

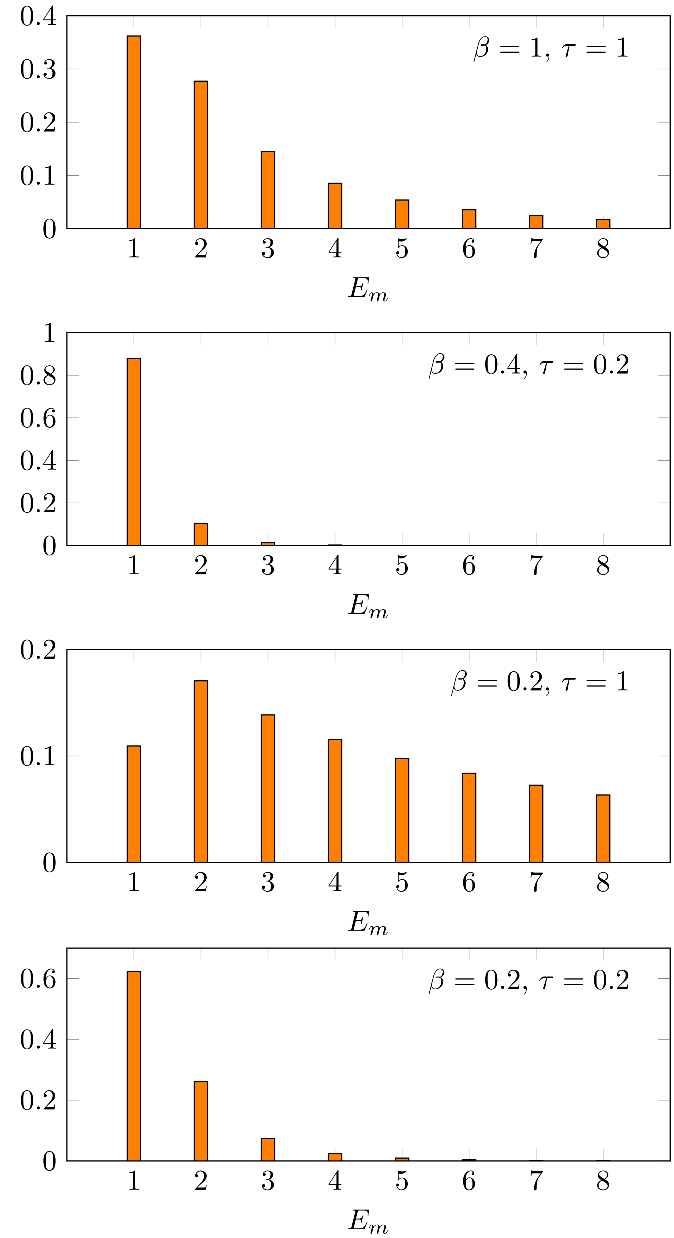

FIG. 1. The normalized number density for different energy states E_m at different values of the parameters $\beta = \mu_B B_0/m$ and $\tau = k_{BT}/m$.

V. LINEAR WAVES

The operator in the square root in Eq. (25) gives the impression that it is very technical and complex to apply the model in studying, e.g., waves in plasma. In this section we consider electrostatic waves in a homogeneous plasma by using Eq. (25). We consider the wave vector $\mathbf{k} = k\mathbf{e}_z$ and express the momentum $\mathbf{p}$ in cylindrical coordinates $p_\perp$, φ_p, and p_z. To linearize Eq. (25), we separate variables according to $W_\pm = W_\pm^0(p_\perp, p_z) + W_\pm^1(z, p_\perp, p_z, t)$, $\mathbf{E} = E_1\mathbf{e}_z$, and $\mathbf{B} = B_0\mathbf{e}_z$, where the subscripts 0 and 1 denote unperturbed and perturbed quantities, respectively. Moreover, the perturbed quantities follow the wave plane ansatz according to $W_\pm^1 = \tilde{W}_\pm^1 e^{ikz - i\omega t}$. Equation (25) is now

$$\left(\omega - \frac{kp_z}{\epsilon'_\pm}\right) W_\pm^1 = -iqE_1 \frac{\partial W_\pm^0}{\partial p_z}. \tag{42}$$

The unperturbed Wigner function $W_\pm^0$ is in general given by a static solution of the form (37). Since the operators in $\epsilon'_\pm$ have been shown to be energy eigenvalues when acting on $W_\pm^0$, we act on both sides by $(\omega - kp_z/\epsilon'_\pm)^{-1}$, such that the perturbed distribution function becomes

$$W_\pm^1 = \frac{-iqE}{\omega - kp_z/\epsilon'_\pm} \frac{\partial W_\pm^0}{\partial p_z} = \sum_n \frac{-iqE}{\omega - kp_z/E_{n\pm}} \frac{\partial W_{n\pm}^0}{\partial p_z}, \tag{43}$$

where in the second equality we used that $W_\pm^0$ is a sum of eigenfunctions of $\epsilon'_\pm$, and the summation is over the Landau levels indexed by n. Using Poisson's equation, the dielectric tensor for the electrostatic case is

$$D(k, \omega) = 1 + \frac{q^2}{k} \sum_{n,\pm} \int d^3p \frac{1}{\omega - kp_z/E_{n\pm}} \frac{\partial W_{n\pm}^0}{\partial p_z}. \tag{44}$$

Note that if we set $\hbar$ to zero in $E_{n\pm}$ in the dielectric tensor, then the denominator in the second term of the dielectric tensor is the same as for the relativistic Vlasov equation [40].

The background distribution $W^0_{n\pm}$ will be divided into its eigenstates, which for the case of thermodynamic equilibrium, (40), will depend on the temperature and the magnetic field, see Fig. 1. As a result, the dispersion relation (44) is that of a relativistic multispecies plasma where each species has its own rest mass, $E_{n\pm}$.

VI. SUMMARY AND DISCUSSION

In the present paper we have derived a kinetic model for plasmas immersed in a relativistically strong magnetic field, i.e., with a field strength of the order of the critical field. Based on a Foldy-Woythausen transformation and a Wigner transformation, an evolution equation for the spin-up and spin-down components $W_\pm$ has been found. Besides having two components, the main difference to a classical relativistic model is that the γ factor in that theory is replaced by an operator containing momentum derivatives. Since our theory is formulated in phase space, we stress that such operators are fundamentally different from the operators of Hilbert space, and to the best of our knowledge, this effect has not been seen in any previous quantum kinetic models.

Comparing the Wigner function of our model with previous quantum kinetic models [8–13] and [36–38], we note that many of them focus on other physical effects, including short scale effects (of the order the de Broglie length) and/or spin dynamics. The latter type of processes is filtered out in our case, due to the short timescale for spin precession associated with strong fields. Most other works studying the Wigner function for electrons make the assumption that the electromagnetic field strength is well below the critical field strength, and hence the strong-field effects of our paper are not included. A notable exception consists of papers based on the Dirac-Heisenberg-Wigner (DHW) approach [36–38], where a kinetic theory is formulated based on a direct Wigner transformation of the Dirac equation. This is contrasted with our approach, where a Foldy-Wouthuysen transformation is made before the Wigner-transformation, in order to separate particle states from antiparticle states. An obvious disadvantage with our approach, compared to the DHW approach, is that some of the physics of the Dirac equation, notably pair creation, is lost. For cases where the separation of particles and antiparticles is applicable, however, the Foldy-Wouthuysen transformation leads to considerable simplifications. This is illustrated by the 16 coupled components of the Wigner function in the DHW approach [37], as compared to the relative simplicity of the present evolution Eq. (25), where the two spin components are decoupled.

As pointed out above, a noted feature of the present theory is that the energy becomes an operator, which has consequences when studying the background Wigner function. Classically, or in less advanced quantum mechanical models, the evolution equation does not predict the detailed momentum dependence for a given Landau level. Thus the background expression has been put in by hand, and one has to return to the starting point of the theory (e.g., the Pauli or Dirac equations for single particles) to find proper expressions. Here, however, the eigenvalue equation $\varepsilon' W_0 = E_{n,\pm} W_0$ determines the background state for a given Landau level (bar a constant for the number density), and there is no need to go outside the kinetic theory itself to find proper initial conditions.

To avoid some technical difficulties related to the operator orderings, we have here divided the magnetic field into an ultrastrong but constant part ($B_0\hat{\mathbf{z}}$) and a fluctuating part $\delta\mathbf{B}$. This approach allows for the treatment of large classes of problems in magnetar atmospheres, for example linear and nonlinear wave propagating in homogeneous backgrounds, even up to relativistic wave amplitudes, as long as the flutuating part fulfills $\mu_B|\delta\mathbf{B}| \ll m$.

Of course, a full modeling of the magnetar surroundings, covering the dipole nature of the background source field, is beyond the scope of such a theory. Moreover, in order to focus on the physics due to ultrastrong magnetic fields, the present theory excludes effects such as the magnetic dipole force, the spin magnetization, and the spin-orbit interaction included in some previous models [10,11], which can be justified for the long scale lengths and moderate frequencies that we focus on here. In this context, however, it should be noted that omission of the spin-orbit interaction is closely related to a correction term of the free current density (see, e.g., Eq. (21) of Ref. [10]). While the additional term kept by Ref. [10] is a small correction for the conditions studied in this paper, it can contribute with currents perpendicular to $\mathbf{B}_0$ that may be of importance for certain problems, specifically for geometries where the (otherwise larger) parallel currents vanish. This and other possible extensions of the current theory is a project for future research.

To illustrate the usefulness of the present theory, we have computed the dispersion relation for Langmuir waves in a strong magnetic field for a relativistic temperature. To a large extent, we find that the electrons behave as if they are divided into different species. More concretely, each Landau level of the background plasma contributes to the susceptibility with a term similar to the classical relativistic expression, but with its own effective mass $m_{n\pm} = m[1 + (2n + 1 \pm 1)\hbar\omega_{ce}/m]^{1/2}$. We expect this result to generalize to some other problems, but not be completely general, as for certain problems, the difference between the standard γ factor and the energy expression of the current theory will be apparent. A more complete study of the effects due to ultrastrong magnetic fields is a project for future research.

APPENDIX: COMMUTATORS

The Hamiltonian in (6) contains some commutators of functions of operators. To calculate these commutators, we need to expand them in a series. We present here some of the calculations that were done in order to obtain the result in (11).

First, both $\hat{T}$ and $\hat{\epsilon}$ are functions of $\hat{\mathcal{O}}$, since these functions can be expanded in a Taylor series of $\hat{\mathcal{O}}$, the commutator

$$[\hat{\mathcal{O}}, \hat{\mathcal{O}}^n] = 0, \tag{A1}$$

thus, we have that

$$[\hat{T}, \hat{\mathcal{O}}] = [\hat{\epsilon}, \hat{\mathcal{O}}] = 0. \tag{A2}$$

We can now rewrite (6) as

$$\hat{H}' = \beta\hat{\epsilon} + \hat{\mathcal{E}} + \frac{1}{2\hat{T}}([\hat{T},[\hat{T},\hat{\mathcal{F}}]] - 2\beta[\hat{\epsilon},\hat{\mathcal{F}}]\hat{\mathcal{O}} + 2\beta[\hat{\mathcal{O}},\hat{\mathcal{F}}]\hat{\epsilon} + 2m\beta[\hat{\mathcal{O}},\hat{\mathcal{F}}] - [\hat{\epsilon},[\hat{\epsilon},\hat{\mathcal{F}}]] - [\hat{\mathcal{O}},[\hat{\mathcal{O}},\hat{\mathcal{F}}]])\frac{1}{\hat{T}}. \quad (A3)$$

Looking first at the commutator of $\hat{\epsilon}$ and $\hat{\mathcal{F}}$ (the same result can be used to the commutator of $\hat{T}$ and $\hat{\mathcal{F}}$), we have

$$[\hat{\epsilon},\hat{\mathcal{F}}] = [\hat{\epsilon}, q\phi(\hat{\mathbf{r}})] + i\hbar\partial_t\hat{\epsilon}. \quad (A4)$$

To calculate the commutator $[\hat{\epsilon},\phi(\hat{\mathbf{r}})]$, we expand the functions in the commutators in a series [33],

$$[\hat{\epsilon},\phi(\hat{\mathbf{r}})] = -\sum_{k=1}^{\infty}\frac{(i\hbar)^k}{k!}\hat{\epsilon}^{(k)}\phi^k(\hat{\mathbf{r}}) \approx -i\hbar\frac{\partial\hat{\epsilon}}{\partial\hat{\pi}_i}\nabla_i\phi(\hat{\mathbf{r}}), \quad (A5)$$

where in the last equality, higher derivative terms were neglected in accordance with the long-scale approximation. Thus, we have

$$[\hat{\epsilon},\hat{\mathcal{F}}] = i\hbar q\frac{\hat{\pi}}{\hat{\epsilon}}\cdot\boldsymbol{E}. \quad (A6)$$

Next, we calculate the commutator

$$[\hat{\epsilon},[\hat{\epsilon},\hat{\mathcal{F}}]] = i\hbar q\frac{1}{\hat{\epsilon}}[\hat{\epsilon},\hat{\pi}\cdot\boldsymbol{E}] = i\hbar q\frac{1}{\hat{\epsilon}}([\hat{\epsilon},\hat{\pi}_j]E_j + \hat{\pi}_j[\hat{\epsilon},E_j]) = i\hbar q\frac{1}{\hat{\epsilon}}[\hat{\epsilon},\hat{\pi}_j]E_j, \quad (A7)$$

where in the last equality, we neglected the commutator of $\hat{\epsilon}$ and E_j in accordance with the long-scale approximation. For the commutator of $\hat{\epsilon}$ and $\hat{\pi}_j$, we need to expand it in a series. However this time, we have a commutator of functions that depend on both $\hat{p}$ and $\hat{r}$. Using the result from Ref. [33], we expand the commutator in a series

$$[\hat{\epsilon},\hat{\pi}_j]E_j = \sum_{k=1}^{\infty}\frac{(i\hbar)^k}{k!}\left(\frac{\partial^k\hat{\pi}_j}{\partial^k r_i}\frac{\partial^k\hat{\epsilon}}{\partial^k p_i} - \frac{\partial^k\hat{\epsilon}}{\partial^k r_i}\frac{\partial^k\hat{\pi}_j}{\partial^k p_i}\right)E_j \approx i\hbar q\frac{\hat{\boldsymbol{\pi}}}{\hat{\epsilon}}\cdot(\mathbf{E}\times\mathbf{B}), \quad (A8)$$

where in the last equality we only kept up to $k = 1$ in the summation since higher-order terms vanish in the long-scale approximation. Finally, we have

$$[\hat{\epsilon},[\hat{\epsilon},\hat{\mathcal{F}}]] = (i\hbar q)^2\frac{\hat{\boldsymbol{\pi}}}{\hat{\epsilon}^2}\cdot(\mathbf{E}\times\mathbf{B}). \quad (A9)$$

Note that $[\hat{T},[\hat{T},\hat{\mathcal{F}}]]$ is calculated in the same way,

$$[\hat{T},[\hat{T},\hat{\mathcal{F}}]] = (i\hbar q)^2\left[\frac{1}{\hat{T}}\left(2+\frac{m}{\hat{\epsilon}}\right)\right]^2\hat{\boldsymbol{\pi}}\cdot(\mathbf{E}\times\mathbf{B}). \quad (A10)$$

Now we will calculate the commutator of $\hat{\mathcal{O}}$ and $\hat{\mathcal{F}}$

$$[\hat{\mathcal{O}},\hat{\mathcal{F}}] = i\hbar q\alpha\cdot\mathbf{E}. \quad (A11)$$

We did not need to do any approximation in calculating this commutator since it is linear in $\hat{\mathcal{O}}$. Finally, we calculate

$$[\hat{\mathcal{O}},[\hat{\mathcal{O}},\hat{\mathcal{F}}]] = i\hbar q(\alpha_i\hat{\pi}_i\alpha_jE_j - \alpha_jE_j\alpha_i\hat{\pi}_i) = -\hbar q\hat{\boldsymbol{\Sigma}}\cdot(\hat{\boldsymbol{\pi}}\times\mathbf{E} - \mathbf{E}\times\hat{\boldsymbol{\pi}}), \quad (A12)$$

where in the last equality we have used

$$\alpha_i\alpha_j = \delta_{ij} + i\varepsilon_{ijk}\Sigma_k.$$

Using (A6) and (A9) to (A12) in (A3), we get the Hamiltonian $\hat{H}'$ in (11).

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