Statistics of residence time for Lévy flights in unstable parabolic potentials

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We analyze the residence time problem for an arbitrary Markovian process describing nonlinear systems without a steady state. We obtain exact analytical results for the statistical characteristics of the residence time. For diffusion in a fully unstable potential profile in the presence of Lévy noise we get the conditional probability density of the particle position and the average residence time. The noise-enhanced stability phenomenon is observed in the system investigated. Results from numerical simulations are in very good agreement with analytical ones.

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I. INTRODUCTION

In the last decade, nonlinear relaxation processes in out-of-equilibrium condensed matter systems and complex systems subject to Gaussian and Lévy noise sources have been largely investigated and of increasing interest [1–6]. In particular, the problem of the statistics of residence times has obtained more attention recently [7–10]. Moreover, the investigation of the stochastic dynamics in highly unstable systems has received great attention from both the experimental [11–18] and the theoretical [14,19–21] points of view.

The investigation of these highly unstable systems, using as minimal models one-dimensional systems, represents a useful first step to understand diverse phenomena such as anomalous transport in complex systems. Free Lévy flights, being one manifestation of anomalous diffusion, represent a special class of discontinuous Markovian processes with infinite mean squared displacement. They can be described within the Langevin framework on the basis of Lévy noises with α-stable distributions. The probability density of Lévy flights evolves according to the fractional Smoluchowski-Fokker-Planck [22–26]. Currently, there is a small amount of rigorous analytical results for statistical characteristics of confined Lévy flights in different potentials, mostly for the probability density functions [27–31] and spectral-correlation characteristics [32–35] at steady state. Moreover, the transient dynamics of this type of anomalous diffusion is insufficiently studied from the analytical point of view. Indeed, in this area of investigation, many papers are concerned with numerical results [36–42]. They all relate mostly to the barrier crossing problem for Lévy flights and indicate that the mean time of transition is inversely proportional to the noise intensity parameter. The main tools for investigation of the barrier crossing problem for Lévy flights are the first passage times, crossing times, arrival times, and residence times [35,43–46]. Investigation of the residence time and nonlinear relaxation time in unstable and metastable potential profiles has been done mainly for Brownian diffusion and interesting noise-induced phenomena have been found such as stochastic resonance, resonant activation, and noise-enhanced stability (NES) [47–59].

In the present paper, first we analyze the residence time statistics for an arbitrary Markovian process describing nonlinear systems without a steady state. We obtain exact results for the statistical characteristics of the residence time. In particular, a closed integral equation for the probability density function of the residence time is derived. Next we consider the anomalous diffusion of a particle in the form of Lévy flights, with an arbitrary Lévy index, in a fully unstable inverse parabolic potential. We obtain the conditional probability density of the particle position and the average residence time in a symmetric interval. This problem was investigated in Refs. [60–67] in the context of Brownian motion.

In particular, here we obtain that the probability of finding the particle in a finite region decreases exponentially and all the moments of the residence time are finite. Moreover,

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the behavior of the mean residence time versus the scale parameter σ and the Lévy index α is discussed in detail. We observe the NES phenomenon [54–59] in the unstable system investigated.

The paper is organized as follows. In the next section (Sec. II), analytical results for the residence time statistics in the case of an arbitrary time-homogeneous Markovian process are derived. In Sec. III, the average residence time of a particle in an unstable parabolic potential with Lévy noise is obtained. The conclusions are drawn in Sec. IV.

II. STATISTICS OF RESIDENCE TIME FOR AN ARBITRARY STATIONARY MARKOVIAN PROCESS

We consider an arbitrary time-homogeneous Markovian process $x(t)$ with the conditional probability density $P(x, t|x_0, t_0) = P(x, t - t_0|x_0, 0)$. According to the definition, if the stochastic process $x(t)$ initially starts from the value x_0 at $t = 0$, the residence time $T(x_0)$ in the given domain G for an infinite observation time reads [35]

$$T(x_0) = \int_0^\infty \mathbb{1}_G(x(t))dt, \tag{1}$$

where $\mathbb{1}_G(y)$ is the indicator of the domain G, defined as

$$\mathbb{1}_G(y) = \begin{cases} 1, & y \in G, \\ 0 & \text{otherwise}. \end{cases} \tag{2}$$

Averaging Eq. (1), we find the mean residence time in domain G,

$$\begin{aligned}\langle T(x_0)\rangle &= \int_0^\infty \mathrm{Pr}(t, x_0)dt \\ &= \int_0^\infty dt \int_G P(x, t|x_0, 0)dx,\end{aligned} \tag{3}$$

where $Pr(t, x_0)$ is the probability of finding the particle in domain G at time t. We suppose that all the integrals in time in Eq. (3) are convergent. This means that the probability $\mathrm{Pr}(t, x_0)$ and the conditional probability density $P(x, t|x_0, 0)$ tend to 0 rapidly enough when $t \to \infty$, and, as a result, the Markovian process $x(t)$ describes a nonlinear system without a steady state. Changing the order of integration, we can write Eq. (3) in the form

$$\langle T(x_0)\rangle = \int_G Y(x, x_0)dx, \tag{4}$$

where

$$Y(x, x_0) = \int_0^\infty P(x, t|x_0, 0)dt. \tag{5}$$

From Eq. (1) the second moment of the residence time can be calculated as

$$\begin{aligned}\langle T^2(x_0)\rangle &= \int_0^\infty dt \int_0^\infty d\tau \langle \mathbb{1}_G(x(t))\mathbb{1}_G(x(\tau))\rangle \\ &= \int_0^\infty dt \int_0^\infty d\tau \int_{G\times G} P(x, t; y, \tau|x_0, 0)dx\, dy.\end{aligned} \tag{6}$$

Using the Markovian property of the stochastic process $x(t)$, $P(x, t_2; y, t_1|x_0, t_0) = P(x, t_2 - t_1|y, 0)P(y, t_1 - t_0|x_0, 0)$, where $t_0 < t_1 < t_2$, we have

$$\begin{aligned}\langle T^2(x_0)\rangle = &\int_0^\infty dt \int_t^\infty d\tau \int_G dx \int_G dy\, P(x, t|x_0, 0) \\ &\times P(y, \tau - t|x, 0) + \int_0^\infty d\tau \int_\tau^\infty dt \int_G dx \\ &\times \int_G dy\, P(y, \tau|x_0, 0)P(x, t - \tau|y, 0).\end{aligned} \tag{7}$$

Changing variables under the integrals and taking into account Eqs. (4) and (5), we finally find

$$\begin{aligned}\langle T^2(x_0)\rangle &= 2\int_G Y(x, x_0)dx \int_G Y(y, x)dy \\ &= 2\int_G Y(x, x_0)\langle T(x)\rangle dx.\end{aligned} \tag{8}$$

The variance of the residence time can be calculated as

$$\mathrm{Var}(x_0) = \langle T^2(x_0)\rangle - \langle T(x_0)\rangle^2. \tag{9}$$

Similarly, the n-th moment of the residence time, (1), can be written as

$$\langle T^n(x_0)\rangle = n\int_G Y(x, x_0)\langle T^{n-1}(x)\rangle dx. \tag{10}$$

Now we derive a closed equation for the characteristic function of the residence time. According to its definition we have

$$\phi(k, x_0) = \langle e^{ikT(x_0)}\rangle = 1 + \sum_{n=1}^\infty \frac{(ik)^n}{n!}\langle T^n(x_0)\rangle. \tag{11}$$

Substitution of Eq. (10) in Eq. (11) gives

$$\begin{aligned}\phi(k, x_0) &= 1 + \sum_{n=1}^\infty \frac{(ik)^n}{(n-1)!}\int_G Y(x, x_0)\langle T^{n-1}(x)\rangle dx \\ &= 1 + ik\int_G Y(x, x_0)\sum_{m=0}^\infty \frac{(ik)^m}{m!}\langle T^m(x)\rangle dx.\end{aligned} \tag{12}$$

Thus, from Eqs. (11) and (12) we arrive at the following integral equation for the characteristic function of the residence time:

$$\phi(k, x_0) = 1 + ik\int_G Y(x, x_0)\phi(k, x)dx. \tag{13}$$

After Fourier transforming we get the integro-differential equation for the probability density function $W_{x_0}(t) = (1/2\pi)\int_{-\infty}^\infty \phi(k, x_0)e^{-ikt}dk$ of the residence time [see Eq. (1)]:

$$W_{x_0}(t) = \delta(t) - \frac{d}{dt}\int_G Y(x, x_0)W_x(t)dx. \tag{14}$$

Equations (13) and (14) describe the full statistics of the residence time and are valid for an arbitrary time-homogeneous Markovian process $x(t)$, which does not have a steady-state probability distribution and whose residence time, for an infinity observation time, is finite. Of course, to solve Eqs. (13) and (14), both analytically and numerically, we need to know the conditional probability density $P(x, t|x_0, 0)$.

Let us explain our statements through a simple example of symmetric Lévy process $L(t)$ with an arbitrary index α (with $0 < \alpha \leqslant 2$), whose characteristic function is given by

$$\vartheta(k,t) = \langle e^{ikL(t)} \rangle = \exp\{ikx_0 - \sigma^\alpha |k|^\alpha t\}, \tag{15}$$

corresponding to the conditional probability density $P(x,t|x_0,0)$, where σ^α is the noise intensity parameter [30,41]. This Markovian random process does not have a steady-state distribution and transforms into the Wiener process for $\alpha = 2$. Applying the inverse Fourier transform to Eq. (15) and substituting the result in Eq. (5), we find the function $Y(x,x_0)$ included in the integral equation, (13):

$$Y(x,x_0) = \frac{1}{\pi\sigma^\alpha}\int_0^\infty \frac{\cos k(x-x_0)}{k^\alpha}dk. \tag{16}$$

The integral in Eq. (16) diverges for the Lévy index $\alpha \geqslant 1$, which means that for these cases, including Brownian diffusion ($\alpha = 2$), the residence time given by Eq. (1) is infinite. At the same time, for $\alpha < 1$ we obtain from Eq. (16) a finite expression,

$$Y(x,x_0) = \frac{|x-x_0|^{\alpha-1}}{2\sigma^\alpha \Gamma(\alpha)\cos(\pi\alpha/2)}, \tag{17}$$

where $\Gamma(\alpha)$ is the Euler gamma function. In particular, from Eqs. (4) and (17) for domain G in the form of the interval $(-L,L)$ we calculate the mean residence time (compare with formula (31) in Ref. [35]):

$$\langle T(x_0)\rangle = \frac{(L-x_0)^\alpha + (L+x_0)^\alpha}{2\Gamma(\alpha+1)\sigma^\alpha\cos(\pi\alpha/2)}. \tag{18}$$

Substituting Eq. (17) into Eq. (13), we can solve it and then, in principle, we can find the probability density function of the residence time. Thus, the residence time of Eq. (1), does not exist for all Markovian processes without a steady-state distribution.

It should be emphasized that the problem of finding the asymptotic probability distribution, when $t \to \infty$, of the occupation time of Markov processes with stationary transitions was considered in Ref. [68]. The principal result of Ref. [68] is the proof that under suitable, but quite general, conditions the limiting distribution must be the Mittag-Leffler distribution.

III. MEAN RESIDENCE TIME OF A PARTICLE IN AN UNSTABLE PARABOLIC POTENTIAL WITH LÉVY NOISE

In this section we investigate the statistical characteristics of the residence time for Lévy flights in the unstable parabolic potential $U(x) = -bx^2/2$ ($b > 0$) (see Fig. 1). An overdamped anomalous diffusion in the form of Lévy flights in a potential profile $U(x)$ can be described by the Langevin equation for the particle coordinate $x(t)$

$$\frac{dx}{dt} = -U'(x) + \xi_\alpha(t), \tag{19}$$

where $\xi_\alpha(t)$ is the symmetric α-stable white noise, characterized by only two parameters, namely, the Lévy index α (with $0 < \alpha \leqslant 2$) and the scale parameter σ [41,69–72]. The limiting case of $\alpha = 2$ corresponds to the Gaussian white noise source in Eq. (19). The stochastic process $\xi_\alpha(t)$ is the

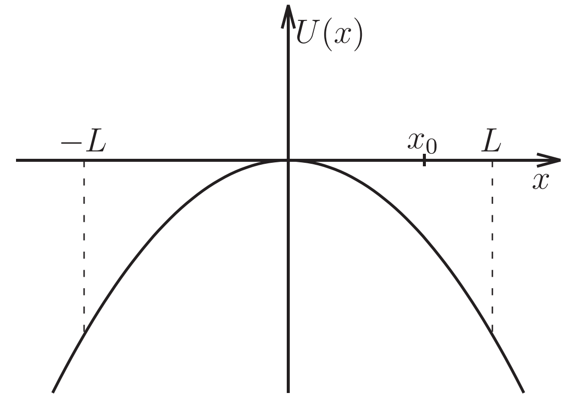

FIG. 1. The unstable parabolic potential.

time derivative of the Lévy process $L_\alpha(t)$: $\xi_\alpha(t) = \dot{L}_\alpha(t)$, with the characteristic function of increments

$$\langle e^{ik[L_\alpha(t)-L_\alpha(0)]}\rangle = e^{-\sigma^\alpha|k|^\alpha t}, \tag{20}$$

where σ^α is the intensity parameter of the Lévy noise [30,41].

The fractional Smoluchowski-Fokker-Planck equation for the probability density function of the particle coordinate, corresponding to the Langevin equation, (19), reads

$$\frac{\partial P}{\partial t} = \frac{\partial}{\partial x}[U'(x)P] + \sigma^\alpha \frac{\partial^\alpha P}{\partial |x|^\alpha}. \tag{21}$$

Substituting the parabolic potential in Eq. (19), we arrive at the following linear differential equation:

$$\frac{dx}{dt} = bx + \xi_\alpha(t). \tag{22}$$

The solution of Eq. (22) can be written in the explicit form

$$x(t) = x_0 e^{bt} + \int_0^t e^{b(t-\tau)}\xi_\alpha(\tau)d\tau. \tag{23}$$

Using the exact solution of Eq. (23), we can find the characteristic function of the particle position x by the definition

$$\vartheta(k,t) = \langle e^{ikx(t)}\rangle. \tag{24}$$

Substituting $x(t)$ from Eq. (23) in Eq. (24), we get

$$\vartheta(k,t) = e^{ikx_0e^{bt}}\left\langle \exp\left\{ik\int_0^t e^{b(t-\tau)}\xi_\alpha(\tau)d\tau\right\}\right\rangle. \tag{25}$$

To calculate the average in Eq. (25) we use the expression for the characteristic functional of the symmetric α-stable white noise (see formula (4) in Ref. [30] and the general result of formula (8), in Ref. [24])

$$\left\langle \exp\left\{i\int_0^t u(\tau)\,\xi_\alpha(\tau)d\tau\right\}\right\rangle = \exp\left\{-\int_0^t |\sigma u(\tau)|^\alpha d\tau\right\}. \tag{26}$$

Replacing $u(\tau)$ with $ke^{b(t-\tau)}$ in Eq. (26) and substituting in Eq. (25), we finally arrive at

$$\vartheta(k,t) = \exp\left\{ikx_0\,e^{bt} - \frac{\sigma^\alpha|k|^\alpha}{\alpha b}(e^{\alpha bt}-1)\right\}. \tag{27}$$

Further, we analyze the residence time of a particle in the symmetric interval $(-L,L)$, for $x_0 \in (-L,L)$ (see Fig. 1). Applying the inverse Fourier transform to Eq. (27), we find the conditional probability density $P(x,t|x_0,0)$ and then the probability of finding a particle in the interval $(-L,L)$:

$$\Pr(t,x_0)=\int_{-L}^{L}P(x,t|x_0,0)dx =\frac{2}{\pi}\int_0^\infty\frac{\sin kL}{k}\cos(kx_0\,e^{bt}) \times\exp\left\{-\frac{(\sigma k)^\alpha(e^{\alpha bt}-1)}{\alpha b}\right\}dk. \quad (28)$$

In the limit of $t\to\infty$ the main contribution to the integral of Eq. (28), comes from the region close to 0 in k so that, using the approximation $\sin kL\simeq kL$ and $\exp(\alpha bt)\gg 1$, we obtain

$$\Pr(t,x_0)\simeq\frac{2L}{\pi}\int_0^\infty\cos\left(kx_0\,e^{bt}\right)\exp\left\{-\frac{(\sigma k\,e^{bt})^\alpha}{\alpha b}\right\}dk$$

or, after setting $q=ke^{bt}$,

$$\Pr(t,x_0)\simeq\frac{2L}{\pi}e^{-bt}\int_0^\infty\cos(qx_0)\exp\left\{-\frac{(\sigma q)^\alpha}{\alpha b}\right\}dq. \quad (29)$$

Thus, the integral in Eq. (3) converges. Moreover, all the moments of the residence time of Eq. (1) are finite.

Substituting Eq. (28) in Eq. (3), after some rearrangements we obtain the exact formula

$$\langle T(x_0)\rangle=\frac{2}{\pi b}\int_0^\infty\frac{\cos(qx_0)}{q}\exp\left\{-\frac{(\sigma q)^\alpha}{\alpha b}\right\}dq \times\int_0^q\frac{\sin kL}{k}\exp\left\{\frac{(\sigma k)^\alpha}{\alpha b}\right\}dk$$

or

$$\langle T(x_0)\rangle=\frac{2}{\pi b}\int_0^\infty\frac{\sin kL}{k}\exp\left\{\frac{(\sigma k)^\alpha}{\alpha b}\right\}dk \times\int_k^\infty\frac{\cos(qx_0)}{q}\exp\left\{-\frac{(\sigma q)^\alpha}{\alpha b}\right\}dq, \quad (30)$$

which gives the average residence time as a function of the initial conditions, the parameters of the system, and the Lévy noise source. This is, together with Eqs. (13) and (14), the main result of this paper.

Let us check the result, (30), in the absence of the noise source $\xi_\alpha(t)$. Putting $\sigma=0$ in Eq. (30) we arrive at

$$T_{\rm dyn}=\frac{2}{\pi b}\int_0^\infty\frac{\cos(qx_0)}{q}dq\int_0^q\frac{\sin kL}{k}dk =\frac{2}{\pi b}\int_0^\infty\frac{\cos(qx_0){\rm Si}(qL)}{q}dq, \quad (31)$$

where ${\rm Si}(x)$ is the sine integral function [73],

$${\rm Si}(x)=\int_0^x\frac{\sin t}{t}dt\,.$$

Using the auxiliary integral $(\alpha,\beta>0)$

$$\int_0^\infty\frac{\cos(\beta x){\rm Si}(\alpha x)}{x}dx=\begin{cases}(\pi/2)\ln(\alpha/\beta), & \alpha>\beta,\\ 0, & \alpha<\beta\end{cases}$$

and taking into account that $|x_0|<L$, we find the dynamical (deterministic) residence time

$$T_{\rm dyn}=\frac{1}{b}\ln\frac{L}{|x_0|}\,. \quad (32)$$

At the same time, the direct integration of Eq. (22) without noise, but with separable variables,

$$\int_{x_0}^{L}\frac{dx}{x}=\int_0^{T_{\rm dyn}}b\,dt,$$

gives the same result.

Let us show that for Cauchy noise ($\alpha=1$) we can write Eq. (30) in the form of a single integral. Substituting $\alpha=1$ in Eq. (30) and changing the order of integration we arrive at

$$\langle T(x_0)\rangle=\frac{2}{\pi b}\int_0^\infty\frac{\sin kL}{k}e^{\sigma k/b}dk\int_k^\infty\frac{\cos(qx_0)}{q}e^{-\sigma q/b}dq. \quad (33)$$

Differentiating Eq. (33) with respect to the parameter x_0 and calculating the internal integral, we get

$$\frac{d\langle T(x_0)\rangle}{dx_0}=-\frac{2}{\pi}\int_0^\infty\frac{(bx_0\cos kx_0+\sigma\sin kx_0)\sin kL}{k\left(\sigma^2+b^2x_0^2\right)}dk. \quad (34)$$

Using the Dirichlet formula

$$\int_0^\infty\frac{\sin\alpha x}{x}dx=\frac{\pi}{2}{\rm sgn}(\alpha)$$

and Frullani formula

$$\int_0^\infty\frac{\cos ax-\cos bx}{x}dx=\ln\left|\frac{b}{a}\right|,$$

where ${\rm sgn}(x)$ is the sign function, we obtain from Eq. (34)

$$\frac{d\langle T(x_0)\rangle}{dx_0}=\frac{\frac{\sigma}{\pi}\ln\left|\frac{L-x_0}{L+x_0}\right|-bx_0 1(L-x_0)}{\sigma^2+b^2x_0^2}, \quad (35)$$

where $1(x)$ is the step function. According to Eq. (30),

$$\lim_{x_0\to\infty}\langle T(x_0)\rangle=0.$$

As a consequence, we find from Eq. (35)

$$\langle T(x_0)\rangle=\int_{x_0}^\infty\left[bz1(L-z)+\frac{\sigma}{\pi}\ln\left|\frac{L+z}{L-z}\right|\right]\frac{dz}{\sigma^2+b^2z^2}$$

or

$$\langle T(x_0)\rangle=\frac{1}{2b}\ln\frac{\sigma^2+b^2L^2}{\sigma^2+b^2x_0^2}+\frac{\sigma}{\pi}\int_{x_0}^\infty\ln\left|\frac{L+z}{L-z}\right|\frac{dz}{\sigma^2+b^2z^2}, \quad (36)$$

where $|x_0|<L$. Of course, in the case where $\sigma=0$, Eq. (36) coincides with Eq. (32).

Using Eqs. (8), (27), and (30) we find the mean squared residence time:

$$\left\langle T^2(x_0)\right\rangle=\frac{4}{\pi^2b^2}\int_0^\infty\exp\left\{\frac{(\sigma k)^\alpha}{\alpha b}\right\}dk \times\int_k^\infty\frac{\cos(qx_0)}{q}\exp\left\{-\frac{(\sigma q)^\alpha}{\alpha b}\right\}dq \times\int_0^\infty\frac{\sin(k_1L)}{k_1}\exp\left\{\frac{(\sigma k_1)^\alpha}{\alpha b}\right\}dk_1 \times\int_{k_1}^\infty\left[\frac{\sin(q_1+k)}{q_1+k}+\frac{\sin(q_1-k)}{q_1-k}\right]e^{-\frac{(\sigma q_1)^\alpha}{\alpha b}}\frac{dq_1}{q_1}. \quad (37)$$

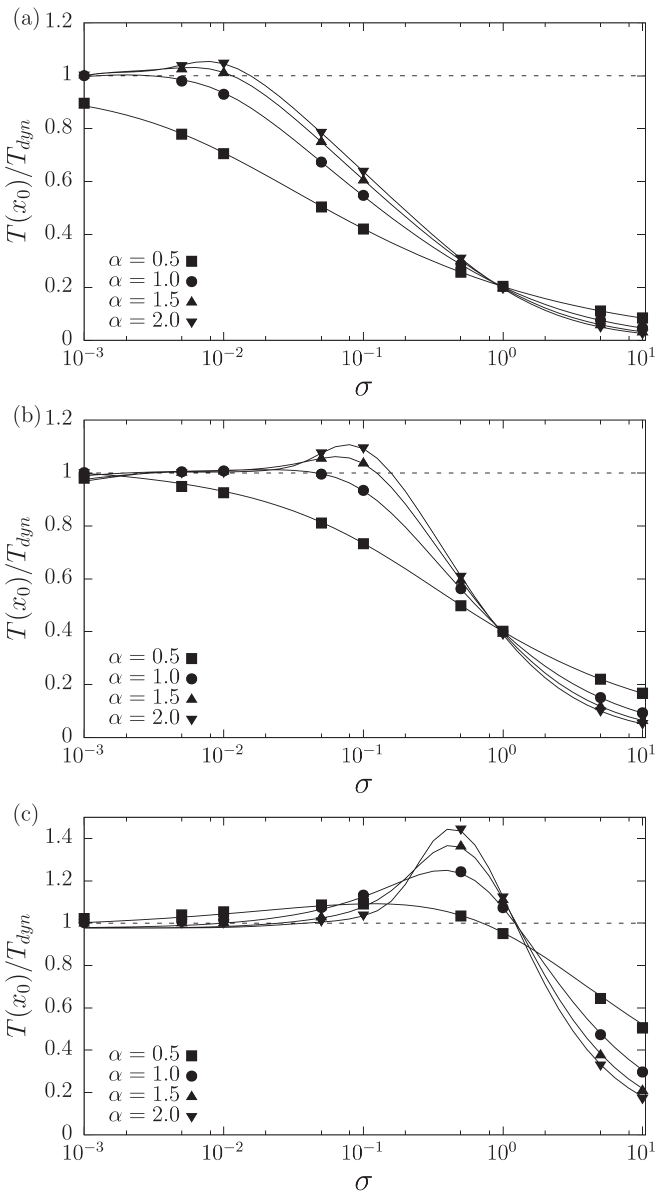

FIG. 2. Normalized mean residence $\langle T(x_0)\rangle/T_{\text{dyn}}$ time as a function of the scale parameter σ for $L = 1$, various initial conditions, namely, $x_0 = 0.01$ (a), $x_0 = 0.1$ (b), and $x_0 = 0.5$ (c), and different values of the Lévy index α, namely, $\alpha = 0.5$, 1.0, 1.5, and 2.0. Symbols are obtained by numerical integration of the Langevin equation, (19); solid lines, by numerical calculation of the analytical formula, (30). Error bars, which are standard deviations of the mean, are within the symbol size.

The plot of the normalized mean residence time $\tau(x_0) = \langle T(x_0)\rangle/T_{\text{dyn}}$ as a function of the scale parameter σ for $L = 1$, under various initial conditions, namely, $x_0 = 0.01$ [Fig. 2(a)], $x_0 = 0.1$ [Fig. 2(b)], and $x_0 = 0.5$ [Fig. 2(c)], and different values of the Lévy index α, namely, $\alpha = 0.5$, 1.0, 1.5, and 2.0, is shown in Fig. 2. Symbols are obtained by numerical integration of the Langevin equation of Eq. (19); solid lines, by numerical calculation of the analytical formula of Eq. (30). The algorithm used in this work to simulate Lévy noise sources is that proposed by Weron [74] for the implementation of the Chambers method [75]. Monte Carlo simulations nicely corroborate analytical results.

In all panels in Fig. 2, it is possible to see a nonmonotonic behavior, with a maximum of the normalized average residence time of the particle in the interval $(-L, L)$ as a function of the scale parameter σ. This is a signature of the noise-enhanced stability phenomenon because the noise increases the average lifetime of the particle in a defined region of the potential profile [54–59] and confirms its first observation in metastable states of short and long Josephson junctions [76–78]. Moreover, the order of magnitude of the scale parameter σ for which we have the maximum is $\sigma \approx (\Delta U)^{(1/\alpha)}$, where ΔU is the height of the potential barrier from x_0 and the maximum of the potential profile. This is in agreement with that we have found for metastable systems and Brownian diffusion (see [4,57,59] and references therein). The NES phenomenon increases as the Lévy index increases and the initial position of the particle approaches the boundary of the interval. Moreover, with an increasing value of x_0, the value of the scale parameter σ for which we have the maximum increases too. This is due to the increase in the height of the potential barrier as the value of x_0 increases. The particle "needs" a larger noise intensity to overcome the potential barrier during its stay in the defined interval, that is, when $x(t) \in (-L, L)$. The decreasing NES effect with decreasing Lévy index is due to the peculiarity of fat tails in the distribution of Lévy noise. In fact, with low values of the α index it is easier for the particle to overcome the barrier of the unstable parabolic potential from one side to the other and to reach the boundaries of the interval $(-L, L)$ more quickly compared to normal Brownian diffusion.

Substituting $\alpha = 2$ in Eq. (27) it is easy to "inverse" it and to find the conditional probability density in the analytical form for Brownian diffusion

$$P(x, t|x_0, 0) = e^{-bt}\sqrt{\frac{b}{2\pi D(1 - e^{-2bt})}} \times \exp\left\{-\frac{b(x_0 - x\, e^{-bt})^2}{2D(1 - e^{-2bt})}\right\}, \quad (38)$$

where $D = \sigma^2$ is the intensity of the Gaussian white noise. According to Eq. (38), this probability distribution has a Gaussian form with the maximum shifted towards one of the sinks $x_{\max} = x_0 e^{bt}$ and exponentially decreases in time. As a result, using Eqs. (4) and (5), we can write Eq. (30) for the mean residence time in another form:

$$\langle T(x_0)\rangle = \frac{1}{\sqrt{2\pi bD}}\int_{-L}^{L} dx \times \int_0^{\pi/2} \exp\left\{-\frac{b(x_0 - x\sin y)^2}{2D\cos^2 y}\right\} dy. \quad (39)$$

In Fig. 2, curves showing the mean residence time, in the case of Brownian diffusion ($\alpha = 2$), are depicted by inverted triangles. For this type of unstable potential they display a similar behavior as for Lévy flights, but the NES effect is more pronounced. This is ascribed to the peculiarity of Lévy flights together with the shape of the potential profile. In fact, with Lévy flight diffusion it is easier for the particle to overcome more quickly the barrier of the unstable parabolic

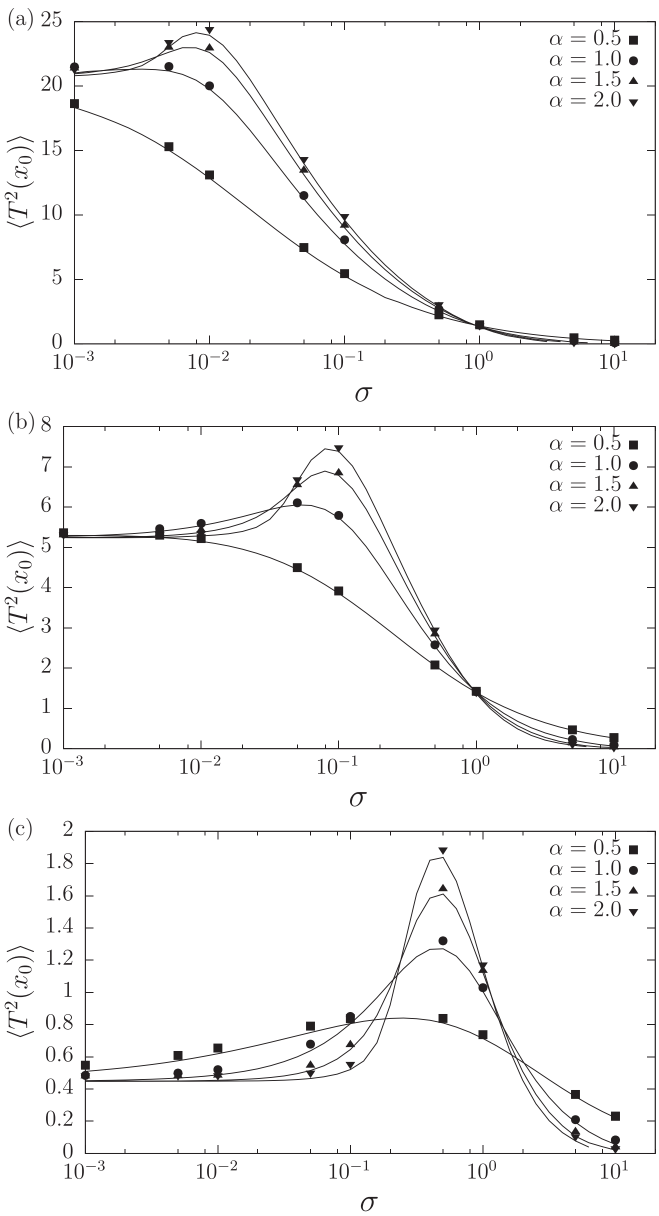

FIG. 3. Mean squared residence time $\langle T^2(x_0)\rangle$ as a function of the scale parameter σ for $L = 1$ under various initial conditions—(a) $x_0 = 0.01$, (b) $x_0 = 0.1$, and (c) $x_0 = 0.5$—and different values of the Lévy index α, namely, $\alpha = 0.5$, 1.0, 1.5, and 2.0. Symbols correspond to the Monte Carlo simulation of the Langevin equation, (19), while lines show the theoretical values given by Eq. (37).

potential and to reach the boundaries of the interval $(-L, L)$ with respect to Brownian diffusion.

Figure 3 shows the dependence of the mean squared residence time versus the scale parameter σ for various initial conditions—$x_0 = 0.01$ [Fig. 3(a)], $x_0 = 0.1$ [Fig. 3(b)], and $x_0 = 0.5$ [Fig. 3(c)]—and different values of the Lévy index α, namely, $\alpha = 0.5$, 1.0, 1.5, and 2.0. Symbols correspond to the Monte Carlo simulation of the Langevin equation, (19), while lines show the theoretical values given by Eq. (37). The agreement between theoretical predictions and computer simulations is very good. Furthermore, the mean squared residence time $\langle T^2(x_0)\rangle$ shows, as a function of the scale parameter σ, the same nonmonotonic behavior as shown by the normalized mean residence time versus σ, with the same peculiarities. Specifically, increasing the value of x_0 also increases the value of σ for which we have the maximum; with a decreasing Lévy index, the NES effect also decreases, and, finally, we have the same order of magnitude of σ for which we have the maximum of $\langle T^2(x_0)\rangle$.

IV. CONCLUSIONS

We study the residence time statistics for an arbitrary time-homogeneous Markov process in nonlinear systems. We derive a closed integral equation for the probability distribution of the residence time. In the particular case of an unstable parabolic potential with a Lévy noise source we obtain the conditional probability density of the particle position and the average residence time in a symmetric interval. We observe the NES phenomenon in the system investigated.

Our theoretical study represents a contribution to the Markov theory of stochastic processes and is the first investigation of the positive role of Lévy noise in the stochastic dynamics of unstable systems. These results stimulate further theoretical and experimental investigations of unstable systems, when a Lévy noise source, including the special case of $\alpha = 2$ (Gaussian thermal noise), is used as a control parameter in nonequilibrium dynamics of classical and quantum systems [20,79,80] or in applications such as Josephson-based noise detection [81–85].

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