Parrondo's paradox in quantum walks with time-dependent coin operators

Marcelo A. Pires [1,*] and Sílvio M. Duarte Queirós [1,2,†]
[1] *Centro Brasileiro de Pesquisas Físicas, Rio de Janeiro, Rio de Janeiro, Brazil*
[2] *National Institute of Science and Technology for Complex Systems, Rio de Janeiro, Rio de Janeiro, Brazil*

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We show that a Parrondo paradox can emerge in two-state quantum walks without resorting to experimentally intricate high-dimensional coins. To achieve such goal we employ a time-dependent coin operator without breaking the translation spatial invariance of the system.

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I. INTRODUCTION

From the so-called Brazil nut effect in granular materials [1]—where the largest particles of a variously sized blend end up on its surface when subjected to (random) shaking—to the boosting of the long-term growth rate of a population by allocating offspring to sink habitats [2], nature has provided us with a myriad of instances which defy common sense and thus are often understood as paradoxical [3]. Within this class of systems yielding counterintuitive results, we can also refer to several thermodynamical approaches that attempted to come up with perpetual machines of both first and second kinds. A canonical instance thereof is the well-known Feynman's ratchet and pawl machine [4] (later scrutinized in Ref. [5]). The concept of ratchet was later employed to Brownian particles in a periodic and asymmetric potential that systematically moves to one of the sides when potential is switched on and off [6]. Such a mechanism was later reinterpreted from a gambling perspective, paving the way to the assertion that the combination of two losing games can yield a winning game when combined. That understanding was later honed to a scenario related to good and bad biased coins which are played more or less frequently when the two games are combined. That recast of a winning combination of losing games case was coined Parrondo's paradox [7–11]. The so-called Parrondian phenomena have lured the information theory community—at first, for their connection with random number generation and game theory—and reached the quantum realm for the development of quantum ratchets, walks, and quantum games that can be translated into a Parrondo framework. As in classical systems, there are different variants of the quantum Parrondo effect [12–19] (for a recent review see Ref. [20]). In Ref. [21], Toral introduced an alternative classical Parrondo walk, the cooperative Parrondo's games, that subsequently gained quantum versions as well [22–24].

In this paper, we aim at implementing an actual Parrondo strategy with quantum walks (QWs) [25] as they are multipurpose models with several possibilities for experimental realizations [26–29] and links with both fundamental [30] and applied [31–34] studies. To the best of our knowledge, the first attempts to set forth a Parrondian QW—considering a capital-dependent rule implemented with a position-dependent potential—were conveyed in Refs. [35–37]. In spite of being successful in the short-run, a long-run analysis shows their Parrondo's paradox can be temporarily suppressed due to periodicity in expected payoffs. Other close attempts at implementing Parrondo's paradox with QWs failed in the asymptotic limit [38,39] as well. Taking a rather different road, it was shown in Ref. [40] the emergence of a Parrondo-like effect consisting of the obtention of an unbiased game from alternating biased games using QWs. However, the issue of a QW-based implementation of a genuine Parrondo game remained pending, irrespective of some proposals [41–46] that demand high-dimensional QWs, which are harder to implement than the qubit-based instances. For example, Ref. [41] used a multicoin approach with history dependence and in Ref. [42,43] the authors opted for a multi register protocol. More recently, a three-state QW was used [44] and in Refs. [45,46] a two-coin QW was used.

That said, we have verified that the implementation of a genuine Parrondo paradox within the scope of two-state QWs with simple alternations between single-parameter coins remains open so far. To solve it, we have resorted to a time-dependent coin operator, which exhibits a very rich phenomenology [47–50] alongside time dependency on the translation operator [51–53]. Hereinafter, we assert that in breaking the temporal constancy of the coin operator in the QW it is possible to successfully implement a quantum Parrondo's paradox.

II. MODEL

Explicitly, our model goes as follows: at a given time $t \in \mathbb{N}$, we consider a QW with a full wave function given by Ψ_t as

$$\Psi_t = \sum_{x\in\mathbb{Z}} \left(\psi_t^U(x)|U\rangle + \psi_t^D(x)|D\rangle\right) \otimes |x\rangle, \qquad (1)$$

where $\{U, D\}$ (standing for up and down, respectively) is the internal degree of freedom of our two-state quantum walker

*piresma@cbpf.br
†sdqueiro@cbpf.br

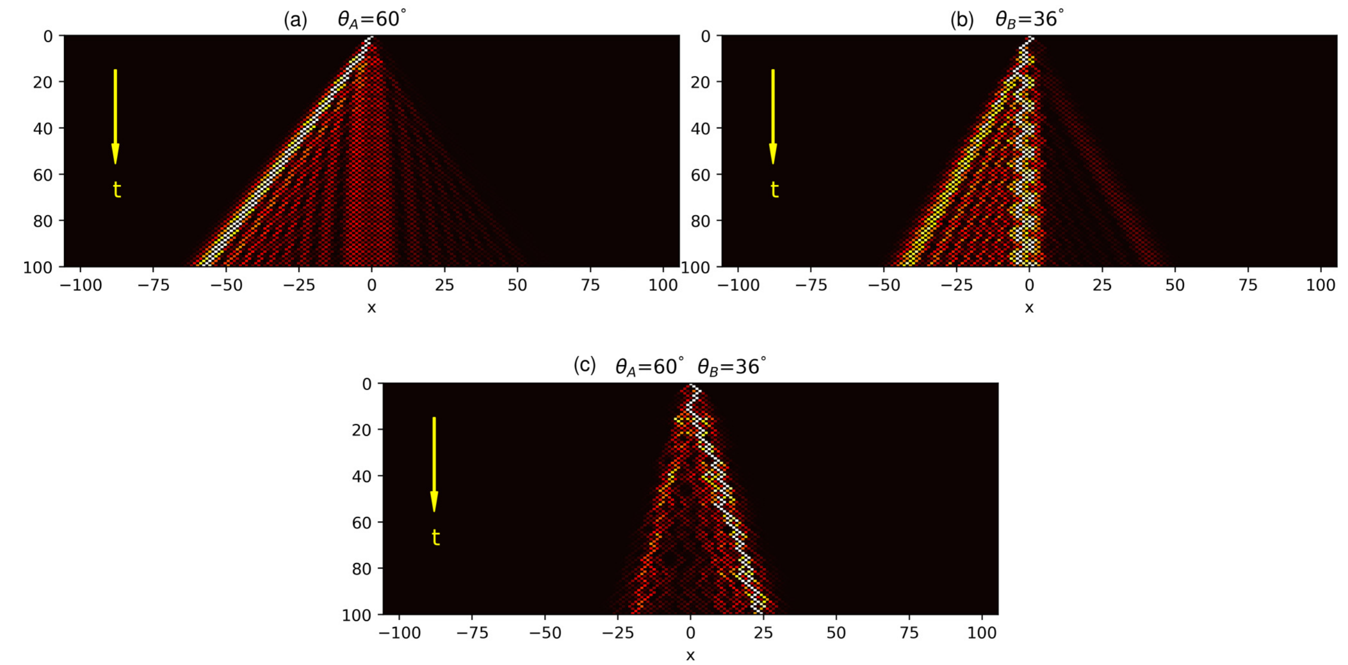

FIG. 1. Spatiotemporal evolution of the normalized $P_t(x)/P_t^{\max}$ for $t = 100$. Genuine Parrondo's paradox with the generalized Hadamard coin with the novel protocol $\theta_t = (t+1)\theta$ for a two-state quantum walk: a combination of losing strategies (left-biased $P_t(x)$ [becomes a winning strategy [right-biased $P_t(x)$]. Different from previous works, this suggests it is not necessary to employ high-dimensional quantum states to implement a Parrondo's game. Top left: $\theta = 60^\circ$. Top right: $\theta = 36^\circ$. Bottom: switching $\theta = 60^\circ$ and $\theta = 36^\circ$.

moving in $x \in \mathbb{Z}$, which corresponds to its external degree of freedom. That is to say, our QW lives in the composite Hilbert space $\mathcal{H}_2 \otimes \mathcal{H}_\mathbb{Z}$. The functions $\psi_t^{U,D}(x)$ are the spatiotemporal amplitudes of probability associated with $\{U, D\}$, respectively. The evolution $t \to t+1$ proceeds with the application of the operator $\hat{W}$ as

$$\Psi_t \xrightarrow{\hat{W}_t} \Psi_{t+1}, \tag{2}$$

$$\hat{W}_t = \hat{T}(\hat{C}_t \otimes \mathcal{I}_\mathbb{Z}), \tag{3}$$

with the identity operator $\mathcal{I}_\mathbb{Z} = \sum_{x\in\mathbb{Z}} |x\rangle\langle x|$ and

(i) the coin operator

$$|x, U\rangle \xrightarrow{\widehat{C}} c_{UU}(t)|x, U\rangle + c_{DU}(t)|x, D\rangle,$$
$$|x, D\rangle \xrightarrow{\widehat{C}} c_{UD}(t)|x, U\rangle + c_{DD}(t)|x, D\rangle, \tag{4}$$

where c_{ij} that are the elements of a rotation matrix that will be described shortly, and

(ii) the state-dependent shift operator

$$|x, U\rangle \xrightarrow{\widehat{T}} |x+1, U\rangle,$$
$$|x, D\rangle \xrightarrow{\widehat{T}} |x-1, D\rangle. \tag{5}$$

For the coin operator, we choose a generalized version of the Hadamard operator,

$$\widehat{C}_H(t) = \cos\theta_t \widehat{\sigma}_z + \sin\theta_t \widehat{\sigma}_x, \tag{6}$$

where $\widehat{\sigma}_z$ and $\widehat{\sigma}_x$ are the standard Pauli matrices. Based on Ref. [54], we choose θ_t as a linear function of time, namely $\theta_t = (t+1)\theta$, the linearity of which has the advantage of being feasible for experiments.

The initial condition is chosen as the localized state:

$$\Psi_0 = \frac{\delta_{x,0}}{\sqrt{2}}(|D\rangle + |U\rangle) \otimes |x\rangle \tag{7}$$

Following the literature related to the Parrondian phenomenon [10,11,20,55] we have assumed the switching rule: for t even we applied θ_A, otherwise we applied θ_B. Concerning x, there are two possible interpretations in this paper that we have used interchangeably. On the one hand, when x was set as capital [35,36], then the QW was interpreted as a game where $x > 0$ means a positive payoff. Taking Refs. [56,57] into account, such interpretation of a capital-dependent game may provide new insights within the context of quantum-like modeling of financial processes [58]. This perspective gives further importance to the development of novel quantum games, as we intend to carry out here. On the other hand, setting x as the position of a particle, we shed light on transport phenomena by which we worked at computing the probability current towards $x < 0$ or $x > 0$.

III. RESULTS AND DISCUSSION

To analyze the outcome of this dynamics we first considered the probability distribution

$$P_t(x) = \left|\psi_t^D(x)\right|^2 + \left|\psi_t^U(x)\right|^2, \tag{8}$$

whence we computed

$$P_t^{\max} = \max_x P_t(x), \tag{9}$$

which is the maximum of $P_t(x)$ over the chain at a time t. In Fig. 1, we show how $P_t(x)/P_t^{\max}$ evolves over time. For

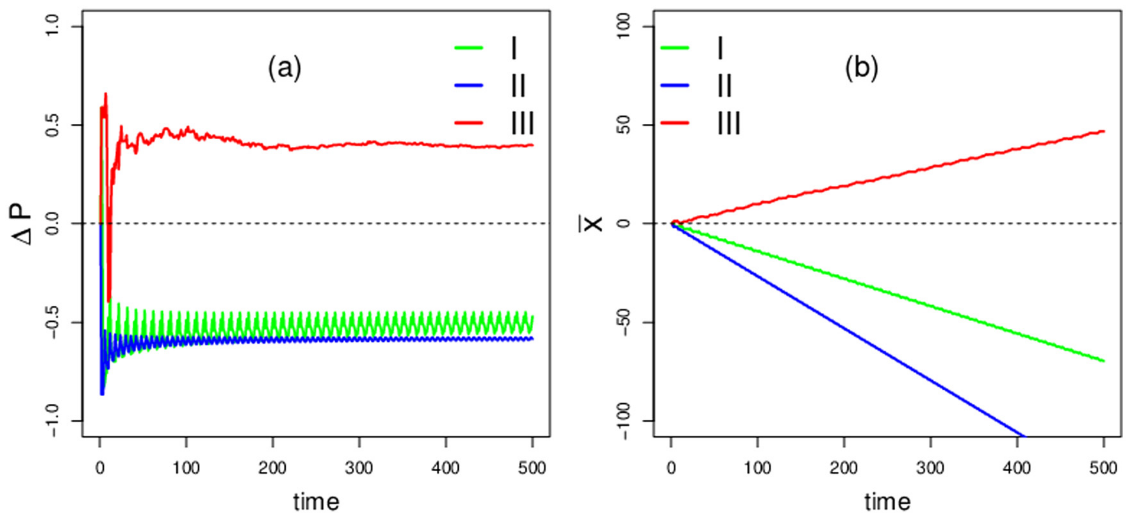

FIG. 2. Time series for the expected payoff $\Delta P = P_L - P_R$ and for the mean position $\bar{x}(t) = m_1(t)$. The net directed current becomes reverted in the opposite direction under the alternating prescription. I: only $\theta = 60^\circ$. II: only $\theta = 36^\circ$. III: alternating $\theta = 60^\circ$ and $\theta = 36^\circ$.

θ_A, we see a left-biased flux of probability. For θ_B, we observe a left-biased current of probability as well. Interestingly, the alternation between θ_A and θ_B leads to a counterintuitive phenomenon corresponding to the very nature of Parrondo's paradox: the combination of two losing games [$P_t(x)$ towards $x < 0$] gives rise to a winning strategy [$P_t(x)$ towards $x > 0$]. In other words, in such quantum carpets, the coupling of two protocols with a left-biased current of probability ends up producing a right-biased current.

In the left panel of Fig. 2, we quantify the current of probability,

$$\Delta P = P_L - P_R = \sum_{x<0} P_t(x) - \sum_{x>0} P_t(x), \tag{10}$$

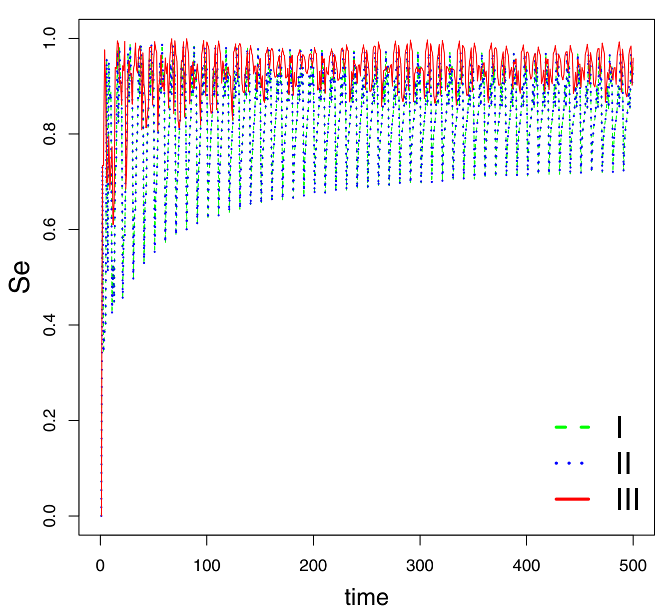

FIG. 3. Time series for the entanglement entropy S_e. I: only $\theta = 60^\circ$. II: only $\theta = 36^\circ$. III: alternating $\theta = 60^\circ$ and $\theta = 36^\circ$.

where in the combined protocol (III) we see a negative peak for $t < 20$. Afterward, the current of the combined game becomes robustly positive.

In the right panel of Fig. 2, we plot the first ($n = 1$)-order statistical moment

$$m_n(t) = \overline{x^n}(t) = \sum_x x^n P_t(x), \tag{11}$$

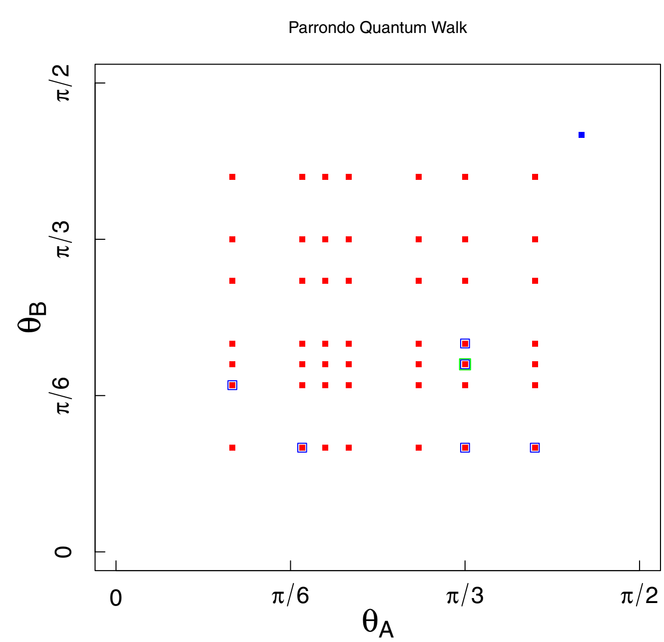

FIG. 4. Diagram for the Parrondo QW. The red squares indicate the left-skewed scenarios with $\Delta P^{\theta_A} < -\epsilon$ and $\Delta P^{\theta_B} < -\epsilon$. The blue squares indicate right-skewed scenarios with $\Delta P^{\text{combined}} > \epsilon$. We set $\epsilon = 1/3$ in order to avoid artifacts. The current reversal takes place in the cases with red and blue squares. The green open square indicates the case corresponding to Figs. 1 and 2. Simulations were performed with $t_{\max} = 1000$. All quantities in this diagram are temporal averages discarding the $t_{\text{transient}} = t_{\max}/2$.

TABLE I. Parameters θ to generate the Parrondo quantum walk.

θ_A	θ_B
20°	36°
36°	20°
60°	20°
60°	36°
60°	40°
72°	20°

from which it is possible to perceive the emergence of the Parrondo effect.

At this point, we still have to answer the question regarding the underlying mechanism yielding the quantum Parrondo effect. First, let us recall the canonical reasoning explaining the aforementioned paradox: blending a game composed of advantageous and disadvantageous ingredients with a losing game inhibits the unfavorable part in the first game. In the standard QW, the central component is quickly suppressed because its probability, $P_t(x)$, concentrates in the borders. Yet, in our protocol, the central component of $P_t(x)$ remains for each isolated game, for instance, with $\theta = 60^\circ$ and $\theta = 36^\circ$. This is similar to the "good" (favorable) and "bad" (unfavorable) components. Therefore, we learn that the alternating protocol ($\theta = 60^\circ$ and $\theta = 36^\circ$) hinders the left-biased borders, which prompts the prevalence of the right component, as shown in Fig. 1.

Following [59,60] we compute the entanglement entropy S_e between the internal and spatial degrees of freedom, Fig. 3,

$$S_e = -\mathrm{Tr}[\rho^c \ln \rho^c], \tag{12}$$

where $\rho^c = \mathrm{Tr}_x(\rho)$ is the reduced density matrix of the particle and ρ is the full density matrix $\rho = |\Psi\rangle\langle\Psi|$ of the QW system. The time series in Fig. 3 highlights the quantum feature of our Parrondo model.

Finally, in Fig. 4 we see a diagram with θ_A versus θ_B for runs until $t_{\max} = 10^3$, where we discard the transient $t < t_{\max}/2$ to compute the temporal average of current of probability ΔP^{single} for the single game (with solely θ_A or θ_B) as well as for $\Delta P^{\text{combined}}$ relative to the combined game (alternating θ_A and θ_B). It is visible that there are specific combinations that lead to the Parrondian effect. From that diagram, we understand that we do not have a smooth Parrondo line, but a set of points wherein a Parrondo strategy emerges. This occurs in other quantum Parrondo-like cases as well and we ascribe it to the complexity of the quantum system we are treating. Contrarily to classical systems, the state of a quantum particle is ruled by the superposition principle, and thus only very specific combinations of the two coins, i.e., θ_A and θ_B allow obtaining an overall winning strategy (See Table I).

IV. CONCLUDING REMARKS

In conclusion, one should acknowledge that Parrondo's paradox contains an important lesson to be learned: one must be careful before labeling a given protocol as useless, because from that it is possible to create a combined protocol—as we have made here—that ends up yielding the features one is aiming at. Apart from that, taking into consideration the assertion in Ref. [45] the need for investigating the class of Parrondian phenomena via QWs is prompted by the research to improve quantum algorithms, namely search algorithms.

The classical Parrondian paradigm can be introduced within the scope of game theory. Similarly the quantum Parrondo's paradox can be introduced in the realm of quantum game theory [61–65]. In this sense, and taking into account that QWs are versatile quantum simulators [66], our new protocol for implementing a Parrondo's game can be employed as a platform to provide further insights intp quantum game theory both theoretically and experimentally.

One advantage of our protocol to achieve directed transport is that we do not require a breaking in the spatial symmetry associated with the coin operator. This approach contrasts with the state of the art, as in Ref. [67] where spatial invariance is broken by introducing a pawl-like effect with position-dependent coin operations. That is, their walk operator is embedded with a local spatial asymmetry. Another advantage of this protocol is the straightforward use of qubits, a feature that makes the model feasible for implementation in photonic architectures [26–29]. More specifically the setups introduced in Ref. [68] are quite appropriate to accommodate this time-dependent coin operator with proper adjustments. From an experimental perspective, our protocol contributes a prospect for laboratory implementation of Parrondo's games in a physical systems beyond the scope of the original model [69], as it fills a gap in that domain by successfully obtaining the actual Parrondo effect with a two-state qubit Parrondian model employing a time-dependent coin. Still, the findings we have reported suit the development of new devices for current reversal without the application of an external gradient. From the point of view of QWs, our work considers the application of time-dependent coin operators.

Note added. Recently, Ref. [70] appeared, in which a Parrondo QW with intricate alternations between three-parameter coins is discussed.

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