

Rate distortion theorem and the multicritical point of a spin glassTatsuto Murayama ^{*}*Graduate School of Science and Engineering, University of Toyama, 3190 Gofuku, Toyama-shi, Toyama 930-8555, Japan*Asaki Saito [†]*Department of Complex and Intelligent Systems, Future University Hakodate, 116-2 Kamedanakano-cho, Hakodate, Hokkaido 041-8655, Japan*Peter Davis [‡]*Telecognix Corporation, 58-13 Yoshida Shimooji-cho, Sakyo-ku, Kyoto-shi, Kyoto 606-8314, Japan*

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A spin system can be thought of as an information coding system that transfers information of the interaction configuration into information of the equilibrium state of the spin variables. Hence it can be expected that the relations between the interaction configuration and equilibrium states are consistent with the known laws of information theory. We show that Shannon's rate distortion theorem can be used to obtain a universal constraint on neighboring spin correlations for a broad range of Ising spin systems with two-body spin interactions. Remarkably, this constraint gives a bound for the multicritical point in the phase diagram, when a mean-field behavior for the neighboring spin pairs can be expected in the paramagnetic phase.

DOI: [10.1103/PhysRevE.102.042122](https://doi.org/10.1103/PhysRevE.102.042122)**I. INTRODUCTION**

Understanding the experimental observations of disordered materials has been a challenge to theoretical physicists. This triggered the rise of a special area of statistical mechanics that deals with a variety of statistical models with frozen disorder, where a series of mathematical techniques has become a common language for the systematic analysis [1,2]. Moreover, these techniques of statistical mechanics have been applied to the study of communication and information systems [3,4], including noisy channel coding [5–7], recursive data compression [8–10], code-division multiple-access (CDMA) multiuser detection [11–13], modern cryptography [14], and some combinatorial optimization problems and methods for them [15–17]. Overall, the physicist's toolbox has successfully been applied to solve issues of information science, but *not vice versa*. To our knowledge, no classical theorem in information theory has been used to analyze the physics of complex condensed matter such as spin glass.

This paper shows that information theory can be effectively applied to the analysis of spin-glass systems. In our scenario, each of the equilibrium states of the Ising spins is regarded as one encoding of the interaction configuration [18,19]. This scenario enables us to apply the Shannon rate distortion theorem of information coding theory [20], which then allows us to develop a method for investigating fundamental restrictions on the phase diagram. As a result, we obtain a general bound

for the location of the multicritical point for Ising spin glasses, where paramagnetic, ferromagnetic, and spin-glass phases merge [21]. Remarkably, our argument is independent of the detailed structure of the lattice. Numerical studies of problems related to the location of the multicritical point for specific lattice models have been carried out by many physicists [22,23]. However, we still have little knowledge about these significant issues from a theoretical point of view [24,25].

This paper is organized as follows. Section II introduces the Hamiltonian of our target models. Section III briefly summarizes the statement of results for the multicritical point, provided that the system exhibits a mean-field behavior in the paramagnetic state. For the reader's convenience, we outline the proof of the statement in Sec. IV. Subsequently, in Sec. V, we invoke Shannon's rate distortion theorem and give a thorough mathematical proof. Finally, Sec. VI concludes the paper.

II. SPIN-GLASS MODEL

In our spin-glass model, we assign a binary spin $S_i = \pm 1$ to each site i and the local energy $-J_{ij}S_iS_j$ to a set of pairwise bonds (i, j) with a binary interaction $J_{ij} = \pm 1$. We investigate a class of Ising spin systems with the Hamiltonian

$$\mathcal{H}\{\mathcal{S}\}\{\mathcal{J}\} = - \sum_{(i,j)} J_{ij} S_i S_j, \quad (1)$$

only assuming that the total number of the sites i and the bonds (i, j) are N and M , respectively. Specifically, we do not restrict the range of the sum (i, j) in Eq. (1). This sum could be over nearest neighbors, or it might include farther pairs, etc. Special features of each lattice will be reflected only

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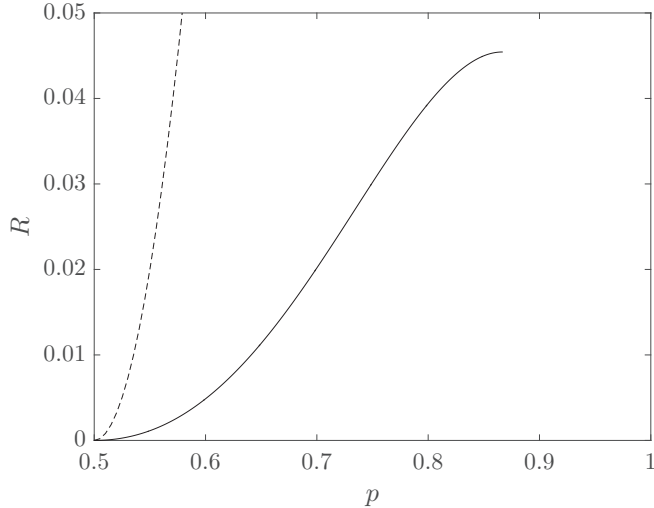


FIG. 1. Theoretical bound for the multicritical point in the phase diagram for disorder parameter p and ratio $R = N/M$. The solid line represents the upper bound p^* for the transition point p_c for a given R . Shannon's rate distortion theorem gives the constraint for R smaller than 0.0454. The dashed line represents the exact value of the transition point p_c for a family of the Bethe lattice spin glass.

through the ratio $R = N/M$. For simplicity, this work deals with a Hamiltonian with two-body interactions to elucidate the benefit of Shannon's rate distortion theorem, although the same arguments apply to other multibody spin systems.

Each J_{ij} is supposed to be distributed independently according to the common distribution

$$P(J_{ij}) = p\delta(1, J_{ij}) + (1 - p)\delta(-1, J_{ij}),$$

for a value of disorder parameter p in the interval $1/2 < p \leq 1$. Here, δ denotes the Kronecker's delta function and the set of interaction coefficients $\mathcal{J} = \{J_{ij}\}$ is called the Bernoulli (p) random variables. In general, we write the inverse temperature as β and then the phase diagram of the system can be depicted in the space of disorder parameter p and temperature $1/\beta$. Now, we consider the Nishimori temperature $1/\beta_p$ for the spin system, defined to be

$$e^{2\beta_p} = \frac{p}{1-p}.$$

Notice that the above equation specifies a line, the Nishimori line, in the space of p and $1/\beta$; see Appendix A. It has been shown that the multicritical point can be always found on this line. And so, we can specify the multicritical point by giving a value for the disorder parameter p , say, p_c . Moreover, since a spin-glass phase does not exist on the Nishimori line, the multicritical point can be characterized as a ferromagnetic transition along the line [26].

III. STATEMENT OF RESULTS

In this paper, we present a general bound for the location of the multicritical point of spin systems on any lattice with a Hamiltonian (1). The solid line in Fig. 1 shows the upper bound p^* of p_c for a given R , only below which we find the multicritical point. Notice that we can use Shannon's rate

distortion theorem to obtain this remarkable constraint when a mean-field behavior can be expected in the paramagnetic phase. More precisely, on the Nishimori line, we assume that

$$P(S_i, S_j | \mathcal{J}) \simeq \exp(\beta_p J_{ij} S_i S_j) \quad (2)$$

holds in the paramagnetic phase, where the $P(S_i, S_j | \mathcal{J})$ denotes the joint distribution of S_i and S_j in the whole complex system and the \simeq means equality up to a normalization constant. This implies that the effect of the rest of the lattice on local marginals should not be dominant and our potential target systems have a certain mean-field property in the paramagnetic state, at least on the Nishimori line. However, we insist that no further physical assumption is required for the Ising spin system. As an example, the dashed line in Fig. 1 represents the exact value p_c of the multicritical point for a family of spin glass on a Bethe lattice [25]. Here, all the bonds (i, j) are chosen randomly to give a diluted lattice with the fixed connectivity of $2/R$. The standard cavity analysis shows that relation (2) holds at any temperature in the paramagnetic phase [27]; see Appendix B. As is expected, we can confirm that p^* upper bounds p_c for this specific model.

IV. OUTLINE OF PROOF

In the remainder of the work we will explain how this general bound can be obtained using the rate distortion theorem. For the reader's convenience, we now outline the proof and then go into specific details afterwards. We first define an average of local correlation functions

$$u = \left[\left\langle \frac{1}{M} \sum_{(i,j)} S_i S_j \right\rangle_{\beta} \right],$$

where we assume that $\langle \cdot \rangle_{\beta}$ represents the expectation value in the equilibrium state of the Hamiltonian (1) at temperature $1/\beta$, and suppose that a bracket $[\cdot]$ indicates averaging over an ensemble of configurations \mathcal{J} . If assumption (2) holds within the paramagnetic phase, we always get $u = (2p - 1)^2$ at $1/\beta_p$ for all $p < p_c$. However, if R is small enough, $u = (2p - 1)^2$ derived from the paramagnetic assumption is smaller than the lower bound $u^*(p)$, which is imposed by Shannon's rate distortion theorem. This implies that such p for a given R indicates the ferromagnetic state, otherwise it is a contradiction. The infimum p^* of such ferromagnetic p , therefore, gives an upper bound for the transition point p_c . Figure 2 illustrates a typical example with ratio $R = 0.03$.

We first show that the local correlation function at $1/\beta_p$ is

$$u = (2p - 1)^2,$$

if the system is in the paramagnetic state. Since relation (2) implies the explicit form

$$P(S_i, S_j | \mathcal{J}) = \frac{1}{4 \cosh \beta_p} \exp(\beta_p J_{ij} S_i S_j),$$

it is an easy matter to check that $\langle S_i S_j \rangle_{\beta_p} = \tanh(\beta_p J_{ij})$ and averaging over $\mathcal{J} = \{J_{ij}\}$ gives $u = [\tanh(\beta_p J_{ij})] = (2p - 1) \tanh \beta_p = (2p - 1)^2$.

Now, we will explain how to obtain the theoretical bound $u^*(p)$ for u based on Shannon's rate distortion theorem. Let us first consider a virtual communication channel where the

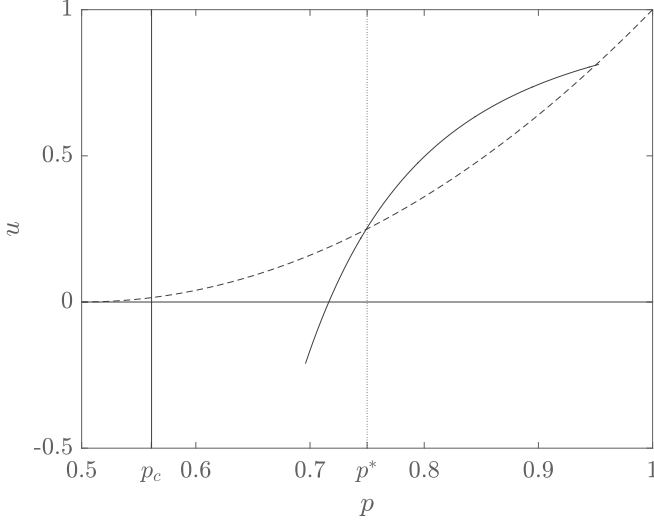


FIG. 2. Theoretical constraints for local spin product u as a function of disorder parameter p for ratio $R = 0.03$. The solid curve represents the universal lower bound of u imposed by Shannon's rate distortion theorem. The dashed parabolic curve indicates the calculation of u based on a mean-field assumption for neighboring spin marginals. The paramagnetic solution contradicts our universal lower bound for p greater than the intersection point $p^* = 0.750$. This means that for $p > p^*$ the paramagnetic solution is no longer the stable solution, implying that $p_c \leq p^*$. The vertical solid line shows $p_c = 0.561$ for a family of the Bethe lattice spin glass.

interaction configuration sequence $\mathcal{J} = \{J_{ij}\}$ is a set of the Bernoulli (p) random variables to be compressed, the set of spins $\mathcal{S} = \{S_i\}$ is its compressed representation or code word, and the spin products $\hat{\mathcal{J}} = \{S_i S_j\}$ are its reproduction at Nishimori temperature $1/\beta_p$. This choice of communication channel is motivated by the fact that at the Nishimori temperature, the Hamming distortion, or the normalized Hamming distance, $D = (1/M) \sum_{(i,j)} [\delta(-1, J_{ij} S_i S_j)]_{\beta_p}$ between the \mathcal{J} and its reproduction $\hat{\mathcal{J}}$ can be easily obtained as $D = 1 - p$ [26]. This specific distortion measure defines the goodness of $\hat{\mathcal{J}}$ as a representation of a set of given Bernoulli (p) random variables \mathcal{J} . The basic problem in Shannon's rate distortion theory can then be stated as follows. *What is the minimum description ratio $R = N/M$ required to achieve a given Hamming distortion D between the two sequences?* Shannon's rate distortion theorem gives the lower bound, say, $R_p(D)$, as a function of the distortion measure D for the theoretically achievable ratio $R = N/M$. The ratio, or rate, $R_p(D)$ is called the rate distortion function for the Bernoulli (p) random variables. However, the distortion $D = 1 - p$ only gives a trivial lower bound $R_p(D) = R_p(1 - p) = 0$ and results in no restrictions for this specific channel [20].

We thus introduce a coding "trick," a set of the Bernoulli (α) random variables $\tilde{\mathcal{J}} = \{\tilde{J}_{ij}\}$ with $1/2 \leq \alpha < p$, which allows us to tighten the bound on $R = N/M$. In the communication channel picture, the manipulation of the Bernoulli (α) sequence $\tilde{\mathcal{J}}$ to get the sequence \mathcal{J} corresponds to a preprocessing step in the encoding operation. After we preprocess $\tilde{\mathcal{J}}$ to get \mathcal{J} , the \mathcal{J} is not Bernoulli (p) assumed in the Nishimori's theory. However, this difference becomes negligible when we

take the large system limit of $N \rightarrow \infty$. As a result, we can use the Nishimori's theory to calculate the Hamming distortion between $\tilde{\mathcal{J}}$ and $\hat{\mathcal{J}}$, which then offers a positive minimum ratio of $R = N/M$. Since distortion D redefined for the new pair depends on p and u , a positive bound on R for the D , if any, imposes a constraint on u as a function of p and R . Hence, we obtain the theoretical lower bound $u^*(p)$ on u for a given ratio $R = N/M$. Notice here that we require no physical assumptions such as (2) in this argument. In Sec. V we explain the essential details of this universal analysis.

V. RATE DISTORTION ANALYSIS

We first introduce a set $\tilde{\mathcal{J}}$ of Bernoulli (α) random variables for some α satisfying $1/2 \leq \alpha < p$. Define the set T_α of all configurations with relative frequency of 1s equal to α . For sufficiently large M , we can consider $\tilde{\mathcal{J}} \in T_\alpha$ and $\mathcal{J} \in T_p$, respectively [28]. So we suppose that any $\tilde{\mathcal{J}}$ configuration can be switched to a \mathcal{J} configuration by flipping $(p - \alpha)M$ elements from -1 to 1 . We consider the set of spin products $\hat{\mathcal{J}}$ as an estimate of the original $\tilde{\mathcal{J}}$.

Here, we evaluate the normalized Hamming distance between the samples $\tilde{\mathcal{J}}$ and $\hat{\mathcal{J}}$, i.e., $(1/M) \sum_{(i,j)} \delta(-1, \tilde{J}_{ij} S_i S_j)$. We first notice that the identity $\tilde{J}_{ij} S_i S_j = J_{ij} \tilde{J}_{ij} \cdot J_{ij} S_i S_j$ leads to

$$\sum_{(i,j)} \delta(-1, \tilde{J}_{ij} S_i S_j) \leq \sum_{(i,j)} \delta(-1, J_{ij} \tilde{J}_{ij}) + \sum_{(i,j)} \delta(-1, J_{ij} S_i S_j).$$

The equality holds if and only if there is no chance of getting $J_{ij} \tilde{J}_{ij} = -1$ and $J_{ij} S_i S_j = -1$ simultaneously. By definition, the preprocessing gives $\sum_{(i,j)} \delta(-1, J_{ij} \tilde{J}_{ij}) = (p - \alpha)M$. The second term on the right-hand side would be

$$\left[\left\langle \sum_{(i,j)} \delta(-1, J_{ij} S_i S_j) \right\rangle_{\beta_p} \right] = (1 - p)M, \quad (3)$$

since the gauge theory tells us that the internal energy becomes $[\langle \mathcal{H}\{\mathcal{S}\}\{\mathcal{J}\} \rangle_{\beta_p}] = -M \tanh \beta_p$ on the Nishimori line; see Appendix C. Assume that the bracket $[\cdot]$ also indicates averaging over an ensemble of configurations $\tilde{\mathcal{J}}$ as well as \mathcal{J} . Then we have

$$\left[\left\langle \sum_{(i,j)} \delta(-1, \tilde{J}_{ij} S_i S_j) \right\rangle_{\beta_p} \right] \leq (1 - \alpha)M. \quad (4)$$

To directly calculate the Hamming distance between the samples $\tilde{\mathcal{J}}$ and $\hat{\mathcal{J}}$ on the Nishimori line, we introduce a pair of auxiliary variables $Q_{1 \rightarrow -1}$ and $Q_{-1 \rightarrow 1}$ defined to be

$$\begin{aligned} Q_{-1 \rightarrow 1} (1 - p)M + Q_{1 \rightarrow -1} pM &= (1 - p)M, \\ (1 - Q_{-1 \rightarrow 1}) (1 - p)M + Q_{1 \rightarrow -1} pM &= (1 - q)M, \end{aligned}$$

where $Q_{x \rightarrow y}$ is the empirical probability of $S_i S_j = y$ when $J_{ij} = x$ and q denotes a frequency of 1s at the random variables $\hat{\mathcal{J}}$. Notice that the former equation just counts up every difference $J_{ij} \neq S_i S_j$, while the latter indicates the total number of $S_i S_j = -1$ in the reconstruction. By solving the two equations, we have

$$Q_{1 \rightarrow -1} = \frac{1 - q}{2p}, \quad Q_{-1 \rightarrow 1} = 1 - \frac{1 - q}{2(1 - p)}.$$

It is easy to check that these formulas are well defined as probabilities in the interval $2p - 1 \leq q \leq 1$ for a given $p \neq 1$. Notice also that $u = 2q - 1$. Then it follows that

$$\left[\left\langle \sum_{(i,j)} \delta(-1, \tilde{J}_{ij} S_i S_j) \right\rangle_{\beta_p} \right] = (1 - \alpha)M - 2Q_{1 \rightarrow -1}(p - \alpha)M \quad (5)$$

(see Appendix D). In other words, the normalized Hamming distance between $\tilde{\mathcal{J}}$ and $\hat{\mathcal{J}}$ on the Nishimori line can be estimated by the formula

$$d_\alpha(p, q) = (1 - \alpha) - 2Q_{1 \rightarrow -1}(p - \alpha),$$

which is non-negative for the relevant intervals.

Lastly, it is possible to invoke Shannon's rate distortion theorem for the Bernoulli (α) random variables [20]. In this communication channel picture with preprocessing, we first write $D = (1/M) \sum_{(i,j)} [\delta(-1, \tilde{J}_{ij} S_i S_j)]_{\beta_p}$ and focus on the Hamming distortion between the original $\tilde{\mathcal{J}}$ and its reproduction $\hat{\mathcal{J}}$. Define the rate distortion function for the Bernoulli (α) random variables as

$$R_\alpha(D) = H_2(\alpha) - H_2(D),$$

where we denote $H_2(\alpha) = -\alpha \log_2(\alpha) - (1 - \alpha) \log_2(1 - \alpha)$. For the ratio $R = N/M$ and the distortion D , the theorem states that

$$R_\alpha(D) < R.$$

This inequality provides a bound on the compression ratio R , dependent only on distortion D . By letting $D = d_\alpha(p, q)$, we can use the formula $R_\alpha(p, q) = H_2(\alpha) - H_2(d_\alpha(p, q))$ to lower bound the ratio as $R_\alpha(p, q) < R$ for every α in the relevant interval $1/2 \leq \alpha < p$. Now write

$$R^*(p, q) = \sup_{1/2 \leq \alpha < p} R_\alpha(p, q).$$

It is obvious that we can still lower bound R as

$$R^*(p, q) \leq R. \quad (6)$$

Notice also that the $R^*(p, q)$ is a nonincreasing continuous function of q . Suppose that the ratio $R = N/M$ is small enough to satisfy an inequality $R < R^*(p, 2p - 1)$. Here, the $R^*(p, 2p - 1)$ is the largest value of $R^*(p, q)$ for q over the interval $2p - 1 \leq q \leq 1$. Since $d_\alpha(p, 1) = 1 - \alpha$, it is an easy matter to check that $R^*(p, 1) = 0$ for every p . Then, by the intermediate value theorem, there exists a number $q^*(p)$ in the closed interval $2p - 1 \leq q \leq 1$ such that

$$R^*(p, q^*(p)) = R. \quad (7)$$

We compare formulas (6) and (7) to conclude that

$$q^*(p) \leq q,$$

i.e., the $q^*(p)$ lower bounds q .

For small enough $R = N/M$, we numerically examine Eq. (7) which implicitly determines $q^*(p)$ for a given pair of p and R . Evaluation of the equation $R^*(p, 2p - 1) = R$ shows that there exists such a solution $q^*(p)$ for some p for every R smaller than 0.0541. Notice that the lower bound $q^*(p)$ for the Bernoulli parameter q gives the lower bound

$u^*(p) = 2q^*(p) - 1$ for local spin product u . Figure 2 compares this universal lower bound $u^*(p)$ with the preceding paramagnetic solution $u = (2p - 1)^2$. However, in this figure, $u = (2p - 1)^2$ violates our lower bound $u^*(p)$ for p larger than the intersection point p^* . Hence, the p larger than p^* implies the ferromagnetic phase, in which the paramagnetic solution could break down. In other words, the multicritical transition point p_c should be smaller than the intersection point p^* . For a given R , this p^* offers an upper bound for p_c as is shown by the solid line in Fig. 1, which is identified with $R^*(p, 2p^2 - 2p + 1) = R$.

VI. CONCLUSION

In this paper, we considered the “ N -bit” spin state of the Ising spin-glass model as compressed representations of a set of M Bernoulli (p) binary random variables encoded in the interaction configuration. We showed that the Shannon rate distortion theorem, which provides a bound on the compression ratio dependent only on distortion, can give an upper bound p^* for the location of the multicritical point p_c for a sufficiently small compression ratio $R = N/M$. Remarkably, our argument is independent of the detailed structure of the lattice and only requires a mean-field assumption for the joint marginals of neighboring spins in the paramagnetic phase. Results obtained here for a certain class of lattice models with two-body Ising spin interactions will motivate applications of Shannon's rate distortion theorem to other Ising spin systems.

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APPENDIX A: NISHIMORI LINE

In this Appendix we illustrate a schematic picture for the location of the multicritical point, Nishimori line, and the three phase boundaries in the space of p and $T = 1/\beta$. As is depicted in Fig. 3 the ferromagnetic phase transition along the Nishimori line coincides with the multicritical point, which enables us to identify the critical value p_c for p in the phase diagram.

APPENDIX B: BETHE LATTICE

In this Appendix we show that the Bethe lattice spin-glass models satisfy relation (2) providing a pedagogical example. We first notice that the Hamiltonian (1) gives

$$\begin{aligned} \mathcal{H}\{\mathcal{S}\}\{\mathcal{J}\} &= - \sum_{(k,l)} J_{kl} S_k S_l \\ &= - \sum_{j \in \partial i} J_{ij} S_i S_j - \sum_{(k,l)|k,l \neq i} J_{kl} S_k S_l, \end{aligned}$$

for any site i on the lattice. The ∂i represents a collection of all sites j connected to i with an interaction J_{ij} . The second term on the right-hand side denotes the Hamiltonian of a

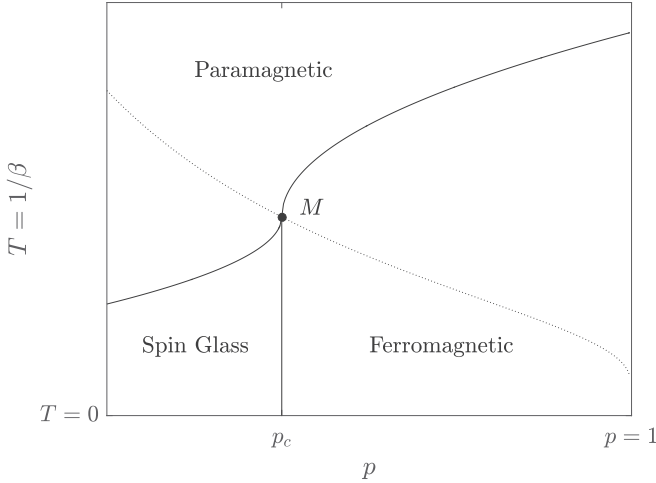


FIG. 3. A schematic illustration of Nishimori line in the Ising spin-glass models. The M denotes the location of the multicritical point on the Nishimori line, where paramagnetic, ferromagnetic, and spin-glass phases merge.

smaller Ising spin system without the spin variable S_i . Then the effective Hamiltonian for the site i and its vicinity is found to be

$$\widehat{\mathcal{H}}(S_i, \{S_j\}_{j \in \partial i}) = - \sum_{j \in \partial i} J_{ij} S_i S_j - \sum_{j \in \partial i} h_{j \rightarrow i} S_j,$$

replacing the effects of the subsystem without S_i with some terms involving the variables $h_{j \rightarrow i}$. These auxiliary variables are called the cavity fields. It follows that

$$\begin{aligned} P(S_i | \mathcal{J}) &= \frac{1}{\mathcal{Z}_1} \sum_{\{S_j\}_{j \in \partial i}} \exp[-\beta \widehat{\mathcal{H}}(S_i, \{S_j\}_{j \in \partial i})] \\ &= \frac{1}{\mathcal{Z}_1} \sum_{\{S_j\}_{j \in \partial i}} \prod_{j \in \partial i} \exp(\beta J_{ij} S_i S_j + \beta h_{j \rightarrow i} S_j) \\ &= \frac{1}{\mathcal{Z}_1} \prod_{j \in \partial i} \sum_{S_j = \pm 1} \exp(\beta J_{ij} S_i S_j + \beta h_{j \rightarrow i} S_j). \end{aligned}$$

Here, \mathcal{Z}_1 denotes the normalization constant. Since $S_j = \pm 1$, the identity

$$\begin{aligned} \exp\{\beta S_j (J_{ij} S_i + h_{j \rightarrow i})\} &= 2 \cosh\{\beta (J_{ij} S_i + h_{j \rightarrow i})\} \\ &\times \frac{1 + S_j \tanh\{\beta (J_{ij} S_i + h_{j \rightarrow i})\}}{2} \end{aligned}$$

gives

$$\begin{aligned} &\frac{1}{\mathcal{Z}_1} \prod_{j \in \partial i} \sum_{S_j = \pm 1} \exp(\beta J_{ij} S_i S_j + \beta h_{j \rightarrow i} S_j) \\ &= \frac{1}{\mathcal{Z}_1} \prod_{j \in \partial i} \{2 \cosh\{\beta (J_{ij} S_i + h_{j \rightarrow i})\}\}. \end{aligned}$$

Now we define the effective field

$$\hat{h}_{j \rightarrow i} = \frac{1}{\beta} \tanh^{-1} \{ \tanh(\beta J_{ij}) \tanh(\beta h_{j \rightarrow i}) \}.$$

Together with another identity

$$\begin{aligned} &\cosh\{\beta (J_{ij} S_i + h_{j \rightarrow i})\} \\ &= 2 \cosh(\beta J_{ij}) \cosh(\beta h_{j \rightarrow i}) \\ &\times \frac{1 + S_i \tanh(\beta J_{ij}) \tanh(\beta h_{j \rightarrow i})}{2}, \end{aligned}$$

we have

$$P(S_i | \mathcal{J}) = \frac{1}{\mathcal{Z}_2} \exp\left(\beta S_i \sum_{j \in \partial i} \hat{h}_{j \rightarrow i}\right),$$

with a new normalization constant \mathcal{Z}_2 . Since the magnetization at site i is

$$m_i = \frac{1}{\mathcal{Z}_2} \sum_{S_i = \pm 1} S_i \exp\left(\beta S_i \sum_{j \in \partial i} \hat{h}_{j \rightarrow i}\right),$$

it is an easy matter to check that $h_{j \rightarrow i} = 0$ for all j gives $m_i = 0$. By letting $h_{j \rightarrow i} = 0$, the summation over the spin variables other than S_i and S_j results in

$$P(S_i, S_j | \mathcal{J}) = \frac{1}{\mathcal{Z}_3} \sum_{\{S_k\}_{k \in \partial i \setminus j}} \exp\left(\beta \sum_{k \in \partial i} J_{ik} S_i S_k\right),$$

where \mathcal{Z}_3 denotes the normalization constant of the joint marginal distribution. Finally, simple algebra gives

$$P(S_i, S_j | \mathcal{J}) = \frac{1}{4 \cosh \beta} \exp(\beta J_{ij} S_i S_j).$$

This indicates that relation (2) holds at any temperature in the paramagnetic phase for the Bethe lattice spin-glass models and, hence, the location of their multicritical points in the phase diagram should be consistent with our results.

APPENDIX C: DERIVATION OF EQ. (3)

In this Appendix we show the details of the calculation for (3) given in Sec. V. First, the gauge theory tells us that the internal energy becomes

$$\frac{-1}{M} [\langle \mathcal{H} \{S\} \{ \mathcal{J} \} \rangle_{\beta_p}] = \tanh \beta_p = 2p - 1$$

at the Nishimori temperature $1/\beta_p$. Notice that the above formula holds for any lattice. Together with the definition of the Hamiltonian (1), i.e.,

$$\begin{aligned} \mathcal{H} \{S\} \{ \mathcal{J} \} &= - \sum_{(i,j)} J_{ij} S_i S_j \\ &= - \sum_{(i,j)} \delta(1, J_{ij} S_i S_j) + \sum_{(i,j)} \delta(-1, J_{ij} S_i S_j), \end{aligned}$$

we get

$$\begin{aligned} &\left[\left\langle \sum_{(i,j)} \delta(1, J_{ij} S_i S_j) \right\rangle_{\beta_p} \right] - \left[\left\langle \sum_{(i,j)} \delta(-1, J_{ij} S_i S_j) \right\rangle_{\beta_p} \right] \\ &= (2p - 1)M. \end{aligned}$$

Notice also that

$$\left[\left\langle \sum_{(i,j)} \delta(1, J_{ij} S_i S_j) \right\rangle_{\beta_p} \right] + \left[\left\langle \sum_{(i,j)} \delta(-1, J_{ij} S_i S_j) \right\rangle_{\beta_p} \right] = M$$

holds. Then the conclusion follows, i.e.,

$$\left[\left\langle \sum_{(i,j)} \delta(-1, J_{ij} S_i S_j) \right\rangle_{\beta_p} \right] = (1-p)M.$$

APPENDIX D: DERIVATION OF EQ. (5)

In this Appendix we show the details of the calculation for (5) given in Sec. V. We put

$$C = \{(i, j) | \tilde{J}_{ij} = -1 \text{ or } 1\},$$

$$D = \{(i, j) | \tilde{J}_{ij} = -1, J_{ij} = 1, S_i S_j = -1\}.$$

Obviously, we have

$$\sum_{(i,j) \in D} \delta(-1, \tilde{J}_{ij} S_i S_j) = 0,$$

$$\sum_{(i,j) \in D} \delta(-1, J_{ij} \tilde{J}_{ij}) = |D|,$$

$$\sum_{(i,j) \in D} \delta(-1, J_{ij} S_i S_j) = |D|,$$

where $|D|$ is the number of the elements in D . As we discussed in the derivation of Eq. (4), we have

$$\begin{aligned} & \sum_{(i,j) \in C \setminus D} \delta(-1, \tilde{J}_{ij} S_i S_j) \\ &= \sum_{(i,j) \in C \setminus D} \delta(-1, J_{ij} \tilde{J}_{ij}) + \sum_{(i,j) \in C \setminus D} \delta(-1, J_{ij} S_i S_j). \end{aligned}$$

Thus, we have

$$\begin{aligned} & \sum_{(i,j) \in C} \delta(-1, \tilde{J}_{ij} S_i S_j) \\ &= \sum_{(i,j) \in C \setminus D} \delta(-1, \tilde{J}_{ij} S_i S_j) + \sum_{(i,j) \in D} \delta(-1, \tilde{J}_{ij} S_i S_j) \\ &= \sum_{(i,j) \in C \setminus D} \delta(-1, J_{ij} \tilde{J}_{ij}) + \sum_{(i,j) \in C \setminus D} \delta(-1, J_{ij} S_i S_j) \\ &= \sum_{(i,j) \in C} \delta(-1, J_{ij} \tilde{J}_{ij}) + \sum_{(i,j) \in C} \delta(-1, J_{ij} S_i S_j) - 2|D|. \end{aligned}$$

Taking the average, we obtain

$$\left[\left\langle \sum_{(i,j)} \delta(-1, \tilde{J}_{ij} S_i S_j) \right\rangle_{\beta_p} \right] = (1-\alpha)M - 2Q_{1 \rightarrow -1}(p-\alpha)M.$$

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