Hitting times in turbulent diffusion due to multiplicative noise

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We study a distribution of times of the first arrivals to absorbing targets in turbulent diffusion, which is due to a multiplicative noise. Two examples of dynamical systems with a multiplicative noise are studied. The first one is a random process according to inhomogeneous diffusion, which is also known as a geometric Brownian motion in the Black-Scholes model. The second model is due to a random processes on a two-dimensional comb, where inhomogeneous advection is possible only along the backbone, while Brownian diffusion takes place inside the branches. It is shown that in both cases turbulent diffusion takes place as the one-dimensional random process with the log-normal distribution in the presence of absorbing targets, which are characterized by the Lévy-Smirnov distribution for the first hitting times.

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I. INTRODUCTION

The quest for an optimal search strategy is central for diverse fields ranging from biology to robotics and from physics to computer science [1–3]. Search strategies have been widely observed, for example, when foraging in an environment with scarce resources [4,5], in a reaction pathway of DNA-binding proteins [6], as well as in intracellular transport [7].

An important characteristic of a search process is a first arrival time distribution. In a random search equation it plays a role of a strength of a δ-sink term, which is considered as an absorbing target. For example, a Brownian random search process is governed by the diffusion equation with the δ-sink of an unknown strength $\wp(t)$,

$$\frac{\partial}{\partial t}f(x,t)=\mathcal{D}\frac{\partial^2}{\partial x^2}f(x,t)-\wp_{\rm fa}(t)\,\delta(x-X), \qquad (1)$$

where $f(x,t)$ is the non-normalized density function, $\mathcal{D}$ is the diffusion coefficient, and the initial position of the searcher is given at $x=x_0$ by $f(x,t=0)=\delta(x-x_0)$ [8]. The equation describes a random Brownian motion of a searcher positioned at $x=x_0$ at the beginning, which will be removed at the first arrival at $x=X$, i.e., $f(x=X,t)=0$. Therefore, $\wp_{\rm fa}(t)$ represents the *first arrival time distribution* (FATD) [8,9], obtained from Eq. (1),

$$\wp_{\rm fa}(t)=-\frac{d}{dt}\int_{-\infty}^{\infty}f(x,t)\,dx=-\frac{d}{dt}\mathcal{S}(t), \qquad (2)$$

where $\mathcal{S}(t)=\int_{-\infty}^{\infty}f(x,t)\,dx$ is the *survival probability*. The search *reliability* is defined by [9]

$$\mathcal{P}=\int_0^{\infty}\wp_{\rm fa}(t)\,dt=\lim_{s\to 0}\wp_{\rm fa}(s), \qquad (3)$$

where $\wp_{\rm fa}(s)=\mathcal{L}[\wp_{\rm fa}(t)](s)=\int_0^t\wp_{\rm fa}(t)\,e^{-st}\,dt$, and the search *efficiency*, which is the average over inverse search times, is defined by [9]

$$\mathcal{E}=\left\langle\frac{1}{t}\right\rangle=\int_0^{\infty}\wp_{\rm fa}(s)\,ds. \qquad (4)$$

In this research we consider the FATD for problems related to a multiplicative noise. We consider two models of the one-dimensional turbulent diffusion related to the log-normal distributions. The first one is inhomogeneous diffusion, which is also known as a geometric Brownian motion in the Black-Scholes model [10]. The latter is well established in a stock price process. The second one is a comb model, which is a continuation of our previous studies [11,12]. This model can be applied in tubular processes like neurons [7,13–15]. A comb model has been suggested as a simplified toy model, which reflects this property of anomalous diffusion resulting from the geometry and where turbulent diffusion can take place [16]. Therefore, our main interest here relates to generic properties of hitting times, which can be also related to search tasks due to the multiplicative noise and correspondingly due to the log-normal distribution, since the latter has a variety of realizations [17] and can be extremely important for future search applications. It is worth mentioning that both examples considered here are specified by exponentially fast spreading. In random processes it is also known as a geometric or

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exponential Brownian motion [18]. We specify these processes as turbulent diffusion, since both phenomena are strongly overlapped (in the field of statistical turbulence). However, the first one is mostly used in mathematical literature for economics and population growth [18], where inhomogeneous advection-diffusion results from multiplicative noise.[1]

The structure of the paper is as follows. In Sec. II we introduce geometric Brownian motion as a search mechanism due to a log-normal distributed random process. Exact results for the FATD, reliability, and efficiency due to this search is presented in Sec. III. In Sec. IV we consider turbulent diffusion in the framework of the comb geometry. We show that this process is also due to the multiplicative noise, which is described by the log-normal distribution. We also calculate the corresponding FATD. Section V is devoted to the fractional transport due to the geometric Brownian motion in the comb geometry. The FATD and its characteristics are considered here as well. The obtained results are discussed in some detail in Sec. VI. Technical details of calculations are presented in the five Appendixes.

II. GEOMETRIC BROWNIAN MOTION AND TURBULENT DIFFUSION

It is well known that at turbulent diffusion a contaminant spreads very fast; for example, in the case of Richardson diffusion, the mean-squared displacement (MSD) behaves as t^3 [19]. Dating back to work by Kolmogorov and Obukhov, it suggests this turbulent acceleration by means of a Gaussian δ-correlated noise η, added to the dynamical system [20], $\ddot{x}+\eta(t)=0$. This spread, however, can be essentially increased due to multiplicative noise [10,16], where the MSD grows exponentially with time.

As an example, a geometric Brownian motion (GBM) is represented by the Langevin equation [10,18]

$$dx(t)=\mu\, x(t)\, dt+\sigma\, x(t)\, dB(t), \quad x_0=x(0), \tag{5}$$

where $x(t)$ is the particle position, μ is the drift, $\sigma>0$ is the volatility, and $B(t)$ is the standard Brownian motion. This stochastic process corresponds to the well-known Black-Scholes model where $x(t)>0$ is an asset price,

$$x(t)=x_0\, e^{\sigma B(t)+\mu t}, \quad x_0=x(0)>0. \tag{6}$$

Depending on the interpretation of Eq. (5) the corresponding Fokker-Planck equation can have different forms. For the Stratonovich interpretation,[2] the corresponding Fokker-Planck equation for the probability density function (PDF) reads

$$\frac{\partial}{\partial t}\bar{P}(x,t)=-\mu\left(\frac{\partial}{\partial x}x\right)\bar{P}(x,t)+\frac{\sigma^2}{2}\left(\frac{\partial}{\partial x}x\right)^2\bar{P}(x,t). \tag{7}$$

In the GBM, the mean value $\langle x(t)\rangle$ and the MSD $\langle x^2(t)\rangle$ have exponential dependence on time, and the solution of the Fokker-Planck equation follows a log-normal distribution [10] (for details see also Ref. [21,23] and see Appendix A):

$$\bar{P}(x,t)=\frac{1}{x\sqrt{2\pi\sigma^2 t}}\times\exp\left\{-\frac{\left(\log\frac{x}{x_0}-\mu t\right)^2}{2\sigma^2 t}\right\}. \tag{8}$$

That is one-dimensional turbulent diffusion, governed by the diffusion-advection equation,[3]

$$\partial_t P(x,t)=\mathcal{D}\,(\partial_x x)^2 P(x,t), \tag{9}$$

and described by the log-normal distribution of Eq. (8) with $\mu=0$ and $\frac{\sigma^2}{2}=\mathcal{D}$,

$$P(x,t)=\frac{1}{x\sqrt{4\pi\mathcal{D}t}}\times\exp\left(-\frac{\log^2\frac{x}{x_0}}{4\mathcal{D}t}\right). \tag{10}$$

Taking modulus $|x|$, one extends this result to $x\in R^-$.

From the solution one can find the moments, $\langle x^n(t)\rangle=\int_0^\infty x^n P(x,t)\,dx$, which are given by

$$\langle x^n(t)\rangle=x_0^n e^{n^2\mathcal{D}t}. \tag{11}$$

From here directly follows the normalization condition $\langle x^0(t)\rangle=x_0^0=1$. The mean value becomes $\langle x(t)\rangle=x_0 e^{\mathcal{D}t}$, while the MSD has the form $\langle x^2(t)\rangle=x_0^2 e^{4\mathcal{D}t}$.

III. FATD IN ONE-DIMENSIONAL TURBULENT DIFFUSION

We consider the FATD in the one-dimensional turbulent diffusion in the framework of a diffusion equation for the non-normalized density function $F=F(x,t)$ with a δ-sink of the strength $\wp_{\rm fa}(t)$,

$$\partial_t F(x,t)=\mathcal{D}\,(\partial_x x)^2 F(x,t)-\wp_{\rm fa}(t)\,\delta(x-X). \tag{12}$$

It also describes heterogeneous diffusion search in inhomogeneous media. We assume here that the initial position is at $x=x_0>0$, taking $F(x,t=0)=\delta(x-x_0)$. The δ-sink means that a random searcher starting at $x=x_0$ will be removed at the first arrival or hitting at $x=X>0$, that means $F(x=X,t)=0$. From solution (10), we obtain $F(x,t)$ as follows:

$$F(x,t)=\frac{e^{-\frac{\log^2\frac{x}{x_0}}{4\mathcal{D}t}}}{x\sqrt{4\pi\mathcal{D}t}}-\int_0^t\wp_{\rm fa}(t')\,\frac{e^{-\frac{\log^2\frac{x}{X}}{4\mathcal{D}(t-t')}}}{x\sqrt{4\pi\mathcal{D}(t-t')}}\,dt'. \tag{13}$$

The sink at $x=X$ represents the target, i.e., the random walker is removed when the target is hit. Therefore, $\wp_{\rm fa}(t)$ is the FATD. After the Laplace transform, the hitting condition $F(x=X,s)=0$ yields the FATD as follows:

$$\wp_{\rm fa}(s)=e^{-\sqrt{\frac{s}{\mathcal{D}}}\left|\log\frac{X}{x_0}\right|}. \tag{14}$$

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[1] In ensuing sections we use "turbulent diffusion," since it relates to a possibility of considering a more sophisticated geometry like a comb model and also opens a window to use "genuine turbulence," like Richardson diffusion, for search processes.

[2] There are also Itô and Klimontovich-Hänggi interpretations of the equation; see Refs. [21,22].

[3] It can be also considered as an Ornstein-Uhlenbeck process with multiplicative noise.

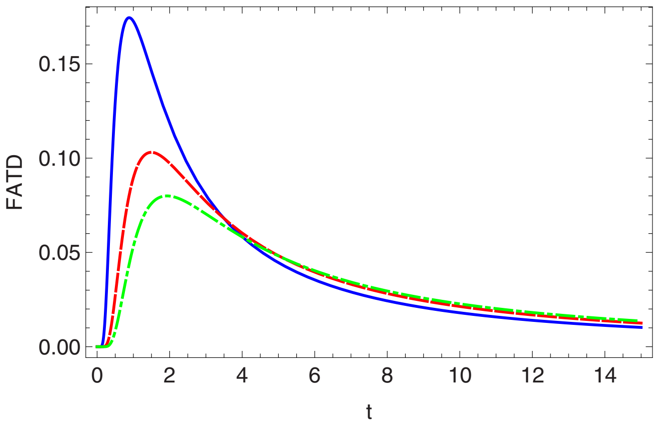

FIG. 1. Graphical representation of the FATD, Eq. (15), for $x_0 = 1$, $\mathcal{D} = 1$, and $X = 10$ (blue solid line), $X = 20$ (red dashed line), and $X = 30$ (green dot-dashed line).

The inverse Laplace transform of Eq. (14) yields

$$\wp_{\rm fa}(t) = \frac{\left|\log\frac{X}{x_0}\right|}{\sqrt{4\pi\mathcal{D}t^3}} \times e^{-\frac{\log^2\frac{X}{x_0}}{4\mathcal{D}t}}, \tag{15}$$

which is the Lévy-Smirnov distribution. By analogy with the log-normal distribution, it is the log-Lévy-Smirnov distribution with respect to $\log\frac{X}{x_0}$. However, with respect to the time, it is the well-known Lévy-Smirnov distribution [24]. Its long-time asymptotic scales as

$$\wp_{\rm fa}(t) \simeq \frac{\left|\log\frac{X}{x_0}\right|}{\sqrt{4\pi\mathcal{D}}} \times t^{-3/2}.$$

Graphical representation of the FATD is given in Fig. 1.

From Eq. (14), the reliability is $\mathcal{P} = \wp_{\rm fa}(s=0) = 1$, and the efficiency becomes

$$\mathcal{E} = \int_0^\infty \wp_{\rm fa}(s)\,ds = \frac{2\mathcal{D}}{\log^2\frac{X}{x_0}}. \tag{16}$$

The result is also valid for entire real axis by taking modulus, $\log^2\frac{|X|}{x_0}$.

IV. TURBULENT DIFFUSION DUE TO COMB GEOMETRY

Another example where turbulent diffusion can be realized due to the multiplicative noise is a comb model [16]. In this case, considering relative diffusion of a pair of particles (see Fig. 2), namely, the distance between them, one finds that the relative diffusivity of two particles grows with the interparticle distance. The mechanism of turbulence due to the comb geometry is important here and plays an essential role when the two nearest particles move into different directions, which contributes to the exponential growth of the interparticle distance.

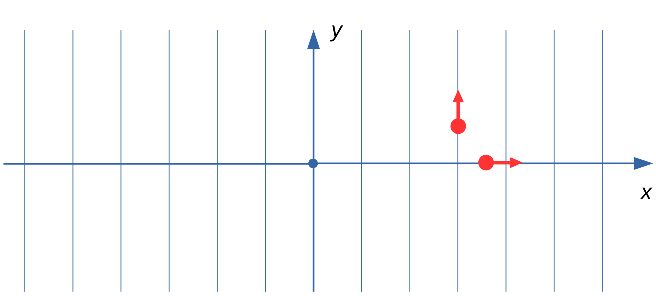

FIG. 2. Comb model with two tracers moving in different directions.

To understand this process, let us consider a Langevin equation in a so-called Matheron–de Marsily form [25]

$$\dot{x}_1 = v(x_2)x_1 \quad \dot{x}_2 = \eta(t), \quad v(x_2) = v\,\delta(x_2). \tag{17}$$

Here $\eta(t)$ is a random Gaussian δ-correlated process (white noise) $\langle\eta(t)\eta(t')\rangle = 2\mathcal{D}\delta(t-t')$, where $\mathcal{D}$ is a diffusion coefficient. The specific form of Eq. (17) corresponds exactly to the comb geometry in Fig. 2, when a random exponential spread $\exp[\int v(B(\tau)\,d\tau]$ is possible along the x axis in complete analogy with Eq. (6), since $x_2 = B(t) = \int^t \eta(\tau)\,d\tau$ is the Brownian process (functional), taking place along the y axis. The specific property of the comb is that the random exponential spread is possible only along the backbone at $y = 0$ in the form of randomly inhomogeneous advection, while normal Brownian diffusion in the y axis is homogeneous. Introducing a distribution function $P(x,y,t) = \langle\delta(x_1 - x)\delta(x_2 - y)\rangle$, we obtain the Fokker-Planck equation in the form of a two-dimensional comb model:

$$\partial_t P(x,y,t) = -v\,\delta(y)\,\partial_x\{x\,P(x,y,t)\} + \mathcal{D}\,\partial_y^2 P(x,y,t). \tag{18}$$

This Fokker-Planck equation conserves the probability flow, $\int P(x,y,t)\,dx\,dy = 1$.

It must be admitted that the diffusion coefficients in Eqs. (9) and (18) with the same nomenclature $\mathcal{D}$ are of a different dimension. Therefore, to avoid any contradictions in the nomenclature, we consider all equations in the dimensionless form when all parameters and variables are dimensionless.[4]

The solution of Eq. (18), for an initial condition $P(x,y,t=0) = \delta(x-x_0)\delta(y)$ and $x > x_0 > 0$, can be obtained by the Laplace transform, which yields (see Appendix B)

$$P(x,y,t) = \left(\frac{2\sqrt{\mathcal{D}}}{v}\log\frac{x}{x_0} + \frac{|y|}{\sqrt{\mathcal{D}}}\right)\Big/\left(x\sqrt{4\pi\,v^2\,t^3}\right) \times \exp\left[-\frac{\left(\frac{2\sqrt{\mathcal{D}}}{v}\log\frac{x}{x_0} + \frac{|y|}{\sqrt{\mathcal{D}}}\right)^2}{4t}\right]\theta(x-x_0), \tag{19}$$

where $\theta(z)$ is the Heaviside function.

Anomalous transport along the backbone can be described by the marginal PDF, which is

$$p_1(x,t) = \int_{-\infty}^{\infty} P(x,y,t)\,dy. \tag{20}$$

[4]In particular, to carry out this dimensionless procedure for the comb model, the time and space scaling parameters can be introduced. Let $\bar{v}$ and $\bar{D}$ be unite velocity and diffusivity correspondingly, and their dimensionality is $[\bar{v}] = LT^{-1}$, $[\bar{D}] = L^2T^{-1}$. That is, $[\bar{t}] = T$, $[\bar{x}] = [\bar{y}] = L$ are dimension variables as well. Then $\bar{D}/\bar{v}^2$ and $\bar{D}/\bar{v}$ are the time and space scaling parameters. This yields the dimensionless procedure as follows: $x\cdot[\bar{v}/\bar{D}] \to x$, $y\cdot[\bar{v}/\bar{D}] \to y$, $t\cdot[\bar{v}^2/\bar{D}] \to t$, $v/\bar{v} \to v$, and $D/\bar{D} \to D$.

From Eq. (19) it reads

$$p_1(x,t) = \frac{2\,\theta(x-x_0)}{x\sqrt{4\,\pi\left(\frac{v}{2\sqrt{\mathcal{D}}}\right)^2 t}} \times \exp\left[-\frac{\log^2\frac{x}{x_0}}{4\left(\frac{v}{2\sqrt{\mathcal{D}}}\right)^2 t}\right], \tag{21}$$

which is a log-normal distribution. It is the solution of the fractional Fokker-Planck equation, which reads

$$\partial_t^{1/2} p_1(x,t) = -\frac{v}{2\sqrt{\mathcal{D}}}\,\partial_x\{x\,p_1(x,t)\}, \tag{22}$$

where $\partial_t^\alpha f(t)$ is the Caputo fractional derivative of order $0<\alpha<1$ [26],[5] and the initial condition is $p_1(x,t=0)=\delta(x-x_0)$. Here we note that this equation can be written in an equivalent form with the Riemann-Liouville fractional derivative from the right-hand side of the equation [27]. Equations (18) and (22) are invariant with respect to inversion $x\to -x$ ($x_0\to -x_0$). Therefore, solutions (19) and (21) can be symmetrically extended on the entire x axis, taking modulus $|x|$.

For the MSD we obtain[6] (see Appendix B)

$$\langle x^2(t)\rangle = x_0^2\,E_{1/2}\left(2\frac{v}{2\sqrt{\mathcal{D}}}\,t^{1/2}\right), \tag{23}$$

where

$$E_\alpha(z) = \sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\alpha n+1)}$$

is the one-parameter Mittag-Leffler function [28], which relates to the Laplace transform as follows:

$$\mathcal{L}[E_\alpha(a\,z^\alpha)] = \frac{s^{\alpha-1}}{s^\alpha - a}.$$

Its asymptotic behavior in the long-time limit is the exponential growth (see Appendix B) according to $\langle x^2(t)\rangle \sim 2\,x_0^2\,e^{4(\frac{v}{2\sqrt{\mathcal{D}}})^2 t}$, while the short-time behavior is subdiffusive, $\langle x^2(t)-x_0^2\rangle \sim 2x_0^2(\frac{v}{2\sqrt{\mathcal{D}}})\frac{t^{1/2}}{\Gamma(3/2)}$, where $\Gamma(z)$ is the Gamma function.

First arrival time distribution

The FATD can be considered for turbulent diffusion on a comb, as well. In this case, the random process is described by the Fokker-Planck equation in the presence of a sink term of strength $\wp_{\rm fa}(t)$,

$$\partial_t F(x,y,t) = -\,v\,\delta(y)\,\partial_x\{x\,F(x,y,t)\} + \mathcal{D}\,\partial_y^2 F(x,y,t) - \wp_{\rm fa}(t)\,\delta(x-X)\,\delta(y), \tag{24}$$

where $F(x,y,t)$ is a non-normalized density function. The initial condition is $F(x,y,t=0)=\delta(x-x_0)\delta(y)$, and the δ-sink means that $F(x=X,y=0,t)=0$. By analogy with the previous sections, the random walker is removed when the target is hit. Therefore, $\wp_{\rm fa}(t)$ is the FATD. By the Laplace transform of Eq. (24), we find the FATD in the Laplace domain[7] (see Appendix D)

$$\hat{\wp}_{\rm fa}(s) = e^{-\frac{2\sqrt{\mathcal{D}}}{v}\sqrt{s}\,\log\frac{X}{x_0}}. \tag{25}$$

By the inverse Laplace transform of Eq. (25), for $X>x_0$, for the FATD we finally obtain [cf. Eq. (15)]

$$\wp_{\rm fa}(t) = \frac{\log\frac{X}{x_0}}{\sqrt{4\,\pi\left(\frac{v}{2\sqrt{\mathcal{D}}}\right)^2 t^3}} \times \exp\left[-\frac{\log^2\frac{X}{x_0}}{4\left(\frac{v}{2\sqrt{\mathcal{D}}}\right)^2 t}\right], \tag{26}$$

which is the Lévy-Smirnov distribution. Its long-time asymptotic scales as

$$\wp_{\rm fa}(t) \simeq \frac{\log\frac{X}{x_0}}{\sqrt{4\pi\left(\frac{v}{2\sqrt{\mathcal{D}}}\right)^2}} \times t^{-3/2}.$$

Equation (26) can be also extended to the negative part of the backbone $x\in R^-$. In this case $X<-x_0<0$. For details of the calculation of the FATD in the case of the search in both directions starting at $\pm x_0$, we refer to Appendix D. It is worth nothing that the obtained result corresponds to Eq. (15) for the one-dimensional turbulent diffusion search, where $X>x_0$ in both cases.

V. GMB IN COMB GEOMETRY

To complete our consideration of the turbulent diffusion search due to the multiplicative white noise, we consider a fractional aspect of the GMB. Recently this issue attracted some attention in connection with a continuous time random walk for a particle on a geometric lattice in the presence of a logarithmic potential [29]. Therefore, it is instructive to consider this task in the framework of the comb geometry. In this case Eq. (18) is modified as follows:

$$\partial_t P(x,y,t) = \mathcal{D}_x\,\delta(y)(\partial_x x)^2 P(x,y,t) + \mathcal{D}_y\,\partial_{yy}P(x,y,t), \tag{27}$$

with the same initial condition $P(x,y,t=0)=\delta(x-x_0)\delta(y)$, $x_0>0$. Therefore, this equation can be considered as the GBM in the comb geometry. Following Ref. [29], the GBM can be described in the framework of CTRW theory with multiplicative jump density in a logarithmic potential. The side branches of the comb represent traps for the particle, leading to power-law waiting time density. These waiting times result in a time fractional Fokker-Planck equation for the PDF along the backbone, and thus the corresponding motion along the backbone can be considered as a fractional GBM.

The marginal PDF along the backbone satisfies the fractional Fokker-Planck equation, obtained in Appendix E, which reads

$$\partial_t^{1/2} p_1(x,t) = \frac{\mathcal{D}_x}{2\sqrt{\mathcal{D}_y}}(\partial_x x)^2 p_1(x,t). \tag{28}$$

[5]The Caputo fractional derivative of order $0<\alpha<1$ is defined by $\partial_t^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-t')^{-\alpha}\frac{d}{dt'}f(t')\,dt'$.

[6]The marginal PDF $p_2(y,t)=\int P(x,y,t)dx$ along the fingers corresponds to normal diffusion, $\langle y^2(t)\rangle = 2\mathcal{D}t$.

[7]For initial condition at (x_0,y_0), $y_0\neq 0$ in the FATD (25) there will appear an additional multiplication term $e^{-\sqrt{\frac{s}{\mathcal{D}}}|y_0|}$. If the sink is at $(x_0,0)$ the FATD becomes the Lévy-Smirnov distribution $\wp_{\rm fa}(t) = \frac{|y_0|}{\sqrt{4\pi\mathcal{D}t^3}}\exp(-y_0^2/4\mathcal{D}t)$.

From the subordination approach described in Appendixes B and E, we find the solution for the marginal PDF as follows:

$$p_1(x,t) = \frac{1}{x\sqrt{\mathcal{D}_1\, t^{1/2}}} H_{1,1}^{1,0}\left[\frac{\log^2 \frac{x}{x_0}}{\mathcal{D}_1\, t^{1/2}}\middle| \begin{matrix} (3/4,\, 1/2) \\ (0,\, 2) \end{matrix}\right], \quad (29)$$

where $H_{p,q}^{m,n}(z)$ is the Fox H function (see Appendix E), and $\mathcal{D}_1 = \frac{\mathcal{D}_x}{2\sqrt{\mathcal{D}_y}}$. The MSD then becomes

$$\langle x^2(t)\rangle = x_0^2 E_{1/2}(4\,\mathcal{D}_1 t^{1/2}), \quad (30)$$

and its asymptotes in the short- and long-time limit read $\langle x^2(t)\rangle = x_0^2[1 + 4\mathcal{D}_1 t^{1/2}/\Gamma(3/2)]$ and $\langle x^2(t)\rangle = 2\,x_0^2 e^{4(\mathcal{D}_x^2/\mathcal{D}_y)t}$, respectively.

Fractional GBM search

Correspondingly, the turbulent diffusion search, due to the fractional GMB, is described by

$$\partial_t F(x,y,t) = \mathcal{D}_x\,\delta(y)\,(\partial_x x)^2 F(x,y,t) + \mathcal{D}_y \frac{\partial^2}{\partial y^2} F(x,y,t) - \wp_{\rm fa}(t)\,\delta(x-X)\,\delta(y), \quad (31)$$

with initial position $F(x,y,t=0) = \delta(x-x_0)\delta(y)$, $x_0 > 0$, and δ-sink at $(X > 0, y = 0)$, $F(x = X, y = 0, t) = 0$. Performing the Laplace transformation of Eq. (31), and looking for the solution to it in the form $F(x,y,s) = g(x,s)\, e^{-\sqrt{\frac{s}{\mathcal{D}}}|y|}$, we find the corresponding Fokker-Planck equation for the non-normalized PDF along the backbone, $f_1(x,t) = \int_{-\infty}^{\infty} F(x,y,t)\, dy$,

$$\partial_t f_1(x,t) = \frac{\mathcal{D}_x}{2\sqrt{\mathcal{D}_y}}\, \partial_{t,\rm RL}^{1/2} (\partial_x x)^2 f_1(x,t) - \wp_{\rm fa}(t)\,\delta(x-X), \quad (32)$$

where $\partial_{t,\rm RL}^{\alpha} f(t)$ is the Riemann-Liouville fractional derivative of order $0 < \alpha < 1$ [26][8]. Therefore, the considered process along the backbone represents the fractional GBM search. For the FATD, we find

$$\wp_{\rm fa}(s) = e^{-\frac{s^{1/4}}{\sqrt{\mathcal{D}_1}}\left|\log \frac{X}{x_0}\right|}, \quad (33)$$

which yields

$$\wp_{\rm fa}(t) = \frac{2}{t} H_{1,1}^{1,0}\left[\frac{\log^2 \frac{X}{x_0}}{\mathcal{D}_1\, t^{1/2}}\middle| \begin{matrix} (0,\, 1/2) \\ (0,\, 2) \end{matrix}\right]. \quad (34)$$

The long-time asymptotic of the FATD scales as

$$\wp_{\rm fa}(t) \simeq \frac{\left|\log \frac{X}{x_0}\right|}{4\,\Gamma(3/4)\sqrt{\mathcal{D}_1}} \times t^{-5/4}.$$

Graphical representation of the FATD (34) is given in Fig. 3.

The search reliability $\mathcal{P} = \wp_{\rm fa}(s=0) = 1$ indicates that the target will be found with the probability 1, while the search

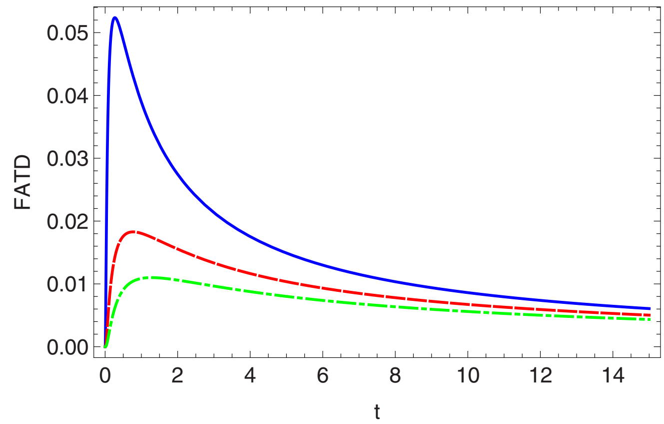

FIG. 3. Graphical representation of the FATD, Eq. (34), for $x_0 = 1$, $\mathcal{D}_x = 1$, $\mathcal{D}_y = 1$, and $X = 10$ (blue solid line), $X = 20$ (red dashed line), and $X = 30$ (green dot-dashed line).

efficiency reads

$$\mathcal{E} = \frac{24\,\mathcal{D}_1^2}{\log^4 \frac{X}{x_0}}. \quad (35)$$

It should be admitted that the quadratic decrease of the efficiency in Eq. (40) in comparison with Eq. (16) is due to subdiffusive growth of $\log x$. As follows from Eq. (29), the geometric Brownian motion taking place in the comb is subdiffusive with respect to $\log x$, such that $\langle \log^2 x\rangle \sim t^{1/2}$. The obtained results also can be extended to the negative x axis, $-x_0 < 0$ by taking modulus $|X|$.

VI. DISCUSSIONS AND CONCLUSIONS

We study a distribution of the first arrival times to absorbing targets in turbulent diffusion, which is due to a multiplicative noise. This phenomenon that combines Brownian diffusion with exponential spreading according to advection and diffusion processes was studied in the framework of two examples of dynamical systems with a multiplicative noise. The first one is inhomogeneous diffusion, which is also known as a geometric Brownian motion in the Black-Scholes model [10]. The second model is due to a random processes on a two-dimensional comb. The comb mechanism of its (advection) interplaying with a white noise in the form of Brownian motion is just a modification of the inhomogeneous advection-diffusion equation (9) and leads to the same turbulent diffusion described by just the same form of the log-normal distribution in Eq. (21). However, there is an essential difference: their local motions are completely different.

As admitted, the first arrival time distribution (FATD), or first hitting time distribution, $\wp_{\rm fa}(t)$, which has been the main characteristic in task, is completely determined by the underlined one-dimensional random process described by the PDF $P(x,t)$ as follows:

$$\partial_t F = L_{FP} F - \wp_{\rm fa}(t)\,\delta(x-X), \quad (36)$$

where L_{FP} is a Fokker-Planck operator, and the equation is a general form of a search problem. In particular, in Eq. (12) it corresponds to inhomogeneous diffusion, in Eq. (18) it is the comb model with inhomogeneous advection, while in Eq. (27)

[8] The Riemann-Liouville fractional derivative of order $0 < \alpha < 1$ is defined by $\partial_{t,\rm RL}^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-t')^{-\alpha} f(t')\, dt'$.

it is the comb model with inhomogeneous diffusion. In the Laplace space $\mathcal{L}[F(t)](s) = \tilde{F}(s)$, we have

$$\tilde{F}(x,s) = \frac{1}{s - \hat{L}_{FP}}\delta(x - x_0) - \frac{\wp_{\rm fa}(s)}{s - \hat{L}_{FP}}\delta(x - X) \equiv g(x,s;x_0) - \wp_{\rm fa}(s)\, g(x,s;X). \quad (37)$$

Here $g(x,s;x_0) = \mathcal{L}[P(x,t)](s)$. Therefore, at the condition of the first hitting at $x = X$, when $\tilde{F}(x = X, s) = 0$ we obtain for the FATD

$$\wp_{\rm fa}(s) = \frac{g(X,s;x_0)}{g(X,s;X)}, \quad (38)$$

which is in a complete agreement with the first passage theory; see, e.g., Ref. [30]. According to this expression, one controls any one-dimensional random process with a variety of distributions $P(x,t)$. It should be stressed that any mechanism which can be responsible for this distribution is a multiplicative noise in the Langevin equation. In particular for the one-dimensional motion it can be $\dot{x} = x\eta(t)$ as in Eq. (5), which leads to inhomogeneous diffusion with $\hat{L}_{FP} \sim (\partial_x x)^2$. Another situation, considered here, is the one-dimensional realization of the log-normal distribution in the two-dimensional comb model. It follows from the Langevin equation (17), which also relates to the multiplicative noise.

To show this, let us consider the Langevin Eq. (17) in the form

$$\dot{x} = -x\, v(y), \quad \dot{y} = \eta(t), \quad (39)$$

where $v(y)$ is an arbitrary function such that $v(0) \neq 0$ and $v'(0) \neq 0$. Differentiating the first equation with respect to the time, and after a little algebra we obtain

$$\ddot{x}x - (\dot{x})^2 = x^2\, v'(y)\,\eta(t), \quad (40)$$

where $v'(y) \equiv \frac{dv(y)}{dy}$. By transition from Eq. (17) to the comb Fokker-Planck equation (18), which is produced from the Langevin equation (17) exactly, the Fokker-Planck equation has the same structure for any $v(y)$ including $v(y) = v\delta(y)$. So, following a particle motion according to the microscopic picture, we conclude that its motion along the backbone is random due to the multiplicative noise. The log-normal distribution $p_1(x,t)$ in Eq. (21) just reflects this property of Eq. (40). It is also interesting to admit that for a specific choice of $v(y) = v_1 y$, Eq. (40) corresponds to the Kolmogorov-Obukhov mechanism of turbulent (Richardson) diffusion [20] for the logarithmic variable $w = \log x$, such that $\ddot{w} + v_1\eta(t) = 0$.

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APPENDIX A: SOLUTION OF THE FOKKER-PLANCK EQUATION FOR GBM

Here we find the solution of the Fokker-Planck equation (7), which is

$$\frac{\partial}{\partial t}\bar{P}(x,t) = -\mu\left(\frac{\partial}{\partial x}x\right)\bar{P}(x,t) + \frac{\sigma^2}{2}\left(\frac{\partial}{\partial x}x\right)^2\bar{P}(x,t), \quad (A1)$$

with the initial condition $\bar{P}(x, t = 0) = \delta(x - x_0)$ and the boundary conditions $\bar{P}(x,t) = \partial_x\bar{P}(x,t) = 0$ at $x = \infty$. Considering $x > 0$, we introduce the new variable $y = \log x$ and the derivatives with respect to y:

$$\frac{\partial}{\partial x} = e^{-y}\frac{\partial}{\partial y}, \quad \frac{\partial^2}{\partial x^2} = -e^{-2y}\frac{\partial}{\partial y} + e^{-2y}\frac{\partial^2}{\partial y^2}.$$

Then Eq. (A1) reads

$$\frac{\partial}{\partial t}\bar{P} = \left(\frac{\sigma^2}{2} - \mu\right)\bar{P}(y,t) + \left(\sigma^2 - \mu\right)\frac{\partial}{\partial y}\bar{P}(y,t) + \frac{\sigma^2}{2}\frac{\partial^2}{\partial y^2}\bar{P}(y,t). \quad (A2)$$

Looking for the solution in the form $\bar{P}(y,t) = e^{-y}f(y,t)$, we obtain the Fokker-Planck equation for the function $f(y,t)$ as follows:

$$\frac{\partial}{\partial t}f(y,t) = -\mu\frac{\partial}{\partial y}f(y,t) + \frac{\sigma^2}{2}\frac{\partial^2}{\partial y^2}f(y,t). \quad (A3)$$

The solution is the Galilei shifted Gaussian [27],

$$f(y,t) = \frac{1}{\sqrt{2\pi\sigma^2 t}} \times \exp\left\{-\frac{[(y - y_0) - \mu t]^2}{2\sigma^2 t}\right\}, \quad (A4)$$

which yields

$$\bar{P}(y,t) = \frac{e^{-y}}{\sqrt{2\pi\sigma^2 t}} \times \exp\left\{-\frac{[(y - y_0) - \mu t]^2}{2\sigma^2 t}\right\}. \quad (A5)$$

Eventually, the solution of Eq. (A1) reads [21]

$$\bar{P}(x,t) = \frac{1}{x\sqrt{2\pi\sigma^2 t}} \times \exp\left\{-\frac{[(\log x - \log x_0) - \mu t]^2}{2\sigma^2 t}\right\}. \quad (A6)$$

For $\mu = 0$, the solution (A6) reads

$$\bar{P}(x,t) = \frac{1}{x\sqrt{2\pi\sigma^2 t}} \times \exp\left(-\frac{\log^2\frac{x}{x_0}}{2\sigma^2 t}\right). \quad (A7)$$

Note that the Mellin transform $\mathcal{M}[f(x)](q) = \int_0^\infty x^{q-1}f(x)\,dx$ can be applied to obtain Eq. (A7) as well. Therefore, performing the Melling [31] and the Laplace transforms in Eq. (A1), we have

$$\bar{P}(q,s) = x_0^{q-1} \times \frac{1}{s - \left[\frac{\sigma^2}{2}(q-1)^2 + \mu(q-1)\right]}. \quad (A8)$$

Then the inverse Laplace transform yields

$$\bar{P}(q,t) = x_0^{q-1} \times \exp\left(\frac{\sigma^2}{2}\left[q + \left(\frac{\mu^2}{\sigma^2} - 1\right)\right]^2 t - \frac{\mu^2}{2\sigma^2}t\right). \quad (A9)$$

Applying the inverse Mellin transform and looking for the solution in the form of the convolution integral

$$\mathcal{M}^{-1}[f(q)\,g(q)] = \int_0^\infty f(r)\,g(x/r)\frac{dr}{r},$$

we obtain the solution as follows:

$$\bar{P}(q,t) = \int_0^\infty \delta(r-x_0)\times\frac{\exp\left(-\frac{[\log\frac{x}{r}-\mu t]^2}{2\sigma^2 t}\right)}{(x/r)\sqrt{2\pi\sigma^2 t}}\frac{dr}{r}$$
$$= \frac{1}{x\sqrt{2\pi\sigma^2 t}}\times\exp\left(-\frac{[\log\frac{x}{x_0}-\mu t]^2}{2\sigma^2 t}\right). \quad (A10)$$

Here we use that

$$f(x) = \mathcal{M}^{-1}[x_0^{q-1}] = \delta(x-x_0)$$

and

$$g(x) = \mathcal{M}^{-1}\left[\exp\left\{\frac{\sigma^2}{2}\left[q+\left(\frac{\mu^2}{\sigma^2}-1\right)\right]^2 t\right)\right\}$$
$$= \frac{1}{x\sqrt{2\pi\sigma^2 t}}\times\exp\left[-\frac{(\log x-\mu t)^2}{2\sigma^2 t}\right].$$

We also used the properties of the inverse Mellin transform [31] $\mathcal{M}^{-1}[f(q+a)] = x^a\mathcal{M}^{-1}[f(q)]$ and

$$\mathcal{M}^{-1}[\exp(\alpha q^2)] = \frac{1}{\sqrt{4\pi\alpha}}\times e^{-\frac{x^2}{4\alpha}}.$$

APPENDIX B: RANDOMLY INHOMOGENEOUS ADVECTION

Let us find the solution of the Fokker-Planck equation

$$\partial_t P(x,y,t) = -v\,\delta(y)\,\partial_x\{xP(x,y,t)\} + \mathcal{D}\,\partial_y^2 P(x,y,t) \quad (B1)$$

with the initial condition $P(x,y,t=0)=\delta(x-x_0)\delta(y)$ and for $x>0$. This Fokker-Planck equation conserves the probability: $\int P(x,y,t)\,dx\,dy = 1$.

Looking for the solution of Eq. (B1) in the form

$$P(x,y,s) = g(x,s)\times e^{-\sqrt{\frac{s}{\mathcal{D}}}|y|}, \quad (B2)$$

we obtain an equation for the backbone anomalous transport

$$v\,x\,\partial_x g(x,s) = -(v+2\sqrt{\mathcal{D}}\,s^{1/2})\,g(x,s)+\delta(x-x_0). \quad (B3)$$

The solution is

$$g(x,s) = \frac{\theta(x-x_0)}{v\,x}\times e^{-\frac{2\sqrt{\mathcal{D}}}{v}\sqrt{s}\log\frac{x}{x_0}}, \quad (B4)$$

where $\theta(z)$ is the Heaviside theta function. By integration of Eq. (B2) over y we obtain the marginal PDF,

$$p_1(x,s) = \int_{-\infty}^{\infty} P(x,y,s)\,dy = 2\sqrt{\frac{\mathcal{D}}{s}}\,g(x,s), \quad (B5)$$

which reads

$$p_1(x,s) = \frac{2\sqrt{\mathcal{D}}}{v}\,\frac{\theta(x-x_0)}{x}\times s^{-1/2}e^{-\frac{2\sqrt{\mathcal{D}}}{v}\sqrt{s}\log\frac{x}{x_0}}. \quad (B6)$$

The inverse Laplace transform, for the PDF $P(x,y,t)$ yields, $x>x_0>0$,

$$P(x,y,t) = \frac{\left(\frac{2\sqrt{\mathcal{D}}}{v}\log\frac{x}{x_0}+\frac{|y|}{\sqrt{\mathcal{D}}}\right)}{x\sqrt{4\pi\,v^2\,t^3}}\,\theta(x-x_0)$$
$$\times\, e^{-\frac{\left(\frac{2\sqrt{\mathcal{D}}}{v}\log\frac{x}{x_0}+\frac{|y|}{\sqrt{\mathcal{D}}}\right)^2}{4t}}. \quad (B7)$$

Then the marginal PDF is

$$p_1(x,t) = \frac{2\,\theta(x-x_0)}{x\sqrt{4\pi\left(\frac{v}{2\sqrt{\mathcal{D}}}\right)^2 t}}\times\exp\left[-\frac{\log^2\frac{x}{x_0}}{4\left(\frac{v}{2\sqrt{\mathcal{D}}}\right)^2 t}\right]. \quad (B8)$$

Equation (B1) is invariant with respect to inversion $x\to -x$ ($x_0\to -x_0$). Therefore, solutions (B7) and (B8) can be symmetrically extended on the entire x axis, taking modulus $|x|$. Note that $|x|>x_0$, because a tracer cannot move upstream.

The marginal PDF is a solution of the fractional Fokker-Planck equation. Substituting Eq. (B5) in Eq. (B3) and performing the inverse Laplace transform, one obtains

$$\partial_t^{1/2} p_1(x,t) = -\frac{v}{2\sqrt{\mathcal{D}}}\,\partial_x\{x\,p_1(x,t)\}, \quad (B9)$$

where

$$\partial_t^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-t')^{-\alpha} f'(t')\,dt'$$

is the Caputo fractional derivative [26], and the initial condition is $p_1(x,t=0)=\delta(x-x_0)$. Here we used the Laplace transform formula of the Caputo fractional derivative,

$$\mathcal{L}[\partial_t^\alpha f(t)](s) = s^\alpha\mathcal{L}[f(t)](s) - s^{\alpha-1}f(t=0+).$$

For the MSD, $\langle x^2(t)\rangle = \int x^2\,p_1(x,t)\,dx$, we obtain

$$\langle x^2(t)\rangle = x_0^2\,E_{1/2}\left(\frac{v}{\sqrt{\mathcal{D}}}\,t^{1/2}\right). \quad (B10)$$

Here

$$E_\alpha(z) = \sum_{n=0}^\infty \frac{z^n}{\Gamma(\alpha n+1)}$$

is the Mittag-Leffler function [28], which relates to the Laplace transform as follows:

$$\mathcal{L}[E_\alpha(a\,z^\alpha)](s) = \frac{s^{\alpha-1}}{s^\alpha - a},$$

where $s>a^{1/\alpha}$. The asymptotic behavior of the MSD is $\langle x^2(t)\rangle \sim 2\,x_0^2\,e^{\frac{v^2}{\mathcal{D}}t}$ according to the asymptotic behavior of the Mittag-Leffler function,

$$E_\alpha(z)\sim\frac{1}{\alpha}\times e^{z^{1/\alpha}}, \quad z\gg 1.$$

The marginal PDF along the fingers $p_2(y,t) = \int P(x,y,t)\,dx$ corresponds to normal diffusion with the MSD $\langle y^2(t)\rangle = 2\mathcal{D}t$.

APPENDIX C: SOLUTION BY SUBORDINATION APPROACH

Let us first consider ordinary form of Eq. (22),

$$\partial_t f(x,t) = -\nu\,\partial_x\{x\,f(x,t)\}. \quad (C1)$$

The Laplace transform yields

$$s\,f(x,s) - \delta(x-x_0) = -\nu\,\partial_x\{x\,f(x,s)\}, \quad (C2)$$

from which we find the solution in the Laplace domain, for $x_0 > 0$,

$$f(x,s) = \frac{\theta(x-x_0)}{\upsilon\, x} \times e^{-\frac{s}{\upsilon}\log\frac{x}{x_0}}. \tag{C3}$$

The inverse Laplace transform gives the solution

$$f(x,t) = \frac{\theta(x-x_0)}{\upsilon\, x} \times \delta\Big(t - \frac{1}{\upsilon}\log\frac{x}{x_0}\Big). \tag{C4}$$

This result can also be obtained from the solution (8) in the limit $\frac{\sigma^2}{2} = \mathcal{D} \to 0$, by using the limit representation $\delta(z) = \lim_{\epsilon\to 0}\frac{1}{\sqrt{4\pi\epsilon}} \times \exp\left(-\frac{z^2}{4\epsilon}\right)$ and $\delta(\alpha z) = \frac{1}{|\alpha|}\delta(z)$.

Subordination

To consider a subordination approach let us consider the fractional Fokker-Planck equation (22) in the Laplace domain,

$$s^{1/2}\, p_1(x,s) - s^{-1/2}\,\delta(x-x_0) = -\frac{\upsilon}{2\sqrt{\mathcal{D}}}\,\partial_x\{x\, p_1(x,s)\}. \tag{C5}$$

Rewriting Eq. (C5) in the form

$$s\, p_1(x,s) - \delta(x-x_0) = -\frac{\upsilon}{2\sqrt{\mathcal{D}}}\, s[s^{-1/2}\,\partial_x\{x\, p_1(x,s)\}], \tag{C6}$$

we obtain by the inverse Laplace transform

$$\partial_t p_1(x,t) = -\frac{\upsilon}{2\sqrt{\mathcal{D}}}\frac{d}{dt}\int_0^t \frac{(t-t')^{-1/2}}{\Gamma(1/2)}\,\partial_x\{x\, p_1(x,t')\}dt', \tag{C7}$$

which is an equivalent representation of Eq. (22).

Let us now use the subordination approach to find the solution of Eq. (C7). We present the solution of Eq. (C7) in the form [27,32–34]

$$p_1(x,t) = \int_0^\infty f(x,u)\, h(u,t)\, du, \tag{C8}$$

where $f(x,u)$ satisfies Eq. (C1). Note that it satisfies formally, since here $f(x,u)$ is a part of equation for $p_1(x,t)$ and the constant ν must be replaced by $\upsilon^2/2\sqrt{D}$. This also relates to the solution (C4). The function $h(u,t)$ is the PDF subordinating the process governed by Eq. (C7) to the process governed by Eq. (C1). The function $h(u,t)$ subordinates the processes from the timescale t (physical time) to the timescale u (operational time). In the Laplace space, Eq. (C8) reads

$$p_1(x,s) = \int_0^\infty e^{-st}\, p_1(x,t)\, dt = \int_0^\infty f(x,u)\, h(u,s)\, du, \tag{C9}$$

where $h(u,s) = \mathcal{L}[h(u,t)]$. By taking

$$h(u,s) = s^{-1/2}\, e^{-u\, s^{1/2}} \to h(u,t) = \frac{1}{\sqrt{\pi\, t}}\, e^{-\frac{u^2}{4t}}, \tag{C10}$$

we have

$$\begin{aligned} p_1(x,s) &= \int_0^\infty f(x,u)\, s^{-1/2}\, e^{-u\, s^{1/2}}\, du \\ &= s^{-1/2}\int_0^\infty f(x,u)\, e^{-u\, s^{1/2}}\, du \\ &= s^{-1/2}\, f(x, s^{1/2}). \end{aligned} \tag{C11}$$

Substituting Eq. (C11) in Eq. (C6), and performing the variable change $s \to s^{1/2}$, we obtain the equation for the PDF $p_1(x,s)$ as follows:

$$s\, p_1(x,s) - p_1(x,0) = s\Big[-\frac{\upsilon}{2\sqrt{\mathcal{D}}}\, s^{-1/2}\,\frac{\partial}{\partial x} x p_1(x,s)\Big], \tag{C12}$$

which is exactly the same as Eq. (C6). Therefore, from Eqs. (C4), (C8), and (C10), for the solution we find

$$\begin{aligned} p_1(x,t) &= \int_0^\infty \frac{\theta(x-x_0)e^{-\frac{u^2}{4t}}}{\sqrt{\pi\, t}\left(\frac{\upsilon}{2\sqrt{\mathcal{D}}}\right)x}\,\delta\Bigg[u - \frac{1}{\left(\frac{\upsilon}{2\sqrt{\mathcal{D}}}\right)}\log\frac{x}{x_0}\Bigg]du \\ &= \frac{2\,\theta(x-x_0)}{x\sqrt{4\pi\left(\frac{\upsilon}{2\sqrt{\mathcal{D}}}\right)^2 t}} \times \exp\Bigg[-\frac{\log^2\frac{x}{x_0}}{4\left(\frac{\upsilon}{2\sqrt{\mathcal{D}}}\right)^2 t}\Bigg]. \end{aligned} \tag{C13}$$

APPENDIX D: COMB FATD

The comb equation with a sink term of the amplitude $\wp_{\rm fa}(t)$ reads

$$\begin{aligned} \partial_t F(x,y,t) = &-\upsilon\,\delta(y)\,\partial_x\{x\, F(x,y,t)\} + \mathcal{D}\,\partial_y^2 F(x,y,t) \\ &- \wp_{\rm fa}(t)\,\delta(x-X)\,\delta(y). \end{aligned} \tag{D1}$$

The initial condition is

$$F(x,y,t=0) = \frac{1}{2}[\delta(x-x_0) + \delta(x+x_0)]\delta(y).$$

The δ-sink means that $F(x=X, y=0, t) = 0$ and $\wp_{\rm fa}(t)$ is the FATD. By the Laplace transform of Eq. (D1), we have

$$\begin{aligned} s\, F(x,y,s) &- \tfrac{1}{2}[\delta(x-x_0) + \delta(x+x_0)]\delta(y) \\ &= -\upsilon\,\delta(y)\,\partial_x\{x\, F(x,y,s)\} \\ &\quad + \mathcal{D}\,\partial_y^2 F(x,y,s) - \wp_{\rm fa}(s)\,\delta(x-X)\,\delta(y). \end{aligned} \tag{D2}$$

Looking for the solution in the form of Eq. (B2)

$$F(x,y,s) = g(x,s) \times e^{-\sqrt{\frac{s}{\mathcal{D}}}|y|}, \tag{D3}$$

we obtain

$$\begin{aligned} \upsilon\, x\,\partial_x g(x,s) = &-(2\sqrt{\mathcal{D}}s^{1/2} + \upsilon)g(x,s) \\ &+ \tfrac{1}{2}[\delta(x-x_0) + \delta(x+x_0)] - \wp_{\rm fa}(s)\,\delta(x-X) \end{aligned} \tag{D4}$$

with the solution

$$\begin{aligned} g(x,s) = &\frac{1}{2\,\upsilon\,|x|}\, e^{-\frac{2\sqrt{\mathcal{D}}}{\upsilon}\sqrt{s}\log\frac{|x|}{x_0}}[\theta(x-x_0) + \theta(x+x_0)] \\ &- \frac{\wp_{\rm fa}(s)}{\upsilon\,|X|}\, e^{-\frac{2\sqrt{\mathcal{D}}}{\upsilon}\sqrt{s}\log\frac{|x|}{|X|}}\theta(x-X). \end{aligned} \tag{D5}$$

The condition $F(x=X, y=0, s) = 0$ means $g(x=X, s) = 0$, and we obtain the FATD

$$\wp_{\rm fa}(s) = e^{-\frac{2\sqrt{\mathcal{D}}}{\upsilon}\sqrt{s}\log\left|\frac{X}{x_0}\right|}. \tag{D6}$$

By the inverse Laplace transform of Eq. (D6), for $X > x_0 > 0$ or $X < -x_0 < 0$, we finally obtain

$$\wp_{\rm fa}(t) = \frac{\log\frac{|X|}{x_0}}{\sqrt{4\pi\left(\frac{\upsilon}{2\sqrt{\mathcal{D}}}\right)^2 t^3}} \times \exp\Bigg[-\frac{\log^2\frac{|X|}{x_0}}{4\left(\frac{\upsilon}{2\sqrt{\mathcal{D}}}\right)^2 t}\Bigg]. \tag{D7}$$

APPENDIX E: SOLUTION OF TURBULENT DIFFUSION EQUATION ON A COMB

The two-dimensional turbulent diffusion on a comb is given by

$$\partial_t P(x,y,t) = \mathcal{D}_x\,(\partial_x x)^2 P(x,y,t) + \mathcal{D}_y\,\partial_{yy} P(x,y,t), \tag{E1}$$

with an initial condition $P(x,y,t=0)=\delta(x-x_0)\delta(y)$. Using a Laplace transform we obtain

$$sP(x,y,s) - \delta(x-x_0)\delta(y) = \mathcal{D}_x\,(\partial_x x)^2 P(x,y,s) + \mathcal{D}_y\,\partial_{yy}P(x,y,s), \tag{E2}$$

from where, by representing the solution $P(x,y,s)$ as $P(x,y,s)=g(x,s)\times e^{-\sqrt{\frac{s}{\mathcal{D}_y}}|y|}$, one finds $p_1(x,s) = 2\sqrt{\mathcal{D}_y}\times s^{-1/2}\,g(x,s)$, and therefore

$$P(x,y,s) = \frac{s^{1/2}}{2\sqrt{\mathcal{D}_y}}\,p_1(x,s)\times e^{-\sqrt{\frac{s}{\mathcal{D}_y}}|y|}.$$

The equation for the marginal PDF in Laplace space then reads

$$s^{1/2}\,p_1(x,s) - s^{-1/2}\,\delta(x-x_0) = \frac{\mathcal{D}_x}{2\sqrt{\mathcal{D}_y}}(\partial_x x)^2 p_1(x,s). \tag{E3}$$

The inverse Laplace transformation yields the time-fractional turbulent diffusion equation of form

$$\partial_t^{1/2} p_1(x,t) = \mathcal{D}_1(\partial_x x)^2 p_1(x,t), \tag{E4}$$

where ∂_t^{α} is the Caputo fractional derivative of order $\alpha = 1/2$. Following the subordination approach, the solution of Eq. (E4) is given by the subordination integral

$$p_1(x,s) = \int_0^\infty f(x,u)\,h(u,s)\,du,$$

where $f(x,u)$ is the solution (10) of the turbulent diffusion equation (9),

$$f(x,u) = \frac{1}{x\sqrt{4\pi\mathcal{D}_1 u}}\times\exp\left(-\frac{\log^2\frac{x}{x_0}}{4\mathcal{D}_1 u}\right), \tag{E5}$$

and

$$h(u,s) = s^{-1/2}\,e^{-u\,s^{1/2}}. \tag{E6}$$

Therefore, by integration we have

$$p_1(x,s) = \frac{1}{2x\sqrt{\mathcal{D}_1}}\,s^{-3/4}\times\exp\left(-\frac{\left|\log\frac{x}{x_0}\right|}{\sqrt{\mathcal{D}_1}}s^{1/4}\right) = \frac{s^{-3/4}}{x\sqrt{\mathcal{D}_1}}H_{0,1}^{1,0}\left[\frac{\log^2\frac{x}{x_0}}{\mathcal{D}_1}s^{1/2}\,\middle|\,\begin{matrix} - \\ (0,2)\end{matrix}\right], \tag{E7}$$

where $H_{p,q}^{m,n}(z)$ is the Fox H function defined by [35]

$$H_{p,q}^{m,n}\left[z\,\middle|\,\begin{matrix}(a_1,A_1),\dots,(a_p,A_p)\\(b_1,B_1),\dots,(b_q,B_q)\end{matrix}\right] = H_{p,q}^{m,n}\left[z\,\middle|\,\begin{matrix}(a_p,A_p)\\(b_q,B_q)\end{matrix}\right] = \frac{1}{2\pi\imath}\int_\Omega \theta(s) z^s\,ds, \tag{E8}$$

and $\theta(s)$ reads as

$$\theta(s) = \frac{\prod_{j=1}^m\Gamma(b_j - B_j s)\prod_{j=1}^n\Gamma(1-a_j+A_j s)}{\prod_{j=m+1}^q\Gamma(1-b_j+B_j s)\prod_{j=n+1}^p\Gamma(a_j - A_j s)}, \tag{E9}$$

$0\leqslant n\leqslant p$, $1\leqslant m\leqslant q$, $a_i, b_j\in C$, $A_i, B_j\in R^+$, $i=1,\dots,p$, $j=1,\dots,q$. The contour Ω starting at $c-\imath\infty$ and ending at $c+\imath\infty$ separates the poles of the function $\Gamma(b_j+B_j s)$, $j=1,\dots,m$ from those of the function $\Gamma(1-a_i-A_i s)$, $i=1,\dots,n$. In Eq. (E7) we use the following properties of the Fox H function [35]:

$$e^{-z} = H_{0,1}^{1,0}\left[z\,\middle|\,\begin{matrix} - \\ (0,1)\end{matrix}\right] \tag{E10}$$

and

$$H_{p,q}^{m,n}\left[z\,\middle|\,\begin{matrix}(a_p,A_p)\\(b_q,B_q)\end{matrix}\right] = k\,H_{p,q}^{m,n}\left[z^k\,\middle|\,\begin{matrix}(a_p,k\,A_p)\\(b_q,k\,B_q)\end{matrix}\right], \tag{E11}$$

where $k>0$. From the inverse Laplace transform [35]

$$\mathcal{L}^{-1}\left[s^{-\rho}H_{p,q}^{m,n}\left[as^{\sigma}\,\middle|\,\begin{matrix}(a_p,A_p)\\(b_q,B_q)\end{matrix}\right]\right] = t^{\rho-1}H_{p+1,q}^{m,n}\left[\frac{a}{t^\sigma}\,\middle|\,\begin{matrix}(a_p,A_p),(\rho,\sigma)\\(b_q,B_q)\end{matrix}\right], \tag{E12}$$

where $\rho, a, s\in C$, $\mathrm{Re}(s)>0$, $\sigma>0$, $\mathrm{Re}(\rho)+\sigma\max_{1\leqslant i\leqslant n}\{\frac{1}{A_i}-\frac{\Re(a_i)}{A_i}\}>0$, $|\arg a|<\pi\theta/2$, $\theta=\alpha-\sigma$, $\alpha=\sum_{j=1}^n A_j - \sum_{j=n+1}^p A_j + \sum_{j=1}^m B_j - \sum_{j=m+1}^q B_j$, for the marginal PDF we finally obtain

$$p_1(x,t) = \frac{1}{x\sqrt{\mathcal{D}_1\,t^{1/2}}}H_{1,1}^{1,0}\left[\frac{\log^2\left|\frac{x}{x_0}\right|}{\mathcal{D}_1\,t^{1/2}}\,\middle|\,\begin{matrix}(3/4,1/2)\\(0,2)\end{matrix}\right]. \tag{E13}$$

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