Coulomb logarithm accuracy in a Yukawa potential

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(Received 15 April 2020; revised 12 August 2020; accepted 3 September 2020; published 17 September 2020)

We present a complete theory of the scattering of a particle in a Yukawa potential when the screening length is much larger than the classical impact parameter for 90° deflection and than the de Broglie length. The classical limit, the quantum limit, and the intermediate case are investigated, enabling an accurate determination of the argument of the Coulomb logarithm in the general case. The connection with previously published results is made.

DOI: 10.1103/PhysRevE.102.033206

I. INTRODUCTION

In plasmas the Coulomb logarithm plays a major role in transport theory. It is indicative of the effect of numerous small angle collisions compared with few large angle collisions. In a recent paper [1] the determination of the Coulomb logarithm has been questioned and a critical revision has been proposed. As a matter of fact, a rigorous justification of the argument of the Coulomb logarithm is often skipped in the literature, so that a clarification is needed. Though some important results have been obtained, sometimes decades ago [2–5], no complete overview linking the different approaches has been done. In this article, we give the missing links enabling such an overview.

We study a simplified model problem, namely, the scattering of a particle of mass m and nonrelativistic velocity v by a screened Coulomb (Yukawa) potential $V(r) = (\alpha/r)\, e^{-r/\lambda}$, where λ is the screening length. We thus ignore here the polarization of the medium by the particle itself [5–7]. In addition to the screening length λ, two other lengths will appear in our discussion: the classical impact parameter for 90° deflection in a pure Coulomb field, $b_0 = \alpha/mv^2$, and the de Broglie length divided by 2π, $\lambda_{dB} = \hbar/mv$. The Coulomb parameter η is the ratio of these two lengths, $\eta = b_0/\lambda_{dB} = \alpha/\hbar v$. We will limit our discussion to the case where $b_0 \ll \lambda$ and $\lambda_{dB} \ll \lambda$ for which the repulsive and attractive collision integrals are equal, i.e., they do not depend on the sign of α, in contrast to the case where, for instance, b_0 and λ are not much different [8,9].

We are interested in the cross section $\sigma(\theta)$, where θ denotes the scattering angle, in the total cross section given by

$$\sigma = 2\pi \int_0^{\pi} \sigma(\theta)\, \sin\theta\, d\theta, \tag{1}$$

and in the cross section for momentum transfer,

$$\sigma_1 = 2\pi \int_0^{\pi} (1-\cos\theta)\, \sigma(\theta)\, \sin\theta\, d\theta. \tag{2}$$

A closely related problem is the heavy ions stopping in matter [5,10–18], where the stopping power due to free electrons initially at rest is given by an expression proportional to the cross section for momentum transfer (2). Our aim is not to review this problem, but simply to recall the correspondence between the model using a Yukawa potential with screening length λ and the models using an unscreened Coulomb potential between the ion and the atomic electrons, $V(r) = \alpha/r$, but taking into account the binding of the atomic electrons, with $\lambda = v/\omega$, where ω is the characteristic atomic frequency of motion.

In Sec. II, we treat the purely classical limit ($\lambda_{dB} \ll b_0$), Sec. III is devoted to the quantum limit ($b_0 \ll \lambda_{dB}$), and Sec. IV to the general case. In the first case we base our discussion mainly on the impact parameter b, in the second case on the scattering angle θ, and in the last case on the quantum number l corresponding to the angular momentum.

II. THE CLASSICAL LIMIT $\eta \gg 1$

In this section, we totally neglect quantum effects. Formally, it corresponds to $\eta = \infty$.

For large angle scattering (close collisions), because of the condition $b_0 \ll \lambda$, the screening can be neglected and one recovers the usual relation between the scattering angle and the impact parameter,

$$\tan(\theta/2) = b_0/b, \tag{3}$$

and the Rutherford expression for $\sigma(\theta)$, denoted as $\sigma_R(\theta)$,

$$\sigma_R(\theta) = \frac{b_0^2}{4\sin^4(\theta/2)}. \tag{4}$$

For small angle scattering (distant collisions), the screening has to be included, but one may use the straight-line approximation, giving [2,3,19]

$$\frac{\theta}{2} \approx \frac{b_0}{\lambda} K_1(x), \tag{5}$$

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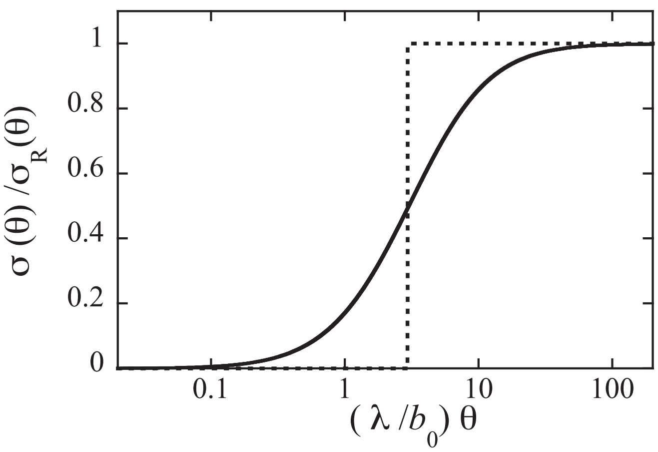

FIG. 1. Cross section $\sigma(\theta)$, divided by the Rutherford cross section $\sigma_R(\theta)$, as a function of $(\lambda/b_0)\theta$, in the purely classical case, with $\lambda \gg b_0$. The dotted line corresponds to a pure Coulomb potential with $\sigma(\theta)$ truncated at a minimum scattering angle $\theta_{\min} \approx 2.9365\, b_0/\lambda$, which is almost equivalent to what would give a Coulomb potential truncated at $r = b_{\max} \approx 0.6811\,\lambda$. The two curves correspond to the same Coulomb logarithm (the area under the curves is the same in this representation where a logarithmic scale is used for θ).

where $x = b/\lambda$, and where K_1 denotes the modified Bessel function of first order,

$$K_1(x) = x \int_1^\infty e^{-xt}\sqrt{t^2-1}\, dt. \tag{6}$$

We recall that $K_1(x) \simeq 1/x$ for $x \ll 1$. Note that Eqs. (3) and (5) are both valid for $b_0 \ll b \ll \lambda$. Furthermore, Eqs. (3) and (5) can be combined into a single expression, valid for all values of b,

$$\tan\left(\frac{\theta}{2}\right) \approx \frac{b_0}{\lambda} K_1(x), \tag{7}$$

and the cross section can then be put into the following semi-implicit form:

$$\sigma(\theta) \approx \sigma_R(\theta) \times \frac{x K_1^3(x)}{|dK_1/dx|}. \tag{8}$$

Figure 1 shows the ratio σ/σ_R as a function of $(\lambda/b_0)\theta$. The effect of the screening is visible for small angles.

The total cross section, which also reads $\sigma = 2\pi\int_0^\infty b\,db$, diverges, as expected in a purely classical theory, for which only potential vanishing at a finite radius, such as the one corresponding to hard balls scattering, can lead to a finite cross section (we will see further on that quantum effects modify this behavior, even for a large but finite value of η).

To compute the cross section for momentum transfer, one can write

$$\sigma_1 = 2\pi \int_0^\infty (1-\cos\theta)\, b\, db \tag{9}$$

and divide the collisions into two groups corresponding to $b < b_i$ and $b > b_i$, the intermediate value b_i belonging to the range $b_0 \ll b_i \ll \lambda$, for which Eqs. (3) and (5) are both valid. Equation (3) is used to evaluate the first term and Eq. (5) to evaluate the second term, and the sum appears independent of the exact value of b_i. This procedure is equivalent to the one used by other authors, including Bohr [10] or Liboff [3]. As noticed by Jackson [5], this is totally equivalent to keeping only the second term and setting $b_i = b_0$, with finally

$$\sigma_1 \approx 4\pi b_0^2 \int_{b_0/\lambda}^\infty x K_1^2(x)\, dx. \tag{10}$$

Now we use the following property of modified Bessel functions [20],

$$xK_1^2 = -\frac{d}{dx}\left[xK_0K_1 - \frac{x^2}{2}\left(K_1^2 - K_0^2\right)\right], \tag{11}$$

where

$$K_0(x) = \int_1^\infty \frac{e^{-xt}}{\sqrt{t^2-1}}\, dt. \tag{12}$$

The integral (9) can then be readily done. Using the asymptotic forms of the modified Bessel functions, in particular the property $K_0(x) \simeq \ln 2 - \gamma - \ln x$ for $x \ll 1$, where $\gamma \approx 0.5772$ is the Euler constant, and the property $K_{0,1}(x) \simeq (\pi/2x)^{1/2} e^{-x}$ for $x \to \infty$, one obtains

$$\sigma_1 \approx 4\pi b_0^2 \ln \Lambda_c, \tag{13}$$

where $\ln \Lambda_c$ denotes the classical Coulomb logarithm,

$$\ln \Lambda_c = \ln(\lambda/b_0) + \ln 2 - \gamma - 1/2 \tag{14}$$

or

$$\ln \Lambda_c = \ln(b_{\max}/b_0) \tag{15}$$

with

$$b_{\max} = (2/e^{\gamma+\frac{1}{2}})\lambda \approx 0.6811\,\lambda. \tag{16}$$

The integral (9) calculated with the pure Coulomb potential, but truncated for large impact parameters at $b = b_{\max}$, would give the same result. For this particular value of the impact parameter, the resulting minimum scattering angle would be $\theta_{\min} = 2\, b_0/b_{\max} \approx 2.9365\, b_0/\lambda$. The cross section for such a truncated Coulomb potential is represented in Fig. 1 as a dotted line.

We now discuss briefly the correspondence with the results of the Bohr's model for ion stopping in matter. In that model, a heavy ion slows down due to energy exchange with bound electrons of mass m. The interaction between the ion and the electron corresponds to an unscreened Coulomb potential, $V(r) = \alpha/r$, and ω is taken to be the characteristic atomic frequency of motion. Similarly to what happens for the screened potential where the screening matters only for distant collisions, close collisions are unaffected by the binding, in contrast with distant collisions. Then the stopping power of electrons appears to be proportional to a logarithmic term of the form [5,10]

$$\ln \Lambda_{\text{Bohr}} \approx \int_{\omega b_0/v}^\infty x\left[K_1^2(x) + K_0^2(x)\right] dx. \tag{17}$$

Doing the integrals, one obtains

$$\ln \Lambda_{\text{Bohr}} \approx \ln \Lambda_c + 1/2 \tag{18}$$

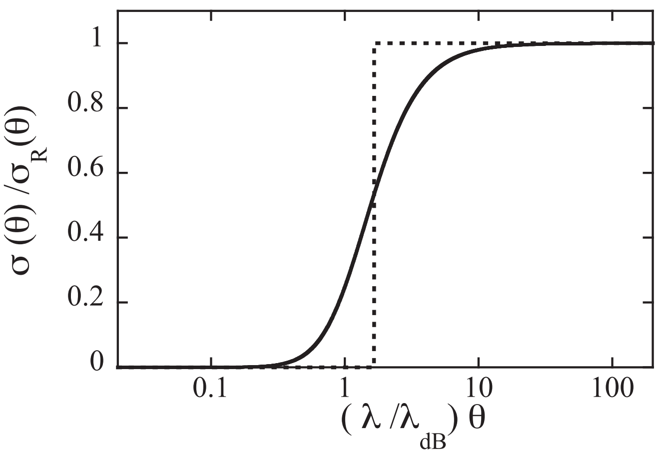

FIG. 2. Cross section $\sigma(\theta)$, divided by the Rutherford cross section $\sigma_R(\theta)$, as a function of $(\lambda/\lambda_{dB})\,\theta$, in the quantum limit, with $\lambda \gg \lambda_{dB}$. The dotted line corresponds to a pure Coulomb potential with $\sigma(\theta)$ truncated at a minimum scattering angle $\theta_{\min} \approx 1.6487\,\lambda_{dB}/\lambda$. The two curves correspond to the same Coulomb logarithm (the area under the curves is the same in this representation where a logarithmic scale is used for θ).

if one sets the adiabatic impact parameter $\lambda = v/\omega$ in the expression of Λ_c. Note that the physical origin of the supplementary term $1/2$ on the right-hand side of (18) has been discussed by Sigmund and Schinner [14].

III. THE QUANTUM LIMIT $\eta \ll 1$

In the preceding section we neglected the quantum effects. As the uncertainty in the impact parameter Δb and the uncertainty in the angle $\Delta\theta$ are linked by the Heisenberg principle, $\Delta b \times mv\Delta\theta \approx \hbar$, the classical approach, which is possible only if $\Delta b \times \Delta\theta \ll b\theta \approx b_0$, implies $\lambda_{dB} \ll b_0$, that is, $\eta \gg 1$.

In the opposite limit, $b_0 \ll \lambda_{dB}$, that is $\eta \ll 1$, the problem can be treated with the Born approximation, which leads to the cross section [4,21]

$$\sigma(\theta) = \frac{b_0^2}{4\left[\sin^2(\theta/2) + (\lambda_{dB}/2\lambda)^2\right]^2}. \tag{19}$$

Figure 2 shows the ratio σ/σ_R as a function of $(\lambda/\lambda_{dB})\,\theta$. The effect of the screening is visible for small angles. Note that the transition is faster than in the classical case; that is, the ratio σ/σ_R decreases faster with $1/\theta$. Due to this fast decreasing, the total cross section is now finite [4,21],

$$\sigma \approx 4\eta^2\pi\lambda^2, \tag{20}$$

while the cross section for momentum transfer is given by

$$\sigma_1 \approx 4\pi b_0^2 \ln \Lambda_B \tag{21}$$

with

$$\ln \Lambda_B = \ln(\lambda/\lambda_{dB}) + \ln 2 - 1/2 = \ln(b_{\max}/b_B), \tag{22}$$

where

$$b_B = \lambda_{dB}/e^{\gamma} \approx 0.5615\lambda_{dB}. \tag{23}$$

Comparing the classical and quantum limits, one obtains

$$\ln \Lambda_B = \ln \Lambda_c + \ln \eta + \gamma. \tag{24}$$

In Eq. (22) the length b_B appears as a minimum impact parameter replacing the quantity b_0, with $b_B \gg b_0$, but the formulation in terms of impact parameter might be misleading, because large angle scattering, classically due to impact parameters comparable to b_0, is not suppressed in the quantum regime [it can be seen in Eq. (19) that the cross section does not differ significantly from the Rutherford one for large angles]. On the other hand, the Coulomb logarithm can also be put under the form $\ln \Lambda_B = \ln(2/\theta_{\min})$ with $\theta_{\min}/2 = b_B/b_{\max}$. Classically this minimum scattering angle would correspond to a maximum impact parameter $b'_{\max} \propto \eta\lambda \ll \lambda$ [the same length $b'_{\max}$ appears, within a numerical factor, as an effective radius of the total cross section (20)], but again it would be incorrect to conclude that the potential is somehow truncated for impact parameters larger than $\eta\lambda$. In fact, the description in terms of impact parameters is inappropriate in the quantum limit, because of the Heisenberg principle, as discussed by Bohr [12].

IV. THE GENERAL CASE

A. The partial wave decomposition approach

In the general case, one could simply use a Coulomb logarithm of the form

$$\ln \Lambda = \ln\left[\frac{b_{\max}}{\max(b_0, b_B)}\right], \tag{25}$$

but a better accuracy is possible. First, we note that the Born approximation requires $\eta \ll 1$, so that it cannot be used in the general case. On that point we disagree with the authors of Ref. [1], who consider that the Born approximation is valid unless $\eta \gg 1$. The cross section is given by [21]

$$\sigma(\theta) = |f(\theta)|^2, \tag{26}$$

where $f(\theta)$ is the scattering amplitude, given by the standard partial wave decomposition,

$$f(\theta) = \frac{\lambda_{dB}}{2i}\sum_{l=0}^{\infty}(2l+1)(e^{2i\delta_l} - 1)P_l(\cos\theta), \tag{27}$$

where δ_l is the partial wave phase shift and P_l the Legendre polynomial of order l.

For moderate values of l ($l \ll \lambda/\lambda_{dB}$), one can use the purely Coulomb result [21],

$$\delta_l = \arg\Gamma(l + 1 + i\eta) - \eta\ln(2r/\lambda_{dB}). \tag{28}$$

The last term in (28) corresponds to the plane wave distortion due to the Coulomb potential at radius r and diverges as $r \to \infty$. This divergent term does not depend on l, so that it does not play any role when one calculates the cross section for momentum transfer, which depends on the phase shift differences $(\delta_{l+1} - \delta_l)$ [13]. But it is essential to treat it correctly when one calculates the total cross section, which depends on the individual values of δ_l.

For the screened potential, one can remark that the phase shift should be limited to a finite value of the radius r, of the order of the range λ of the potential. One takes advantage of the fact that that value does not depend on l. As will be found further on by studying the quasiclassical limit, which is relevant for large values of l, the correct value is $r = \lambda/e^{\gamma}$, so that

$$\delta_l = \arg\Gamma(l+1+i\eta) - \eta\,[\,\ln(\lambda/\lambda_{dB}) + \ln 2 - \gamma\,]. \quad (29)$$

For $\max(1, \eta) \ll l$, one has $\arg\Gamma(l+1+i\eta) \simeq \eta \ln l$, and thus for $\max(1, \eta) \ll l \ll \lambda/\lambda_{dB}$, one has

$$\delta_l \simeq \eta\,[\,\ln l - \ln(\lambda/\lambda_{dB}) - \ln 2 + \gamma\,]. \quad (30)$$

On the other hand, for $\max(1, \eta) \ll l$ and in the case of the screened potential, one can use the quasiclassical form of the phase shift [21],

$$\delta_l \simeq -\int_{r_0}^{\infty} \frac{mV(r)\,dr}{\hbar^2\sqrt{k^2 - l^2/r^2}}, \quad (31)$$

with $r_0 = l/k$ and $k = 1/\lambda_{dB}$, so that [18]

$$\delta_l \simeq -\eta K_0(x), \quad (32)$$

where $x = l\,\lambda_{dB}/\lambda$. Note that for $x \ll 1$, i.e., for $l \ll \lambda/\lambda_{dB}$, one has $K_0(x) \simeq \ln 2 - \gamma - \ln x$, and one recovers Eq. (30), which is valid in the range $\max(1, \eta) \ll l \ll \lambda/\lambda_{dB}$, where the two expressions (29) and (32) have a common range of validity.

B. The total cross section

The total cross section is easily calculated from Eqs. (26) and (27). Using the following property of Legendre polynomials:

$$\int_0^{\pi} P_l(\cos\theta)P_{l'}(\cos\theta)\sin\theta\, d\theta = \frac{2}{2l+1}\,\delta_{ll'}, \quad (33)$$

one obtains [21]

$$\sigma = 4\pi\lambda_{dB}^2 \sum_{l=0}^{\infty} (2l+1)\sin^2\delta_l. \quad (34)$$

The dominant terms in the sum correspond to values of l satisfying the condition $\max(1, \eta) \ll l$, so that one can replace the sum by an integral and use the asymptotic form (32), leading to

$$\sigma \approx 4\pi\lambda_{dB}^2 \int_0^{\infty} (2l+1)\sin^2[\eta K_0(x)]\, dl \quad (35)$$

$$\approx 4\pi\lambda^2 \int_0^{\infty} 2x\sin^2[\eta K_0(x)]\, dx. \quad (36)$$

In the purely quantum limit, $\eta \ll 1$, the argument of the sine is small compared to 1 (except for very small values of x which, in any case, do not contribute significantly to the integral), and

$$\sigma \approx 4\pi\eta^2\lambda^2 \int_0^{\infty} 2xK_0^2(x)\, dx \quad (37)$$

$$\approx 4\eta^2\pi\lambda^2, \quad (38)$$

an expression that we already obtained directly via the Born approximation [Eq. (20)]. We remark here that using the quasiclassical form of the phase shift, Eq. (32), is sufficient to get that result.

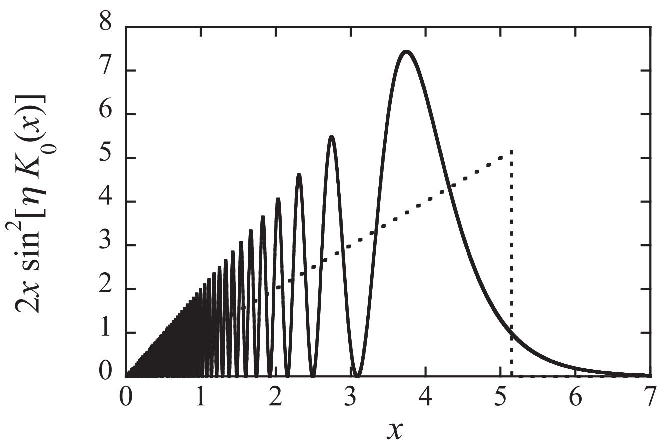

FIG. 3. Function $2x\sin^2[\eta K_0(x)]$ for $\eta = 100$. This function appears in the integral (36) giving σ. The dotted line corresponds to its averaged value x up to the point $x_{\max} \approx 5.15$ for which $\eta K_0(x_{\max}) \approx \pi/10$.

In the general case, σ has to be calculated numerically. Note that, when $1 \ll \eta$, which corresponds to the classical limit, one here obtains a finite value of the total cross section. As noticed, for instance, by Landau and Lifshitz [21], this result holds whenever the potential is decreasing faster than $1/r^2$. The $\sin^2$ term in (36) quickly oscillates, with an average value of $1/2$, between $x = 0$ and the point $x_{\max}$ for which $\eta K_0(x_{\max}) \approx \pi/10$. Figure 3 shows the function $2x\sin^2[\eta K_0(x)]$ for $\eta = 100$, together with its averaged value x up to the point $x_{\max} \approx 5.15$. For $x > x_{\max}$, the argument of the sine goes rapidly to 0 and the contribution to the integral can be neglected, so that

$$\sigma \approx 2\pi\lambda^2 x_{\max}^2. \quad (39)$$

A crude approximation is $x_{\max} \approx \ln\eta$, leading to $\sigma \approx (2\ln^2\eta)\,\pi\lambda^2$, significantly larger than $\pi\lambda^2$, but finite.

C. The cross section for momentum transfer

To compute the cross section for momentum transfer, one has to use the property

$$\cos\theta\, P_l = \frac{1}{2l+1}[(l+1)P_{l+1} + l\,P_{l-1}], \quad (40)$$

together with Eq. (33), to obtain [13]

$$\sigma_1 = 4\pi\lambda_{dB}^2 \sum_{l=0}^{\infty} (l+1)\sin^2(\delta_l - \delta_{l+1}). \quad (41)$$

In this sum, it is necessary to treat separately moderate values of l ($l \ll \lambda/\lambda_{dB}$), and large values of l [$\max(1, \eta) \ll l$]. At the end we will connect the two ranges of values of l by their common range of validity, namely, $\max(1, \eta) \ll l \ll \lambda/\lambda_{dB}$.

For $l \ll \lambda/\lambda_{dB}$, by using (28), one easily gets [13]

$$\delta_{l+1} - \delta_l = \arg(l+1+i\eta) \quad (42)$$

and

$$\sin^2(\delta_{l+1} - \delta_l) = \frac{\eta^2}{(l+1)^2 + \eta^2}, \quad (43)$$

while for $\max(1,\eta) \ll l$, using (32) and $\delta_{l+1} - \delta_l \approx d\delta_l/dl$ with $dK_0(x)/dx = -K_1(x)$, one gets

$$\sin^2(\delta_{l+1} - \delta_l) \approx \sin^2[b_0 K_1(x)/\lambda] \approx (b_0/\lambda)^2 K_1^2(x). \tag{44}$$

Expressions (43) and (44) both give

$$\sin^2(\delta_{l+1} - \delta_l) \approx \eta^2/l^2 \tag{45}$$

in the range $\max(1,\eta) \ll l \ll \lambda/\lambda_{dB}$. To compute the cross section for momentum transfer, we split the infinite sum of Eq. (41) into two terms σ_{1-} and σ_{1+}, corresponding, respectively, to $0 \leqslant l < l_i$ and to $l_i \leqslant l$. The intermediate value l_i is chosen so that $\max(1,\eta) \ll l_i \ll \lambda/\lambda_{dB}$. The splitting is similar to the one made in the purely classical case when calculating the integral (9). Each term gives a contribution of the form

$$\sigma_{1\pm} = 4\pi b_0^2 \ln \Lambda_\pm \tag{46}$$

with

$$\ln \Lambda_-(\eta) = \sum_{l=1}^{l_i} \frac{l}{l^2 + \eta^2}, \tag{47}$$

where we have shifted l to $l_1 = l + 1$ and dropped the subscript 1, and

$$\ln \Lambda_+ \approx \int_{x_i}^{\infty} x K_1^2(x)\, dx, \tag{48}$$

where we have defined $x_i = l_i \lambda_{dB}/\lambda$ and replaced the second sum, which is independent of η, by an integral.

For $\eta \to 0$,

$$\ln \Lambda_-(0) = \sum_{l=1}^{l_i} \frac{1}{l} \approx \ln l_i + \gamma, \tag{49}$$

while

$$\ln \Lambda_+ \approx -\ln x_i - \gamma + \ln 2 - 1/2. \tag{50}$$

The sum of the two terms does not depend on l_i, and

$$\begin{aligned} \ln \Lambda(0) &= \ln \Lambda_-(0) + \ln \Lambda_+ \\ &\approx \ln(\lambda/\lambda_{dB}) + \ln 2 - 1/2. \end{aligned} \tag{51}$$

It corresponds to the result of the Born approximation used in Sec. III. Note that the same result would have been obtained by keeping only the first term, i.e., $\ln \Lambda = \ln \Lambda_-$ and replacing l_i by $l_{\max} = b_{\max}/\lambda_{dB}$.

In the general case one can focus on the first term $\ln \Lambda_-(\eta)$ since the second one does not depend on η. One denotes $\Delta(\eta)$ the difference

$$\Delta(\eta) = \ln \Lambda(0) - \ln \Lambda(\eta) = \eta^2 \sum_{l=1}^{l_i} \frac{l}{l(l^2 + \eta^2)}. \tag{52}$$

As $l_i \gg 1$, the sum can be extended to ∞ without significantly changing the result [13]. This expression corresponds to the Bloch correction [11] and is related to the Digamma function [20] $\psi(z) = [d\Gamma(z)/dz]/\Gamma(z)$ by the equation

$$\Delta(\eta) = \mathrm{Re}\psi(1 + i\eta) + \gamma. \tag{53}$$

For $\eta \gg 1$, one has $\mathrm{Re}\psi(1 + i\eta) \simeq \ln \eta$, and

$$\Delta(\eta) \simeq \ln \eta + \gamma, \tag{54}$$

which corresponds to Eq. (24). For intermediate values of η, the function $\mathrm{Re}\psi(1 + i\eta)$ can be approximated by [13]

$$\Delta(\eta) \approx \ln \sqrt{1 + (e^\gamma \eta)^2}. \tag{55}$$

Using Eq. (22) and the approximation (55), one can write

$$\ln \Lambda(\eta) = \ln(b_{\max}/b_{\min}) \tag{56}$$

with

$$b_{\min} \approx (b_B^2 + b_0^2)^{1/2}. \tag{57}$$

An even more precise approximation of $\Delta(\eta)$, correct to higher order both for $\eta \to 0$ and for $\eta \to \infty$, gives

$$b_{\min} \approx \left(\frac{b_B^4 + c\, b_B^2 b_0^2 + a\, b_0^4}{b_B^2 + a\, b_0^2} \right)^{1/2} \tag{58}$$

with

$$\begin{aligned} a &= [1 - 2\zeta(3)/C]/(1 - C/6) \approx 0.5138, \\ c &= 1 + aC/6 \approx 1.2716, \end{aligned} \tag{59}$$

where $C = e^{2\gamma} \approx 3.1722$ and $\zeta(z)$ is the Riemann Zeta function, with $\zeta(3) \approx 1.2021$.

We note here that the Bloch correction (52) is related to small values of the quantum number l, i.e., to close collisions, for which the screening can be neglected. This explain why it is the same as for the problem studied in fact by Bloch, which corresponds to the quantum version of the Bohr model, as similarly bounding forces can be neglected for small values of l. As a result, the term 1/2 appearing in the right-hand side of Eq. (18) and characterizing the difference between the screened potential model and the bound electrons model is independent of η.

V. CONCLUSION

We made a complete study of the scattering of a particle in a Yukawa potential when the screening length is much larger than the classical impact parameter for 90° deflection and than the de Broglie length. The argument of the Coulomb logarithm has been accurately established, in the general case. A link with already known results has been given, and unambiguous justifications have been given in every case. In particular it has been demonstrated that the Born approximation is not convenient if the Coulomb parameter η is not small compared to 1, in contrast to the assertion of Ref. [1], but in agreement with other authors [13]. The relation with the problem of ion stopping in matter has been briefly considered. However, our aim was not to discuss all the numerous aspects of ion stopping, which would be far beyond the scope of this paper. Similarly we have ignored the dielectric polarization of the medium by the moving particle.

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