PHYSICAL REVIEW E **102**, 032310 (2020)

Mixing protocols in the public goods game

Maja Duh,[1] Marko Gosak,[1,2] and Matjaž Perc[1,3,4,*]
[1]*Faculty of Natural Sciences and Mathematics, University of Maribor, Koroška cesta 160, 2000 Maribor, Slovenia*
[2]*Faculty of Medicine, University of Maribor, Taborska ulica 8, 2000 Maribor, Slovenia*
[3]*Department of Medical Research, China Medical University Hospital, China Medical University, Taichung, Taiwan*
[4]*Complexity Science Hub Vienna, Josefstädterstraße 39, 1080 Vienna, Austria*

(Received 17 June 2020; accepted 10 September 2020; published 28 September 2020)

If interaction partners in social dilemma games are not selected randomly from the population but are instead determined by a network of contacts, it has far reaching consequences for the evolutionary dynamics. Selecting partners randomly leads to a well-mixed population, where pattern formation is essentially impossible. This rules out important mechanisms that can facilitate cooperation, most notably network reciprocity. In contrast, if interactions are determined by a lattice or a network, then the population is said to be structured, where cooperators can form compact clusters that protect them from invading defectors. Between these two extremes, however, there is ample middle ground that can be brought about by the consideration of temporal networks, mobility, or other coevolutionary processes. The question that we here seek to answer is, when does mixing on a lattice actually lead to well-mixed conditions? To that effect, we use the public goods game on a square lattice, and we consider nearest-neighbor and random mixing with different frequencies, as well as a mix of both mixing protocols. Not surprisingly, we find that nearest-neighbor mixing requires a higher frequency than random mixing to arrive at the well-mixed limit. The differences between the two mixing protocols are most expressed at intermediate mixing frequencies, whilst at very low and very high mixing frequencies the two almost converge. We also find a near universal exponential growth of the average size of cooperator clusters as their fraction increases from zero to one, regardless of whether this increase is due to increasing the multiplication factor of the public goods, decreasing the frequency of mixing, or gradually shifting the mixing from random to nearest neighbors.

DOI: 10.1103/PhysRevE.102.032310

I. INTRODUCTION

The discovery of network reciprocity in an evolutionary prisoner's dilemma game almost three decades ago set in motion a remarkable research trajectory [1], which eventually saw a mutually rewarding fusion of evolutionary game theory and methods of statistical physics [2–7]. It is now thoroughly established that the structure of the network of contacts plays a decisive role in the outcome of evolutionary games. For example, networks with a scale-free degree distribution are strong facilitators of cooperation [8–21], as they can be coevolutionary networks if appropriate rules are applied [22–32]. Much research dedicated to evolutionary games has also been done on small-world networks [33–39], multilayer and interdependent networks [40–53], hierarchical [54,55] and community networks [56], and most recently higher-order networks [57,58]. It is indeed nearly impossible to give a comprehensive account of all research efforts along these lines being made in the past couple of years, but we hope the above references are a good sample.

Only with a slight delay in comparison to the research dedicated to the importance of network structure in evolutionary games, mobility has also featured prominently in its role as a possible catalyst or impediment to cooperation. Already in 2007, Vainstein *et al.* [59] studied the impact of mobility in a spatial prisoner's dilemma game, showing that cooperation is not only possible (which it would not be in a well-mixed population) but may even be enhanced by an "always-move" rule. Their research also found that mobility may increase the capability of cooperators to invade defectors, which could play a fundamental role in the very onset of cooperation. Later on it was shown that random dilution and mobility may suppress the inhibiting factors of the spatial structure in the snowdrift game, while further enhancing the already promoted cooperation (due to network reciprocity) in the prisoner's dilemma and the stag-hunt game [60]. The impact of mobility was also studied in spatial social dilemmas with imitation dynamics [61], as well as in off-lattice models [62–64], in conjunction with punishment [65] and optional participation [66], as well as in the prisoner's dilemma game with influential players [67]. Furthermore, how shortcut links affect the evolution of cooperation in a snowdrift game within the Watts-Strogatz small-world model has been studied in [68], while adaptive dynamical networks, where competing agents can rewire their connections in order to achieve a better competition environment, have been considered as possible facilitators of cooperation in [30,69–71] (see also [32] for a review).

---

[*] matjaz.perc@gmail.com

In general, while mobility in socially inspired models has numerous implications, ranging from affecting pathogen competition [72] and their persistence [73] to driving vector-borne epidemics [74], it is not clear when, if at all, mobility actually leads to well-mixed behavior. In the context of evolutionary games, for example, the question is how much mixing, whereby mixing is akin to mobility (see [67]), makes the game mixed well. To be more specific, in the public goods game, where each individual in a group containing G individuals can either cooperate and contribute $c = 1$ to the common or defect and contribute nothing, and where the sum of all contributions is then multiplied by a multiplication factor $R > 1$ and equally divided amongst all group members irrespective of their contribution, in a well-mixed population cooperators cannot survive if $R < G$ [75]. On a square lattice, however, even if $R/G < 1$ some fraction of cooperators can prevail through the effect of network reciprocity [76]. When then, because of mixing, does $R = G$ become the critical threshold for cooperation survival?

To address this, we study the public goods game on a square lattice with nearest-neighbor and random mixing with different frequencies. We also consider a mixture of both mixing protocols, such that with probability p we exchange two random players, and with probability $1 - p$ two nearest neighbors. As we will show, nearest-neighbor mixing requires a higher frequency than random mixing to arrive at the well-mixed $R = G$ limit. We will also show that the two mixing protocols, exchanging nearest-neighbor and randomly selected pairs, have almost the same effect at very low and very high mixing frequencies, but markedly different effects at intermediate mixing frequencies. Finally, we will also show a near universal exponential growth of the average size of cooperator clusters as their fraction increases from zero to one that is independent of the reason for doing so. This suggests that, at least in the public goods game, mixing has the same effect on the spatiotemporal evolutionary dynamics as simply increasing the social dilemma strength.

In what follows, we first present the public goods game and the mixing protocols. We then continue with presenting the main results, and finally with discussing their implications and possible impact.

II. PUBLIC GOODS GAME WITH MIXING

The public goods game is staged on a $L \times L$ square lattice with von Neumann neighborhood, where each node is occupied by one player. Initially, each player on site x is designated as a cooperator or defector with equal probability. Each player x participates in $G = 5$ overlapping groups, where each group is composed of a focal player and its $G - 1$ nearest neighbors. In the standard public goods game cooperators (C) will contribute a fixed cost ($s_x = 1$) to every group in which they participate, while defectors (D) contribute nothing ($s_x = 0$). Afterwards, the overall contributions are multiplied by the synergy factor $R > 1$. The resulting amount is then redistributed equally among all group members. Accordingly, the accumulated payoff of a player x in every group g is given by

$$P_x^g = \frac{RN_C^g}{G} - s_x, \qquad (1)$$

where N_C^g is the number of cooperators within the group g. Since each player participates in G groups, therefore, the total payoff P_x of player x is the sum of payoffs from all the G groups where he is a member: $P_x = \sum_g P_x^g$. We simulate the evolutionary process in accordance with the established Monte Carlo simulation procedure by the following elementary steps. First, player x is randomly selected among the population, and its payoff P_x as per Eq. (1) can be calculated. Next, one its nearest neighbors y is chosen randomly and acquires its payoff P_y in the same way. Then, player y will adopt the strategy of player x according to the Fermi probability:

$$W(s_x \to s_y) = \frac{1}{1 + \exp[(P_y - P_x)/K]}, \qquad (2)$$

where K quantifies the uncertainty during the strategy imitation. In this paper, conveniently, we set $K = 0.5$ without loss of generality [7]. We repeat the described elementary steps N times, where N is the number of all lattice nodes. Consequently, every player has a chance to adopt one of his neighbors' strategies once on average during one full Monte Carlo step.

To incorporate mixing of players, a mixing process is performed after each mth Monte Carlo step. We consider two mixing protocols: nearest-neighbor and random mixing. In the former, the randomly selected player u exchanges his site with one of its four nearest neighbors z, while in the latter the player u exchanges his site with one of the randomly selected players z on the square lattice. Within one mixing step each player u has a chance to exchange his site with the selected player z once on average. For example, if the mixing frequency m is 4, N randomly selected pairs are switched after every fourth full Monte Carlo step. For the purpose of combining the two mixing protocols, we introduce a probability p that determines the likelihood that the random mixing is selected. Accordingly, if $p = 0$, only nearest-neighbors pairs are exchanged during the game, whereas for $p = 1$ only random mixing will be selected at each mixing step. In case $p = 0.5$, both mixing types are selected with equal probability. Furthermore, for the evaluation of the microdynamics we compute the sizes of spatial clusters formed by cooperators. In particular, if two cooperators are adjacent as specified by the von Neumann architecture, they are considered to belong to the same cluster. The average cluster size $\langle C \rangle$ is then determined by simply averaging over all identified clusters in the lattice.

All results of Monte Carlo simulations in the next section have been obtained on a square lattice with $N = 160\,000$ nodes and periodic boundary conditions. The equilibrium fraction of cooperators (ρ_C) has been determined by averaging the last 10 000 generations after a transient period of 10 000 Monte Carlo time steps. In addition, the final results are averaged over 10 to 100 runs with different initial conditions for each set of parameter values.

III. RESULTS

We begin by showing results obtained with nearest-neighbor and random mixing in Figs. 1(a) and 1(b), respectively. The fraction of cooperators is presented for different mixing frequencies, expressed as the number m of full Monte Carlo steps at which the players are mixed as described in

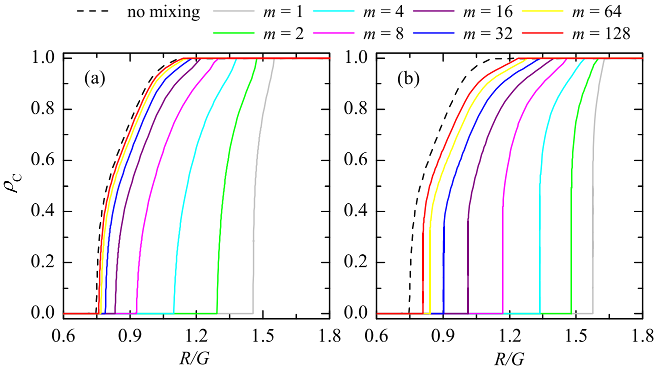

FIG. 1. In the public goods game on a square lattice, a certain mixing frequency m can lead to well-mixed conditions, in which $R = G$ becomes the critical threshold for cooperation survival. To arrive at the well-mixed limit, the nearest-neighbor mixing ($p = 0$) depicted in panel (a) requires a higher frequency than random mixing ($p = 1$) depicted in panel (b). But in both cases the well-mixed limit can also be exceeded, in that $R/G > 1$ is not necessarily enough for cooperator dominance as under well-mixed conditions. Both panels show the fraction of cooperators ρ_C in dependence on the normalized multiplication factor R/G, as obtained on a square lattice for eight different m values (solid lines), where m increases from right to left (see also the legend). The dashed black line shows the result for the public goods game on a square lattice without mixing.

Sec. II above. For example, $m = 2$ and nearest-neighbor mixing means that every two full Monte Carlo steps one player from the population is chosen uniformly at random, one of its nearest neighbors is chosen uniformly at random, and the two players exchange their location—and this is repeated $L \times L$ times. Results shown in Fig. 1 confer that both types of mixing impair the evolutionary success of cooperators, and the more so the smaller the value of m. The fact that mixing does impair cooperation, i.e., that larger values of R are required for cooperators to survive, agrees with the destruction of network reciprocity that comes with approaching, and indeed exceeding, well-mixed conditions.

The fact that mixing random pairs of players, not necessarily nearest neighbors, is more detrimental for cooperation, as can be seen by comparing results in Figs. 1(a) and 1(b), is also understandable, since mixing is in that case more aggressive. Nevertheless, in the $m \to 1$ and ∞ limit the two mixing types almost converge. This is because for $m \to 1$ the mixing is already so fast that the population is thoroughly mixed regardless of whether nearest neighbors and random pairs are mixed. Indeed, we note that well-mixed conditions are exceeded in the sense that the $R/G > 1$ condition for cooperator dominance that is valid for well-mixed conditions is not sustained. In other words, if m is small, having $R/G > 1$ does not ensure $\rho_C = 1$. It can be observed that for nearest-neighbor mixing and $m = 1$ we need $R/G > 1.55$ to have $\rho_C = 1$, while for random mixing and $m = 1$ we even need $R/G > 1.63$.

To quantify the differences between the effects of nearest-neighbor and random mixing more accurately, we introduce a probability p, such that for $p = 0$ only nearest-neighbor pairs are exchanged, whereas for $p = 1$ randomly chosen payers, i.e., not necessarily nearest neighbors, except by chance, are exchanged. In Fig. 2, it can be observed that if the mixing is applied very frequently [$m = 1$, Fig. 2(a)] or infrequently [$m = 64$, Fig. 2(d)] the differences for different values of p remain confined to a rather narrow R/G interval (the no mixing case is shown only for reference and is not meant to be a part of this comparison). Conversely, for intermediate values of m, as shown in Fig. 2(b) for $m = 2$, and even more so in Fig. 2(c) for $m = 8$, the differences become notable as p increases from zero to one. These results thus confirm our reasoning from above in terms of the impact of different mixing protocols and the $m \to 1$ and ∞ limits. At this point we also note that the latter limit is technically fully reached at $m = 128$ for nearest-neighbor mixing, while for random mixing we would still need approximately twice as rare mixing, i.e., $m = 256$, but the computational resources needed in that case to obtain accurate results exceed our options (and are also not merited due to the expected outcome).

Results presented thus far indicate that mixing alone can be used to evoke a phase transition from an absorbing D phase to a mixed $C + D$ phase, and further a phase transition to an absorbing C phase. Naturally, as shown explicitly in Figs. 1 and 2, this can also be achieved by increasing R/G, which enables us to compare the spatiotemporal evolutionary dynamics by means of rescaled parameters. To that effect, we determine the average size of cooperator clusters $\langle C \rangle$ in dependence on the rescaled parameter $n' \in [0, 1]$. Specifically, there are four

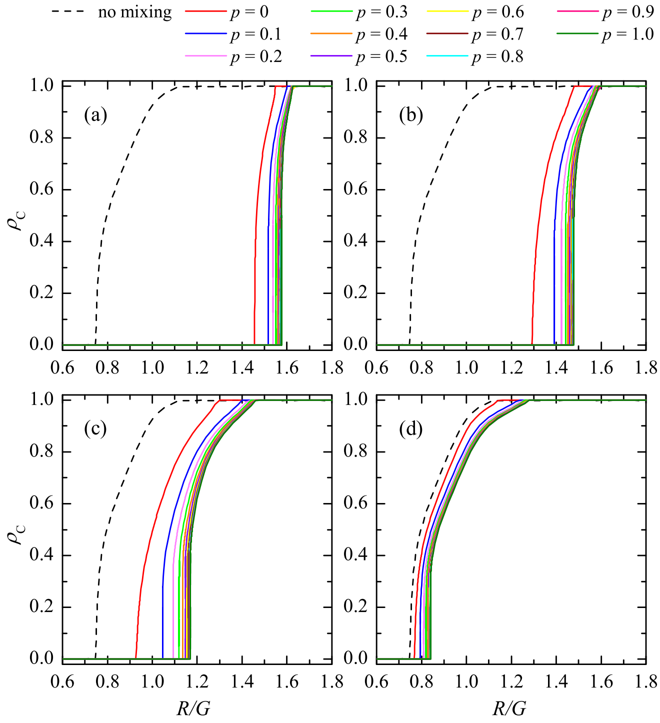

FIG. 2. The effect of the mixture of both mixing protocols on the evolutionary dynamics of the public goods game depends on the mixing frequencies m. Depicted is the fraction of cooperators ρ_C in dependence on the normalized multiplication factor R/G for four different mixing frequencies: $m = 1$ (a), $m = 2$ (b), $m = 8$ (c), and $m = 64$ (d). In all panels results for different combinations of both mixing protocols are presented, as specified by the probability p (as shown in the legend): for $p = 0$ only nearest-neighbor pairs are exchanged, whereas for $p = 1$ randomly chosen payers are exchanged.

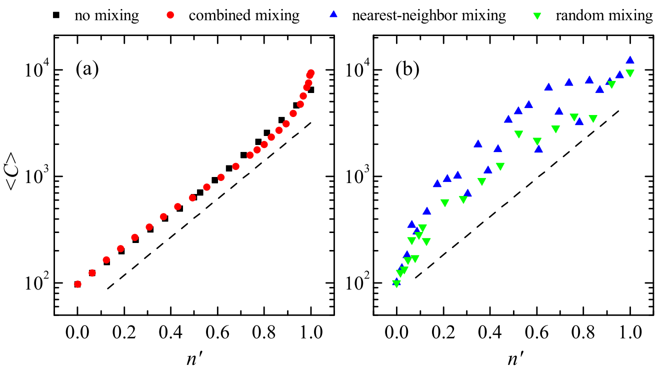

FIG. 3. In the public goods game, mixing of players has the same effect on the evolutionary dynamics as simply increasing the social dilemma strength. The growth of the average size of cooperator clusters $\langle C\rangle$ can be described by a universal exponential growth, regardless of the nature of this increase. In panel (a) the latter is due to increasing the normalized multiplication factor R/G without mixing (no mixing), and due to decreasing the probability p considering a mixture of both mixing protocols obtained for $R/G = 1.56$ and $m = 1$ (combined mixing). In panel (b) the increase is due to decreasing the mixing frequency m for the case of nearest-neighbor mixing obtained for $R/G = 1.1$ (nearest-neighbor mixing), and due to random mixing obtained for $R/G = 1.3$ (random mixing). Both panels show the average size of cooperator clusters $\langle C\rangle$ in dependence on the parameter n', which represents the normalized values either of R/G (no mixing), of m (nearest-neighbor or random mixing), or of p (combined mixing), all rescaled to the unit interval to enable direct comparisons. The slope of the fitted exponential growth (black dashed line) in panels (a) and (b) is 1.81(8) and 1.8(2), respectively. Presented results have been averaged over 100 independent realizations.

different options to go from $\rho_C = 0$ (absorbing D phase) to $\rho_C = 1$ (absorbing C phase) via a mixed $C + D$ phase. We focus on the mixed $C + D$ phase to observe scaling in $\langle C\rangle$, thus considering no mixing and varying $0.86 \leqslant R/G \leqslant 1.02$, and combined mixing at $R/G = 1.56$ and $m = 1$ and varying $0.015 \leqslant p \leqslant 0.34$, both shown in Fig. 3(a). We also consider nearest-neighbor mixing at $R/G = 1.1$ and vary $30 \geqslant m \geqslant 7$, and random mixing at $R/G = 1.3$ and vary $11.5 \geqslant m \geqslant 5.2$, both shown in Fig. 3(b). It can be observed that in all four cases the increase in $\langle C\rangle$ in dependence on n' is described quite well by an exponential fit with a slope ≈1.8. We also note that in all cases all phase transitions are continuous, so that thus there is indeed a universal evolutionary dynamics at play that renders the details of just how these phase transitions are evoked irrelevant. Remarkably, our results show that whether this is due to more favorable game parametrization or less mixing, achieved in whichever way, does not make a difference. This in turn indicates that going from structured to well-mixed populations has the same effect as does increasing the strength of the social dilemma, and this not just in terms of the outcome for cooperation, but also in terms of the spatiotemporal evolutionary dynamics and the governing phase transitions.

IV. DISCUSSION

The classical evolutionary game theory constitutes a mean-field-like approach that considers an infinite, randomly well-mixed population, where defection is known to prevail if $R < G$. Agents placed on a two-dimensional spatial array, however, can interact only with their nearest neighbors. Such a spatial structure can have a promoting effect on the evolution of public cooperation [4,7,76]. Several studies discovered new mechanisms for stabilizing and sustaining cooperation. Particularly mobility has received much attention in the last decade. Sicardi *et al.* [60] extended the results of Vainstein *et al.* [59] and studied the effect of dilution and mobility in the spatial prisoner's dilemma, snowdrift, and stag-hunt evolutionary games. They introduced mobility in a diluted system (a defined fraction of sites is free) where agents jump to a nearest-neighbor site, that is accepted, provided the site is empty, with a given probability. Depending on the population density, mobility may suppress the inhibiting factors of the spatial structure in the snowdrift game, while further enhancing the already promoted cooperation in the prisoner's dilemma and the stag-hunt game. In contrast to this previous research, where two-player games were studied, we studied in the present paper the multiplayer public goods game on a square lattice. Moreover, we introduced mobility in a different manner. A mixing process, where N selected pairs (as described in Sec. II) are switched, was performed after each mth Monte Carlo step. In particular, nearest-neighbor and random mixing were considered, both with varying frequencies separately and combined in different proportions, with the aim of systematically identifying the conditions under which the critical threshold $R = G$ for cooperation survival in well-mixed populations is recovered. We have shown that both types of mixing impair the evolutionary success of cooperators, and the more so the smaller the value of m. In addition, random mixing recovers the well-mixed threshold if N randomly selected pairs are switched after approximately every 16th full Monte Carlo step.

The same effect with randomly selected nearest-neighbor pairs occurs if mixing is applied after approximately every sixth full Monte Carlo step. By considering a mixture of both mixing protocols, such that with probability p we have exchanged two random players, and with probability $1 - p$ we have exchanged two nearest neighbors, we have shown that the greatest difference in the effects of both mixing protocols occurs at intermediate mixing frequencies, while at very low, e.g., after every 64th full Monte Carlo step, and high, e.g., after every full Monte Carlo step, mixing frequencies both effects almost converge, the rationale being that, if the frequency is high, the population cannot be more than mixed, and nearest-neighbor switching, if done frequently enough, achieves this just as well as exchanging random pairs. Conversely, if the frequency is very low, exchanging either nearest neighbors or random pairs simply cannot mix the population at all.

Importantly, these limits are closely related to the typical temporal scale of the evolutionary dynamics of the spatial public goods game. The low-frequency limit implies that, during the time between consecutive mixing, cooperative clusters can already recover and cooperators regroup, such that the next mixing episode is not as damaging. The high-frequency limit, on the other hand, implies that during this time nothing significant happens and that mixing essentially just alters the initial conditions. In future works, it would therefore be interesting to look at other evolutionary games, possibly with very different relaxation times, to study whether and to what extent the inherent temporal scale of evolutionary dynamics affects reaching the well-mixed limit with mixing.

To gain a better understanding of the effect of different frequencies and types of mixing, we have also studied the average size of cooperator clusters, as obtained for different fractions of cooperators on the square lattice. We have first done a benchmark analysis without mixing, simply increasing the multiplication factor R so that the fraction of cooperators ρ_C grew from zero to one. In doing so, we have observed a near-exponential growth of the average size of cooperator clusters in a wide intermediate range of cooperator frequencies, breaking down only close to $\rho_C = 0$ and 1. We have then elicited the same growth in ρ_C by either decreasing the frequency of mixing or increasing the fraction of nearest-neighbor mixing at the expense of random mixing, and we have always observed much the same near-exponential growth of the average size of cooperator clusters. Accordingly, despite intuitively very different effects that the mixing and the strength of the social dilemma might have, they nevertheless evoke the same effect on the spatiotemporal patterns of the evolutionary dynamics, in particular on the emergence and percolation of cooperator clusters.

Since well-mixed conditions underlie the replicator equation, and indeed many other assumptions in evolutionary dynamics [77], it is of utmost importance to know when exactly a population can be considered mixed well. Our research reveals these conditions for the public goods game, which arguably is a central theoretical paradigm for studying cooperation in groups, especially in the realm of physics research [7]. A population being well mixed also implies that the structure of the interaction network ceases to matter. The latter, however, in line with so many preceding findings, plays a central role in determining evolutionary outcomes, and so it is arguably important to know when such a strong and simplifying assumption can be made. We hope that our research will prove useful towards these goals, and inspire future research along similar lines.

ACKNOWLEDGMENTS

We gratefully acknowledge funding from the Slovenian Research Agency (Grants No. J1-2457, No. J1-9112, No. P1-0403, and No. P3-0396).

---

[1] M. A. Nowak and R. M. May, Nature (London) **359**, 826 (1992).

[2] G. Szabó and G. Fáth, Phys. Rep. **446**, 97 (2007).

[3] C. P. Roca, J. A. Cuesta, and A. Sánchez, Phys. Life Rev. **6**, 208 (2009).

[4] M. Perc, J. Gómez-Gardeñes, A. Szolnoki, L. M. Floría, and Y. Moreno, J. R. Soc. Interface **10**, 20120997 (2013).

[5] Z. Wang, S. Kokubo, M. Jusup, and J. Tanimoto, Phys. Life Rev. **14**, 1 (2015).

[6] Z. Wang, L. Wang, A. Szolnoki, and M. Perc, Eur. Phys. J. B **88**, 124 (2015).

[7] M. Perc, J. J. Jordan, D. G. Rand, Z. Wang, S. Boccaletti, and A. Szolnoki, Phys. Rep. **687**, 1 (2017).

[8] F. C. Santos, J. M. Pacheco, and T. Lenaerts, Proc. Natl. Acad. Sci. USA **103**, 3490 (2006).

[9] J. Gómez-Gardeñes, M. Campillo, L. M. Floría, and Y. Moreno, Phys. Rev. Lett. **98**, 108103 (2007).

[10] Z. Rong, X. Li, and X. Wang, Phys. Rev. E **76**, 027101 (2007).

[11] J. Poncela, J. Gómez-Gardeñes, L. M. Floría, and Y. Moreno, New J. Phys. **9**, 184 (2007).

[12] N. Masuda, Proc. R. Soc. B **274**, 1815 (2007).

[13] M. Tomassini, L. Luthi, and E. Pestelacci, Int. J. Mod. Phys. C **18**, 1173 (2007).

[14] A. Szolnoki, M. Perc, and Z. Danku, Physica A **387**, 2075 (2008).

[15] S. Assenza, J. Gómez-Gardeñes, and V. Latora, Phys. Rev. E **78**, 017101 (2008).

[16] F. C. Santos, M. D. Santos, and J. M. Pacheco, Nature (London) **454**, 213 (2008).

[17] J. Poncela, J. Gómez-Gardeñes, and Y. Moreno, Phys. Rev. E **83**, 057101 (2011).

[18] J. Poncela, J. Gómez-Gardeñes, L. M. Floría, Y. Moreno, and A. Sánchez, EPL **88**, 38003 (2009).

[19] M. Brede, EPL **94**, 30003 (2011).

[20] J. Tanimoto, M. Brede, and A. Yamauchi, Phys. Rev. E **85**, 032101 (2012).

[21] F. Pinheiro, J. Pacheco, and F. Santos, PLoS ONE **7**, e32114 (2012).

[22] H. Ebel and S. Bornholdt, Phys. Rev. E **66**, 056118 (2002).

[23] M. G. Zimmermann, V. M. Eguíluz, and M. San Miguel, Phys. Rev. E **69**, 065102(R) (2004).

[24] G. Szabó and J. Vukov, Phys. Rev. E **69**, 036107 (2004).

[25] J. M. Pacheco, A. Traulsen, and M. A. Nowak, Phys. Rev. Lett. **97**, 258103 (2006).

[26] F. C. Santos, J. M. Pacheco, and T. Lenaerts, PLoS Comput. Biol. **2**, e140 (2006).

[27] F. Fu, X. Chen, L. Liu, and L. Wang, Physica A **383**, 651 (2007).

[28] J. Tanimoto, Phys. Rev. E **76**, 021126 (2007).

[29] F. Fu, C. Hauert, M. A. Nowak, and L. Wang, Phys. Rev. E **78**, 026117 (2008).

[30] F. Fu, T. Wu, and L. Wang, Phys. Rev. E **79**, 036101 (2009).

[31] A. Arenas, J. Camacho, J. A. Cuesta, and R. Requejo, J. Theor. Biol. **279**, 113 (2011).

[32] M. Perc and A. Szolnoki, BioSystems **99**, 109 (2010).

[33] G. Abramson and M. Kuperman, Phys. Rev. E **63**, 030901(R) (2001).

[34] B. J. Kim, A. Trusina, P. Holme, P. Minnhagen, J. S. Chung, and M. Y. Choi, Phys. Rev. E **66**, 021907 (2002).

[35] N. Masuda and K. Aihara, Phys. Lett. A **313**, 55 (2003).

[36] M. Tomassini, L. Luthi, and M. Giacobini, Phys. Rev. E **73**, 016132 (2006).

[37] J. Vukov, G. Szabó, and A. Szolnoki, Phys. Rev. E **73**, 067103 (2006).

[38] F. Fu, L.-H. Liu, and L. Wang, Eur. Phys. J. B **56**, 367 (2007).

[39] J. Vukov, G. Szabó, and A. Szolnoki, Phys. Rev. E **77**, 026109 (2008).

[40] Z. Wang, A. Szolnoki, and M. Perc, EPL **97**, 48001 (2012).

[41] J. Gómez-Gardeñes, I. Reinares, A. Arenas, and L. M. Floría, Sci. Rep. **2**, 620 (2012).

[42] J. Gómez-Gardeñes, C. Gracia-Lázaro, L. M. Floría, and Y. Moreno, Phys. Rev. E **86**, 056113 (2012).

[43] B. Wang, X. Chen, and L. Wang, J. Stat. Mech. (2012) P11017.

[44] S. Gómez, A. Díaz-Guilera, J. Gómez-Gardeñes, C. J. Pérez-Vicente, Y. Moreno, and A. Arenas, Phys. Rev. Lett. **110**, 028701 (2013).

[45] Z. Wang, A. Szolnoki, and M. Perc, Sci. Rep. **3**, 1183 (2013).

[46] Z. Wang, A. Szolnoki, and M. Perc, Sci. Rep. **3**, 2470 (2013).

[47] L.-L. Jiang and M. Perc, Sci. Rep. **3**, 2483 (2013).

[48] A. Szolnoki and M. Perc, New J. Phys. **15**, 053010 (2013).

[49] Z. Wang, L. Wang, and M. Perc, Phys. Rev. E **89**, 052813 (2014).

[50] F. Battiston, M. Perc, and V. Latora, New J. Phys. **19**, 073017 (2017).

[51] C. Shen, C. Chu, L. Shi, M. Jusup, M. Perc, and Z. Wang, EPL Europhys. Lett. **124**, 48003 (2018).

[52] C. Xia, X. Li, Z. Wang, and M. Perc, New J. Phys. **20**, 075005 (2018).

[53] L. Shi, C. Shen, Y. Geng, C. Chu, H. Meng, M. Perc, S. Boccaletti, and Z. Wang, Nonlinear Dyn. **96**, 49 (2019).

[54] J. Vukov and G. Szabó, Phys. Rev. E **71**, 036133 (2005).

[55] S. Lee, P. Holme, and Z.-X. Wu, Phys. Rev. Lett. **106**, 028702 (2011).

[56] B. Fotouhi, N. Momeni, B. Allen, and M. A. Nowak, J. R. Soc. Interface **16**, 20180677 (2019).

[57] U. Alvarez-Rodriguez, F. Battiston, G. F. de Arruda, Y. Moreno, M. Perc, and V. Latora, arXiv:2001.10313.

[58] G. Burgio, J. T. Matamalas, S. Gomez, and A. Arenas, Entropy **22**, 744 (2020).

[59] M. H. Vainstein, A. T. C. Silva, and J. J. Arenzon, J. Theor. Biol. **244**, 722 (2007).

[60] E. A. Sicardi, H. Fort, M. H. Vainstein, and J. J. Arenzon, J. Theor. Biol. **256**, 240 (2009).

[61] M. H. Vainstein and J. J. Arenzon, Physica A **394**, 145 (2014).

[62] S. Meloni, A. Buscarino, L. Fortuna, M. Frasca, J. Gómez-Gardeñes, V. Latora, and Y. Moreno, Phys. Rev. E **79**, 067101 (2009).

[63] M. H. Vainstein, C. Brito, and J. J. Arenzon, Phys. Rev. E **90**, 022132 (2014).

[64] A. Antonioni, M. Tomassini, and P. Buesser, J. Theor. Biol. **344**, 40 (2014).

[65] R. Cong, Q. Zhao, K. Li, and L. Wang, Sci. Rep. **7**, 14015 (2017).

[66] L.-X. Zhong, W.-J. Xu, Y.-D. Shi, and T. Qiu, Chaos, Solitons, and Fractals **47**, 18 (2013).

[67] M. Droz, J. Szwabinski, and G. Szabó, Eur. Phys. J. B **71**, 579 (2009).

[68] L.-X. Zhong, D. F. Zheng, B. Zheng, C. Xu, and P. M. Hui, Europhys. Lett. **76**, 724 (2006).

[69] O. Gräser, C. Xu, and P. M. Hui, EPL **87**, 38003 (2009).

[70] O. Gräser, C. Xu, and P. M. Hui, New J. Phys. **13**, 083015 (2011).

[71] W. Zhang, Y. S. Li, P. Du, C. Xu, and P. M. Hui, Phys. Rev. E **90**, 052819 (2014).

[72] C. Poletto, S. Meloni, V. Colizza, Y. Moreno, and A. Vespignani, PLoS Comput. Biol **9**, e1003169 (2013).

[73] A. Aleta, A. N. Hisi, S. Meloni, C. Poletto, V. Colizza, and Y. Moreno, R. Soc. Open Sci. **4**, 160914 (2017).

[74] D. Soriano-Paños, J. H. Arias-Castro, A. Reyna-Lara, H. J. Martínez, S. Meloni, and J. Gómez-Gardeñes, Phys. Rev. Research **2**, 013312 (2020).

[75] K. Sigmund, *The Calculus of Selfishness* (Princeton University, Princeton, NJ, 2010).

[76] A. Szolnoki, M. Perc, and G. Szabó, Phys. Rev. E **80**, 056109 (2009).

[77] M. A. Nowak, *Evolutionary Dynamics* (Harvard University, Cambridge, MA, 2006).