Aging arcsine law in Brownian motion and its generalization

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Classical arcsine law states that the fraction of occupation time on the positive or the negative side in Brownian motion does not converge to a constant but converges in distribution to the arcsine distribution. Here we consider how a preparation of the system affects the arcsine law, i.e., aging of the arcsine law. We derive an aging distributional theorem for occupation time statistics in Brownian motion, where the ratio of time when measurements start to the measurement time plays an important role in determining the shape of the distribution. Furthermore, we show that this result can be generalized as an aging distributional limit theorem in renewal processes.

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I. INTRODUCTION

The classical arcsine law states that the fraction of the time $T_+(t)$ that a random walker spends in the positive side follows the arcsine distribution [1]. The time $T_+(t)$ can be the occupation time for some observables in physical systems. This law can be generalized to the distribution of the occupation time in renewal processes with fat-tailed distributions [2,3] and in the fractional Brownian motion [4]. These laws can be applied to a plethora of systems such as the mean magnetization in spin systems [3], occupation times in fluorescence of quantum dots [5], currents in stochastic thermodynamics [6], α-percentile options in stock prices [7,8], and leads in sports games [9]. Moreover, the propagator in Lévy walk processes can be obtained by the generalized arcsine distribution through a simple transformation [10,11]. Therefore, the arcsine laws play an important role in many physical processes.

Stationarity is one of the most fundamental properties in stochastic processes. In equilibrium, physical quantities fluctuate around a constant value, and the value is given by the equilibrium ensemble. However, statistical properties of physical quantities depend explicitly on time in nonequilibrium processes such as glassy systems and biological systems, where the characteristic timescale diverges [3,5,12–21]. In nonstationary stochastic processes, aging phenomena are essential, which can be observed by changing the start of the observation time or the total measurement time under the same setup [12,22]. In renewal processes, the distribution of the time when the first renewal occurs, i.e., the forward recurrence time, explicitly depends on the time when the observation starts [3,21,23]. Furthermore, the mean-square displacement (MSD) and the diffusion coefficient obtained by single trajectories depend on the start of the observation as well as the total measurement time in some diffusion processes [15–20,24–26]. A typical model that shows aging is a continuous-time random walk (CTRW) with infinite mean waiting time. In the CTRW, the MSD increases nonlinearly [27], i.e., anomalous diffusion,

$$\langle x(t)^2\rangle \sim D_\alpha t^\alpha \quad (t\to\infty), \tag{1}$$

where $x(t)$ is a displacement, D_α is a constant, and $0<\alpha<1$ characterizes the power-law exponent of the waiting time distribution. Moreover, it shows aging; i.e., the MSD explicitly depends on the start of the observation:

$$\langle [x(t_a+t)-x(t_a)]^2\rangle \sim D_\alpha[(t_a+t)^\alpha - t_a^\alpha] \tag{2}$$

for $t_a \gg 1$, where t_a is called the aging time.

Aging phenomena are also observed in weakly chaotic dynamical systems such as the Pomeau-Manniville map [28–30]. In weakly chaotic maps, the invariant measure cannot be normalized, i.e., infinite measure [31]. Moreover, the generalized Lyapunov exponent, which characterizes a dynamical instability of the system, depends explicitly on the aging time [30]. In particular, the dynamical instability becomes weak when the aging time is increased. When the invariant measure of a dynamical system cannot be normalized, the density of a position does not converge to the invariant measure. This situation is similar to nonequilibrium processes exhibiting aging. In dynamical systems with infinite measures, time-averaged observables do not converge to a constant but converge in distribution in the long-time limit [32,33]. In particular, the distribution of time averages of an $L^1(\mu)$ function, i.e., a function integrable with respect to invariant measure μ, converge to the Mittag-Leffler distribution [32,33]. Distributional behaviors of time averages are characteristics of infinite ergodic theory, which includes the Mittag-Leffler distribution, the generalized arcsine distribution, and another distribution [34–40].

The aging distributional limit theorem in renewal processes, i.e., aging of the Mittag-Leffler distribution, has been studied in Refs. [21,23] and has been applied to a weakly chaotic dynamical system [30]. However, aging of the arcsine law has not been considered so far to the best of our

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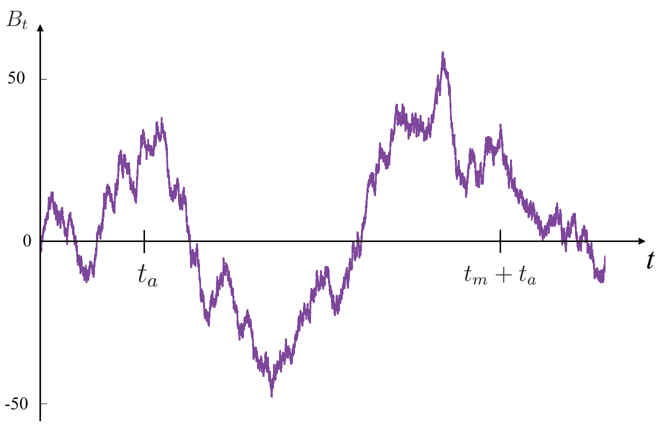

FIG. 1. Trajectory of Brownian motion with $B_0 = 0$. In the aging arcsine law, we measure the occupation time from t_a to $t_a + t_m$.

knowledge. In this paper, we consider aging of the arcsine law. We rigorously prove an aging distributional theorem in Brownian motion. Moreover, we generalize the aging distributional theorem to that in renewal processes in the long-time limit. Finally, we discuss applications of the aging distributional limit theorem to physical systems.

II. PRELIMINARIES

Consider 1D Brownian motion starting from the origin. This fundamental model of a stochastic process is described by

$$\dot{x}(t) = \xi(t),$$

where $\xi(t)$ is a white Gaussian noise:

$$\langle \xi(t)\xi(t')\rangle = \delta(t - t').$$

As is well known, the MSD grows as $\langle x(t)^2\rangle = t$, implying that the diffusion coefficient D is $D = 1/2$. In what follows, we denote Brownian motion at time t by B_t.

Here we recall the first-passage time (FPT) distribution of a Brownian motion starting from position x, the classical arcsine law, and give some notations. Let $P_x(s)$ be the probability density function (PDF) of FPT, which is the time when a Brownian motion starting from position x reaches zero for the first time. It is known that the PDF is given by

$$P_x(s) = \frac{x}{s} p(s, x) \tag{3}$$

for all $x > 0$ and $s > 0$ [41], where $p(s, x)$ is the propagator of a Brownian motion,

$$p(s, x) = \frac{1}{\sqrt{2\pi s}} e^{-\frac{x^2}{2s}}, \tag{4}$$

for $s > 0$ and $x \in \mathbb{R}$.

Lemma 1: For all $t > 0$, the distribution of FPT D_t, which is the time when a Brownian motion reaches zero for the first time after time t passed, i.e., $D_t \equiv \inf\{s > 0; B_{s+t} = 0\}$ follows $\Pr(D_t > s) = \int_s^\infty \psi_t(u)\,du$, where

$$\psi_t(s) = \frac{1}{\pi} \frac{\sqrt{t}}{\sqrt{s}(s + t)}. \tag{5}$$

Proof: Integrating $P_x(s)p(t, x)$ with respect to x, we have

$$\psi_t(s) = \int_0^\infty \frac{x}{s} p(s, x) p(t, x)\,dx = \frac{1}{\pi} \frac{\sqrt{st}}{s(s + t)}. \tag{6}$$

■

We consider an occupation time $T_+(t)$ that a Brownian motion B_t spends on the positive side until time t,

$$T_+(t) = \int_0^t 1_{[B_s > 0]}\,ds, \tag{7}$$

for $t > 0$, where $1_{[B_s>0]} = 1$ if $B_s > 0$ and 0 otherwise. The classical arcsine law states that a ratio between an occupation time of a Brownian motion starting from zero and measurement time t_m follows the arcsine distribution:

$$\Pr\left[\frac{T_+(t_m)}{t_m} \leqslant s\right] = \int_0^s \phi(s')\,ds' = \frac{2}{\pi} \arcsin\sqrt{s}, \tag{8}$$

where

$$\phi(s) \equiv \frac{1}{\pi\sqrt{s(1 - s)}} \tag{9}$$

for $0 < s < 1$. Here we do not represent the initial position of a Brownian motion explicitly, but it is $B_0 = 0$. By the scaling property of a Brownian motion, this statement is equivalent to the following:

$$\Pr[T_+(1) \leqslant s] = \frac{2}{\pi} \arcsin\sqrt{s}. \tag{10}$$

III. AGING ARCSINE LAW

We introduce the aging time t_a, which is a start of the measurement (see Fig. 1). Before t_a we do not track the trajectory although the process was started. In other words, a position of a Brownian motion is not the origin when the measurement is started.

Theorem 1: For all $t_m > 0$ and $t_a > 0$, the ratio of occupation time $T_+(t_m; t_a) \equiv T_+(t_m + t_a) - T_+(t_a)$ to measurement time t_m follows

$$\Pr\left[\frac{T_+(t_m; t_a)}{t_m} \leqslant s \,\middle|\, B_0 = 0\right] = \int_0^s \phi(r; s')\,ds' + q(r) + 1_{[s\geqslant 1]} q(r), \tag{11}$$

where $r \equiv t_a/t_m$ is the aging ratio,

$$\phi(r; s) = \frac{1}{2\pi^2} \int_0^{1/r} \left\{\frac{1}{\sqrt{1 - s(1 + sv)}} + \frac{1}{\sqrt{s}[1 + (1 - s)v]}\right\} \frac{dv}{\sqrt{v(1 - rv)}} \tag{12}$$

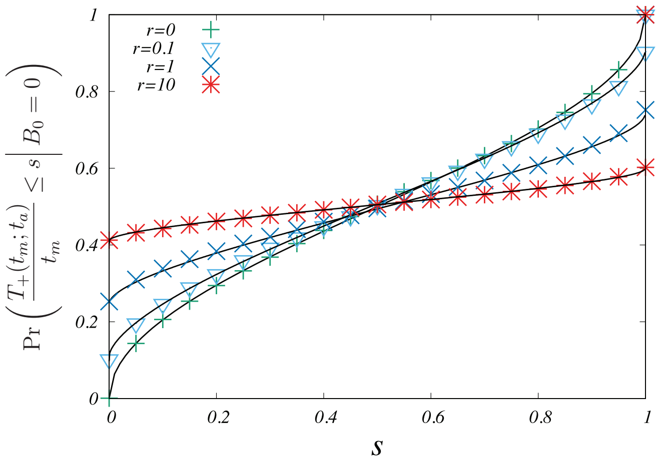

FIG. 2. Distribution of the ratio of occupation time $T_+(t_m;t_a)$ to t_m in Brownian motion for different aging ratio r, where measurement time t_m is fixed as $t_m = 10^3$. Symbols are the results of numerical simulations, and the solid lines represent our theory [Eq. (11)].

and

$$q(r) = \frac{1}{2\pi}\int_{1/r}^{\infty} \frac{dv}{(1+v)\sqrt{v}} = \frac{1}{\pi}\,\mathrm{arccot}(\sqrt{r^{-1}}). \tag{13}$$

The proof of Theorem 1 is given in Appendix A. We note that $\phi(r;s) \to \phi(s)$ for $r \to 0$. In other words, the classical arcsine law is recovered when $t_a \ll t_m$. This is consistent with the arcsine law without aging, $t_a = 0$. Figure 2 shows the effect of aging in the occupation time statistics. In the limit of $r \to 0$, the classical arcsine law is actually recovered. Furthermore,

$$\phi(r;s) \sim c(r)\phi(s) \tag{14}$$

for $s \to 0$ and $s \to 1$, where

$$c(r) = \frac{1}{2\pi}\int_0^{1/r} \frac{dv}{(1+v)\sqrt{v(1-rv)}}. \tag{15}$$

Therefore, constant $c(r)$ explicitly depends on aging ratio r. In particular, $c(r) \to 1/2$ and $c(r) \to 0$ for $r \to 0$ and $r \to \infty$, respectively. We note that the classical arcsine law cannot be recovered when the limit $s \to 0$ or $s \to 1$ is taken in advance, i.e., $c(r)$ does not go to one for $r \to 0$ after $s \to 0$ or $s \to 1$. In other words, the limits of $s \to 0$ and $r \to 0$ are not commutative.

IV. GENERALIZATION OF THE AGING ARCSINE LAW

Here we generalize our result, the aging arcsine law, to occupation time statistics in renewal processes [3,42]. We consider a two-state process $(R_t)_{t\geqslant 0}$, where the state is described by a $+1$ or -1 state (see Fig. 3). Durations for $+1$ and -1 states are independent and identically distributed (IID) random variables. The PDFs of durations for $+1$ and -1 states are denoted by $\rho_+(\tau)$ and $\rho_-(\tau)$, respectively. We assume that the PDFs follow power-law distributions,

$$\rho_\pm(\tau) \sim A_\pm \tau^{-1-\alpha} \quad (\tau \to \infty), \tag{16}$$

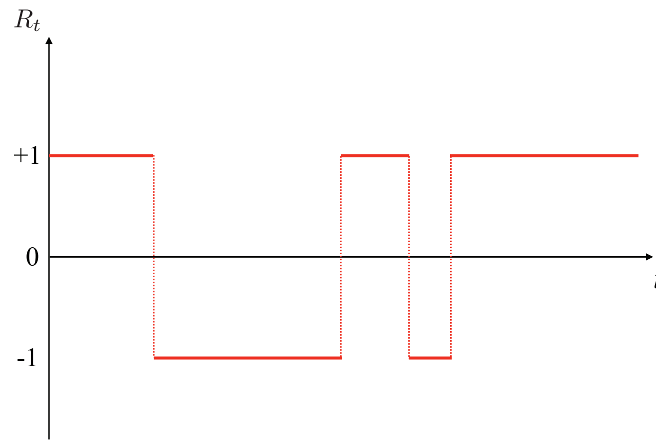

FIG. 3. Trajectory of R_t.

and $\alpha < 1$. In general, the first duration does not follow $\rho_\pm(\tau)$. However, the following results do not depend on the first duration distribution in general. Therefore, in what follows, we do not specify the initial condition. For $0 < \alpha < 1$, the mean duration diverges, and the forward recurrence time $D_t^\pm$, which is a time at which state changes from $\pm$ to $\mp$, respectively, for the first time after time t shows aging. In particular, the PDF of $D_t^\pm$ depends explicitly on time t [43]. Let us define $\psi_t^\pm(\tau)$ as

$$\psi_t^\pm(\tau) = p_\pm \frac{\sin\pi\alpha}{\pi}\frac{t^\alpha}{\tau^\alpha(\tau+t)}, \tag{17}$$

where $p_\pm$ is the probability of finding state is $\pm$ at time t and given by $p_\pm = A_\pm/(A_+ + A_-)$. In the limit of $t_m \to \infty$ with $t_a/t_m = r$ being fixed, we have

$$\Pr\left[\frac{D_{t_a}^\pm}{t_m} \leqslant s, R_{t_a} \gtrless 0\right] \to \int_0^s \psi_r^\pm(\tau)\,d\tau. \tag{18}$$

This is consistent with the Brownian motion result, Eq. (5), where $\alpha = p_\pm = 1/2$ in Brownian motion.

In the renewal process, the classical arcsine law can be generalized. Occupation time of the $+1$ state in the renewal processes follows the generalized arcsine law [2,35]:

$$\Pr\left[\frac{T_+(t)}{t} \leqslant s\right] \to \frac{1}{\pi\alpha}\,\mathrm{arccot}\left\{\frac{[(1-s)/s]^\alpha}{\beta\sin\pi\alpha} + \cot\pi\alpha\right\}$$
$$= \int_0^s \phi_{\alpha,\beta}(s')\,ds' \quad (t \to \infty), \tag{19}$$

where $T_+(t) = \int_0^t 1_{[R_s>0]}\,ds$, $\beta = A_-/A_+$ and

$$\phi_{\alpha,\beta}(s) = \frac{\beta\sin\pi\alpha}{\pi}\frac{s^{\alpha-1}(1-s)^{\alpha-1}}{\beta^2 s^{2\alpha} + 2\beta s^\alpha(1-s)^\alpha\cos\pi\alpha + (1-s)^{2\alpha}}. \tag{20}$$

Theorem 2: In the limit of $t_m \to \infty$ with $t_a/t_m = r$ being fixed, the ratio of occupation time $T_+(t_m;t_a)$ measured from t_a to $t_m + t_a$ to measurement time t_m follows

$$\Pr\left[\frac{T_+(t_m;t_a)}{t_m} \leqslant s\right] \to \Phi_{\alpha,\beta}(r;s)$$
$$\equiv \int_0^s \phi_{\alpha,\beta}(r;s')\,ds' + q_\alpha^-(r) + 1_{[s\geqslant 1]}q_\alpha^+(r), \tag{21}$$

where $T_+(t_m;t_a) = \int_{t_a}^{t_a+t_m} 1_{[R_s>0]}\, ds$,

$$\phi_{\alpha,\beta}(r;s) = \int_0^s \psi_r^+(s')\phi_{\alpha,\beta}\left(\frac{s-s'}{1-s'}\right)\frac{ds'}{1-s'} + \int_0^{1-s} \psi_r^-(s')\phi_{\alpha,\beta}\left(\frac{s}{1-s'}\right)\frac{ds'}{1-s'} \quad (22)$$

and

$$q_\alpha^\pm(r) = \int_1^\infty \psi_r^\pm(\tau)\, d\tau. \quad (23)$$

The proof of Theorem 2 is given in Appendix B.

V. APPLICATION OF THE AGING GENERALIZED ARCSINE LAW TO PHYSICAL SYSTEMS

Here we apply the aging distributional limit theorem in renewal processes to dynamical systems and Lévy walk processes. The 1D map that we consider here is defined on [0,1], $T(x)$:[0, 1] → [0, 1]:

$$T(x) = \begin{cases} x + (1-c)\left(\frac{x}{c}\right)^{1+1/\alpha} & x \in [0, c] \\ x - c\left(\frac{1-x}{1-c}\right)^{1+1/\alpha} & x \in (c, 1] \end{cases}, \quad (24)$$

where c $(0 < c < 1)$ is a parameter characterizing a skewness of the map and $0 < \alpha < 1$ [37]. There are two indifferent fixed points at $x = 0$ and 1, $T(0) = 0$ and $T(1) = 0$ with $T'(0) = T'(1) = 1$. With the aid of the chaotic behaviors outside the two indifferent fixed points, durations on $[0, c]$ or $(c, 1]$ are considered to be independent and identically distributed random variables. Moreover, the duration distributions follow a power law [37,38]. Therefore, the aging distributional limit theorem can be applied to the occupation time statistics in the intermittent map. In the case of no aging, the ordinary generalized arcsine law is shown [35], where parameter β is given by

$$\beta = \frac{\alpha + c}{\alpha + 1 - c}\left(\frac{1-c}{c}\right)^{2\alpha}. \quad (25)$$

Figure 4 shows the distribution of the ratio of occupation time $T_+(t_m;t_a)$ to t_m on $[0, c]$. The shape of the distribution strongly depends on aging ratio r. Moreover, the generalized arcsine distribution can be recovered for small r. This is because the generalized arcsine distribution is obtained by substituting $r = 0$ and $\psi_r^+(s) = p_+\delta(s)$ and $\psi_r^-(s) = p_-\delta(s)$ in Eq. (22).

Occupation time statistics can be applied to Lévy walk processes. In a Lévy walk [10], a particle moves with constant velocity v for a random duration and changes its sign, i.e., $-v$, at the next walk, where we assume that durations are IID random variables and the duration PDFs for $\pm v$ states follow $\rho_\pm(\tau)$ with Eq. (16) and $\alpha < 1$. Therefore, the two-state process R_t is considered to be velocity over v in the Lévy walk. The position X_t in the Lévy walk at time t can be represented by $X_t = \int_0^t vR_{t'}\, dt' = v[2T_+(t) - t]$, where $T_+(t)$ is the time of the positive velocity up to time t. Since the velocity in the Lévy walk, vR_t, is described by a renewal process, our aging distribution limit theorem can be applied to obtain the propagator of the position X_t in the Lévy walk. In particular, the distribution of the displacement X_{t_m,t_a} from time t_a to t_m, $X_{t_m,t_a} \equiv X_{t_m+t_a} - X_{t_a}$, is given by the distribution of $v[2T_+(t_m;t_a) - t_m]$:

$$\Pr(X_{t_m,t_a}/t_m \leqslant x) \to \Phi_{\alpha,\beta}\left(r; \frac{x+v}{2v}\right). \quad (26)$$

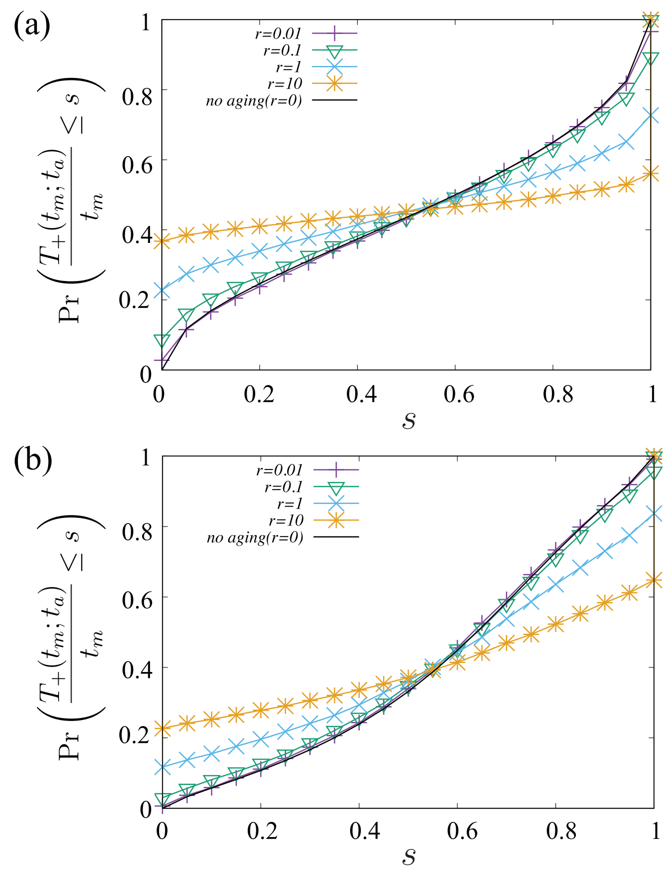

FIG. 4. Distribution of the ratio of occupation time $T_+(t_m;t_a)$ to measurement time t_m in the intermittent map [Eq. (24)] for different aging ratio r, where measurement time t_m is fixed as $t_m = 10^5$ [(a) $\alpha = 0.5$ and (b) $\alpha = 0.7$ $(c = 0.6)$]. Symbols with lines are the results of numerical simulations, and the solid line represent the generalized arcsine distribution without aging, $\Phi_{\alpha,\beta}(s) = \int_0^s \phi_{\alpha,\beta}(s')\, ds'$. We used a uniform distribution as the initial distribution.

VI. CONCLUSION

We have shown an aging distributional theorem of occupation time on the positive side in Brownian motion. FPT distribution $P_x(s)$ of Brownian motion starting from x is a key distribution to derive the theorem. The distribution of the occupation time is described by aging ratio r. The classical arcsine law is recovered when aging time t_a is much smaller than measurement time t_m, $r \to 0$. We have also shown that the aging arcsine law is generalized to the occupation time distribution in renewal processes under the limits of $t_a \to \infty$ and $t_m \to \infty$. The ordinary generalized arcsine law is also recovered in the limit of $r \to 0$. Finally, this generalized aging arcsine law can be successfully applied to the occupation time statistics in intermittent maps with infinite invariant measures and the distribution of the displacement in the Lévy walk.

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APPENDIX A: PROOF OF THEOREM 1

Proof: By the scaling property of Brownian motion, statistical properties of B_{st} are the same as those of $\sqrt{t}B_s$. It follows that statistical properties of occupation time $T_+(t_m;t_a)/t_m$ are the same as those of $T_+(r+1) - T_+(r)$ because

$$\frac{T_+(t_m;t_a)}{t_m} = \frac{1}{t_m}\int_{t_a}^{t_a+t_m} 1_{[B_s>0]}\,ds = \int_r^{r+1} 1_{[B_{st_m}>0]}\,ds. \tag{A1}$$

First, we consider case $B_{t_a} > 0$. Using the scaling property, we have

$$\Pr\left[\frac{T_+(t_m;t_a)}{t_m} \leqslant s, B_{t_a} > 0\right] = \Pr[T_+(r+1) - T_+(r) \leqslant s, B_r > 0]. \tag{A2}$$

Since the probability of $B_{t_a} > 0$ is $1/2$,

$$\Pr\left[\frac{T_+(t_m;t_a)}{t_m} \leqslant s, B_{t_a} > 0\right] = \frac{1}{2}\int_0^s \psi_r(s')\Pr[T_+(1-s') \leqslant s - s']\,ds' \tag{A3}$$

for $s < 1$ and

$$\Pr\left[\frac{T_+(t_m;t_a)}{t_m} = 1, B_{t_a} > 0\right] = \int_1^\infty \frac{\psi_r(s')}{2}\,ds'. \tag{A4}$$

It follows that

$$\begin{aligned}\Pr\left[\frac{T_+(t_m;t_a)}{t_m} \leqslant s, B_{t_a} > 0\right] &= \frac{1}{2}\int_0^s \psi_r(s')\Pr\left[T_+(1) \leqslant \frac{s-s'}{1-s'}\right]ds' + 1_{[s\geqslant 1]}q(r)\\ &= \frac{1}{2\pi^2}\int_0^s ds' \frac{\sqrt{r}}{\sqrt{s'}(s'+r)}\int_0^{\frac{s-s'}{1-s'}} \frac{du}{\sqrt{u(1-u)}} + 1_{[s\geqslant 1]}q(r)\end{aligned} \tag{A5}$$

By a change of variables ($y = s'/r$), we obtain

$$\Pr\left[\frac{T_+(t_m;t_a)}{t_m} \leqslant s, B_{t_a} > 0\right] = \frac{1}{2\pi^2}\int_0^{\frac{s}{r}} \frac{dy}{\sqrt{y}(1+y)}\int_0^{\frac{s-ry}{1-ry}} \frac{du}{\sqrt{u(1-u)}} + 1_{[s\geqslant 1]}q(r). \tag{A6}$$

Moreover, by a change of variables ($u = \frac{v-ry}{1-ry}$), we obtain

$$\begin{aligned}\Pr\left[\frac{T_+(t_m;t_a)}{t_m} \leqslant s, B_{t_a} > 0\right] &= \frac{1}{2\pi^2}\int_0^{\frac{s}{r}} \frac{dy}{\sqrt{y}(1+y)}\int_{ry}^{s} \frac{dv}{\sqrt{(1-v)(v-ry)}} + 1_{[s\geqslant 1]}q(r)\\ &= \int_0^s \frac{1}{2\pi^2}\frac{dv}{\sqrt{1-v}}\int_0^{\frac{v}{r}} \frac{dy}{\sqrt{y(v-ry)}(1+y)} + 1_{[s\geqslant 1]}q(r)\\ &= \int_0^s \frac{1}{2\pi^2}\frac{dv}{\sqrt{1-v}}\int_0^{\frac{1}{r}} \frac{dy'}{\sqrt{y'(1-ry')}(1+vy')} + 1_{[s\geqslant 1]}q(r),\end{aligned} \tag{A7}$$

where the order of integration was interchanged in the second line. By a similar calculation, we have

$$\Pr\left[\frac{T_+(t_m;t_a)}{t_m} \geqslant s, B_{t_a} < 0\right] = \int_s^1 \frac{1}{2\pi^2}\frac{dv}{\sqrt{v}}\int_0^{\frac{1}{r}} \frac{du}{\sqrt{u(1-ru)}[1+(1-v)u]} \tag{A8}$$

for $s > 0$. For $B_{t_a} < 0$ and $s = 0$, the probability is

$$\Pr\left[\frac{T_+(t_m;t_a)}{t_m} = 0, B_{t_a} < 0\right] = \frac{1}{2}\int_1^\infty \psi_r(s')\,ds'. \tag{A9}$$

It follows that aging arcsine distribution is given by Eq. (11), and the PDF $\phi(r;s)$ is given by Eq. (12). ■

APPENDIX B: PROOF OF THEOREM 2

Proof: By a scaling argument, aging occupation time statistics can be obtained by a similar way in Brownian motion. By a change of variables, we have

$$\frac{T_+(t_m;t_a)}{t_m} = \tilde{T}_+(r+1) - \tilde{T}_+(r), \tag{B1}$$

where $\tilde{T}_+(r) = \int_0^r 1_{[R_{st_m}>0]}\, ds$. We note that limits $t_a \gg 1$ and $t_m \gg 1$ are necessary to derive the distribution of occupation time in renewal processes, which is different from the arcsine law in Brownian motion. For $R_{t_a} > 0$ and $t_m \gg 1$ and $t_a = rt_m \gg 1$, we have

$$\Pr\left[\frac{T_+(t_m;t_a)}{t_m} \leqslant s, R_{t_a} > 0\right] = \Pr[\tilde{T}_+(r+1) - \tilde{T}_+(r) \leqslant s, R_{t_a} > 0]. \tag{B2}$$

By a similar calculation as in the aging arcsine law, we obtain

$$\Pr\left[\frac{T_+(t_m;t_a)}{t_m} \leqslant s, R_{t_a} > 0\right] \to \int_0^s dv \int_0^v \frac{\psi_r^+(s')}{1-s'}\phi_{\alpha,\beta}\left(\frac{v-s'}{1-s'}\right)ds' + 1_{[s\geqslant 1]}q_\alpha^+(r). \tag{B3}$$

Similarly,

$$\Pr\left[\frac{T_+(t_m;t_a)}{t_m} \geqslant s, R_{t_a} < 0\right] \to \int_s^1 dv \int_0^{1-v} \psi_r^-(s')\phi_{\alpha,\beta}\left(\frac{v}{1-s'}\right)\frac{1}{1-s'}\,ds' \tag{B4}$$

for $s > 0$ and

$$\Pr\left[\frac{T_+(t_m;t_a)}{t_m} = 0, R_{t_m} < 0\right] \to \int_1^\infty \psi_r^-(s')\,ds'. \tag{B5}$$

It follows that aging arcsine distribution is given by Eq. (21), and the PDF $\phi_{\alpha,\beta}(r;s)$ is given by Eq. (22). ■

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