Coherent vortex in two-dimensional turbulence: Interplay of viscosity and bottom friction

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We examine coherent vortices appearing as a result of the inverse cascade of two-dimensional turbulence in a finite box in the case of pumping with arbitrary correlation time in the quasilinear regime. We demonstrate that the existence of the vortices depends on the ratio between the values of the bottom friction coefficient α and the viscous damping of the flow fluctuations at the pumping scale νk_f^2 (ν is the kinematic viscosity coefficient and k_f is the characteristic wave vector at the pumping scale). The coherent vortices appear if $\nu k_f^2 \gg \alpha$ and cannot exist if $\nu k_f^2 \ll \alpha$. Therefore there is a border value $\alpha \sim \nu k_f^2$ separating the regions. In numerical simulations, $\nu k_f^2/\alpha$ can be arbitrary, whereas in a laboratory experiment $\nu k_f^2/\alpha \lesssim 1$ and the coherent vortices can be observed solely near the border value of $\nu k_f^2/\alpha$.

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Turbulence is a chaotic eddy state excited in fluids due to external forcing at large Reynolds numbers Re, see, e.g., Ref. [1]. In thin fluid films turbulence is effectively two-dimensional on scales larger than the thickness of the film. In the two-dimensional case there is a tendency to form larger and larger eddies due to nonlinearity [2,3]. As a result, in a finite box coherent flows with well-defined mean velocity can appear, containing big vortices. We examine the dependence of the velocity profile of the vortices on the pumping characteristics and dissipation.

Already the first theoretical works [4–6] devoted to two-dimensional turbulence reveal its principal difference from the three-dimensional one. The difference is related to the existence of two quadratic quantities (energy and enstrophy) conserved by the two-dimensional Euler equation. That leads to two different cascades produced by the nonlinear interaction: enstrophy is carried from the pumping scale to smaller scales (direct cascade) whereas energy is carried to larger scales (inverse cascade). The enstrophy is dissipated due to viscosity at scales smaller than the pumping length and the energy is dissipated due to bottom friction at scales larger than the pumping length.

Statistical properties of the velocity fluctuations in the inverse cascade were investigated both experimentally [7] and numerically [8]. Results of the works are in good agreement with the analytical theory developed for an unrestricted system [9]. Interestingly, the normal Kolmogorov scaling of the velocity correlation functions is observed in the inverse cascade [2]. The normal scaling is in contrast to the anomalous scaling observed in three-dimensional turbulence [10]. Some theoretical arguments in favor of the normal scaling in the inverse cascade were presented in Ref. [11], where the feature was related to a leading role of the converging Lagrangian trajectories in the inverse cascade.

In an unrestricted two-dimensional system the inverse cascade is terminated at the scale L_α, which is determined by a balance between the power pumped to the fluid by the external forcing and the energy dissipation. One should equate the energy flux ϵ (per unit mass) toward large scales to the friction loss αV^2 where α is the bottom friction coefficient and V is the characteristic velocity at the scale L_α. Using the Kolmogorov estimate $V \sim (\epsilon L_\alpha)^{1/3}$, one obtains from the balance $L_\alpha = \epsilon^{1/2}\alpha^{-3/2}$. If the box size L is smaller than L_α, then the energy carried by the inverse cascade to scales of the order of the box size L starts to accumulate there. Of course, the box size L should be much greater than the pumping scale l, $L \gg l$, otherwise there is no space for the inverse cascade carrying the energy to the box size L.

The accumulation of the energy at the scale L leads to the appearance of an intensive large-scale motion including big coherent vortices. The tendency towards the formation of the vortices was indicated previously in the first works devoted to two-dimensional turbulence, both experimental [12] and numerical [13–15]. It was established in the numerical case [16], where periodic boundary conditions were utilized, that due to the inverse cascade in a square box a stable structure appears that is the vortex dipole. A bit different coherent vortex structure was generated in laboratory experiments in a square box [17,18]. From the theoretical point of view, the last case corresponded to zero boundary conditions for the flow velocity at the box boundaries.

The authors of Ref. [19] reported results of intensive numerical simulations of two-dimensional turbulence for a periodic setup in a square box with random pumping short correlated in time. The average velocity profile of the coherent vortex appears to be isotropic in a range of distances to the vortex center. In the same work an interval of separations from the vortex center was found where the flat velocity profile was realized and some theoretical arguments toward the flat profile were formulated. In Ref. [20] we demonstrated that the flat velocity profile corresponded to the passive (quasilinear) regime of turbulent fluctuations. In Ref. [21] we calculated the structure function of the velocity fluctuations in the case. In Ref. [22] the coherent velocity profile around a rotating disk immersed into the fluid was analyzed. In Ref. [23] moments of accelerations inside the vortex were calculated.

There is a series of theoretical and numerical works devoted to coherent structures in two-dimensional turbulent flows in nonsquare geometry. In Ref. paper [24] analytical solutions for two-dimensional mean flows generated by an inverse turbulent cascade on a sphere and in planar rectangular domains of different aspect ratios were obtained in the framework of the approach. In Ref. [25] it was demonstrated numerically that for a rectangular box at increasing aspect ratio a system of coinciding jets and vortices was formed. In Ref. [26] correlation functions of velocity were investigated both analytically and numerically for the rectangular box. In Ref. [27] it was demonstrated numerically that in the case of a pressure driven flow a jet slithering between wall vortices appears. In the above works the model of the pumping short correlated in time was used.

Results of numerical simulations for static pumping with different types of the large-scale dissipation were reported in Refs. [28,29]. In Ref. [30] we argued that the flat velocity profile should be observed in the quasilinear regime for static pumping in a square box. The prediction was checked in experiment in Ref. [31], where just static pumping was realized (produced by Lorentz force in electrolytes). Here we examine theoretically the case where the pumping force is random with arbitrary correlation time. The previous results can be obtained as limited cases of our general scheme developed in this paper.

We analyze the case of the pumping statistically homogeneous in time and assume that the coherent vortex is already formed. We examine the statistically stationary case, where all mean values are independent of time. Based on the cited results, we assume that the coherent vortex is isotropic. It is convenient to formulate the equations which determine the vortex structure in the reference system with the origin in the vortex center. The reference frame moves with some random velocity relative to the laboratory system. The movements possess their own statistics. Isotropy of the vortex implies that in the reference frame all the values, characterizing the mean flow, depend solely on the distance to the center of the vortex r.

The mean flow in the reference frame is the differential rotation that is characterized by the polar velocity $U(r)$. The stationary equation for U is written as the momentum balance

$$\alpha U = -(\partial_r + 2/r)(\Pi - \nu\Sigma). \tag{1}$$

Here α is the bottom friction coefficient, Π is the corresponding component of the Reynolds stress tensor, ν is the kinematic viscosity coefficient, and Σ is the local shear rate

$$\Sigma = \partial_r U - U/r, \tag{2}$$

of the mean flow.

The component of the Reynolds stress tensor Π figuring in Eq. (1) is expressed via the simultaneous one-point average of the product of the velocity fluctuations

$$\Pi = \langle v_\varphi v_r \rangle. \tag{3}$$

Here v_r and v_φ are the radial and the polar components of the fluctuating velocity and the angle brackets mean time averaging. In the statistically stationary case the stress Π is independent of time. In addition, it is a function of the distance to the center of the vortex r, to be determined.

We work in the quasilinear approximation, which implies that the average flow is strong enough to suppress the nonlinear interaction of the flow fluctuations. Then we can treat the flow fluctuations as passive. It is convenient to describe the fluctuations in terms of the vorticity $\varpi = \operatorname{curl} \boldsymbol{v}$, where $\boldsymbol{v}$ is the velocity of the flow fluctuations. In the approximation we deal with the linear equation

$$\partial_t \varpi + \boldsymbol{V}\nabla\varpi + (\alpha - \nu\nabla^2)\varpi = \phi, \tag{4}$$

where $\boldsymbol{V}$ is the mean velocity and $\phi = \operatorname{curl} \boldsymbol{f}$ and $\boldsymbol{f}$ is the external force (per unit mass) exiting turbulence. We emphasize that the pressure drops out of Eq. (4).

Note that in our setup the inequality

$$\Sigma \gg \Gamma(k_f) = \alpha + \nu k_f^2, \tag{5}$$

should be satisfied. Here $\Gamma(k) = \alpha + \nu k^2$ is the damping of the fluctuations with the wave vector k and the wave vector k_f is the characteristic wave vector of the pumping ϕ. Thus $\Gamma(k_f)$ is the damping of the flow fluctuations at the pumping scale. The damping should be much less than the shear rate Σ suppressing the nonlinear interaction of the fluctuations. Otherwise the damping itself suppresses the nonlinear interaction, then no inverse cascade and no coherent vortices are possible.

Our immediate task is to express the average (3) in terms of the statistical characteristics of the pumping that excites turbulence. We assume that the external force $\boldsymbol{f}$, exciting turbulence, has homogeneous statistics in space and time. Then the pair correlation function of $\phi = \operatorname{curl} \boldsymbol{f}$ can be written as

$$\langle \phi(t, \boldsymbol{r})\phi(0, \boldsymbol{0})\rangle = \int \frac{d\omega\, d^2k}{(2\pi)^3} k^2 F(\omega, \boldsymbol{k}) e^{i\boldsymbol{kr} - i\omega t}, \tag{6}$$

where F determines the pair correlation function of the external force $\boldsymbol{f}$. The integral over $\boldsymbol{k}$ in Eq. (6) is gained at $k \sim k_f$. We assume also that the pumping is statistically isotropic, that is, F depends solely on $k = |\boldsymbol{k}|$.

Let us stress again that averaging is performed in the reference frame associated with the vortex moving with a variable speed $\boldsymbol{V}_c$ relative to the laboratory reference frame. Therefore the correlation function (6) is written as the double average

$$\langle \phi_{lf}[t, \boldsymbol{r} - \boldsymbol{R}(t)]\phi_{lf}(0, 0)\rangle,$$

where $\phi_{lf}(t, \boldsymbol{r})$ is the curl $\boldsymbol{f}$ in the laboratory reference frame, $\dot{\boldsymbol{R}}(t) = \boldsymbol{V}_c$, and $\boldsymbol{V}_c$ is the velocity of the vortex relative to the laboratory reference frame. The average should be taken both over the force $\boldsymbol{f}$ statistics in the laboratory reference frame and over the velocity $\boldsymbol{V}_c$ statistics. If the external forcing is short correlated in time in the laboratory reference frame then the correlation function of ϕ in the moving reference system is the same. Here we examine the general case of pumping with finite correlation time in the laboratory reference frame. Let us stress that, even the static pumping (in the laboratory reference frame) has a finite correlation time in the reference frame attached to the vortex, determined by the statistics of vortex movements [30]. The characteristic frequency (spectral width) of the pair correlation function (6) after the recalculation will be designated as Ω_f. One expects that $\Omega_f \gtrsim V_c k_f$, where V_c is the characteristic velocity of the vortex motion.

Aiming to reveal universal properties of the coherent vortex, we consider the distances r to the center of the vortex where the inequality $k_f r \gg 1$ is satisfied. At the condition the pair correlation function (3) is formed near the observation point, which we characterize by $r = R$, $\varphi = 0$, where φ is the polar angle. Then we can use the shear flow approximation for the mean velocity near the point. Introducing the variables $x_1 = r - R$ and $x_2 = R\varphi$ we find from Eq. (4) in the main approximation

$$\partial_t \varpi + U\partial_2 \varpi + x_1 \Sigma \partial_2 \varpi - (\nu \nabla^2 - \alpha)\varpi = \phi, \tag{7}$$

where $\nabla^2 = \partial_1^2 + \partial_2^2$ and the values of U, Σ are taken at $r = R$.

One can perform Fourier transform in terms of the coordinates x_1, x_2. We obtain from Eq. (7) for the component of ϖ with the wave vector $\boldsymbol{k} = (k_1, k_2)$:

$$\partial_t \varpi(\boldsymbol{k}) + ik_2 U \varpi(\boldsymbol{k}) - \Sigma k_2 \frac{\partial \varpi(\boldsymbol{k})}{\partial k_1} + \Gamma \varpi(\boldsymbol{k}) = \phi, \tag{8}$$

where $\Gamma = \nu k^2 + \alpha$. The relation (6) leads to

$$\langle \phi(t, \boldsymbol{k})\phi(t', \boldsymbol{q})\rangle = (2\pi)^2 \delta(\boldsymbol{k} + \boldsymbol{q}) k^2 \chi(t - t', \boldsymbol{k}), \tag{9}$$

$$\chi(t, \boldsymbol{k}) = \int \frac{d\omega}{2\pi} F(\omega, \boldsymbol{k}) e^{-i\omega t}. \tag{10}$$

The assumed isotropy of the pumping statistics means that F depends solely on $k = \sqrt{k_1^2 + k_2^2}$.

Equation (8) is of the first order in the derivative over $\boldsymbol{k}$. Therefore it can be solved by the method of characteristics to obtain

$$\varpi(t, \boldsymbol{k}) = \int_0^\infty d\tau \, \exp(-ik_2 U\tau) W(\tau, \boldsymbol{k}) \times \phi(t - \tau, k_1 + \tau\Sigma k_2, k_2), \tag{11}$$

where the factor $W(\tau, \boldsymbol{k})$ is determined by the damping

$$\ln W(\tau, \boldsymbol{k}) = -\int_0^\tau ds \, \Gamma\left(\sqrt{[k_1 + s\Sigma k_2]^2 + k_2^2}\right). \tag{12}$$

We imply that the pumping acts during a long time and therefore the integration over time τ in Eq. (11) can be extended up to ∞.

The simultaneous pair correlation function of the vorticity ϖ can be then found from Eqs. (9) and (11). The explicit expression is

$$\begin{aligned} &\langle \varpi(t, \boldsymbol{k})\varpi(t, \boldsymbol{q})\rangle \\ &= (2\pi)^2 \delta(k_2 + q_2) \\ &\times \int_0^\infty d\tau_1 \int_0^\infty d\tau_2 \, e^{-ik_2 U(\tau_2 - \tau_1)} W(\tau_1, \boldsymbol{k}) W(\tau_2, \boldsymbol{q}) \\ &\times \delta[k_1 + q_1 + \Sigma k_2(\tau_1 - \tau_2)]\left[(k_1 + \tau_1 \Sigma k_2)^2 + k_2^2\right] \\ &\times \chi\left[\tau_1 - \tau_2, \sqrt{(k_1 + \tau_1 \Sigma k_2)^2 + k_2^2}\right]. \end{aligned} \tag{13}$$

At the next step we pass to the Fourier components of the velocity in accordance with the relations

$$v_\varphi(\boldsymbol{k}) = -i\frac{k_1}{k^2}\varpi(\boldsymbol{k}), \quad v_r(\boldsymbol{k}) = i\frac{k_2}{k^2}\varpi(\boldsymbol{k}).$$

Using the relations, one obtains the simultaneous correlation function of the velocity fluctuations from Eq. (13).

Then we pass to the stress (3). Taking it at the point $\boldsymbol{x} = 0$ one finds

$$\begin{aligned} \Pi = \langle v_\varphi v_r \rangle &= \int \frac{d^2k d^2q}{(2\pi)^4} \langle v_\varphi(\boldsymbol{k}) v_r(\boldsymbol{q})\rangle \\ &= -\int \frac{d^2k d^2q}{(2\pi)^4} \frac{k_1 q_2}{k^2 q^2} \langle \varpi(\boldsymbol{k})\varpi(\boldsymbol{q})\rangle, \end{aligned} \tag{14}$$

where the expression (13) has to be substituted. In the expression we change the order of integrations and then substitute $k_1 + \Sigma k_2 \tau_1$ by k_1 and $q_1 + \Sigma q_2 \tau_2$ by q_1. After the substitution we arrive at the factor $\delta(\boldsymbol{q} + \boldsymbol{k})$ in the integrand. Performing then the trivial integration over $\boldsymbol{k}$, we find

$$\begin{aligned} \Pi = &-\int \frac{d^2q}{(2\pi)^2} \int_0^\infty d\tau_1 \int_0^\infty d\tau_2 e^{-iq_2 U(\tau_2 - \tau_1)} \\ &\times \frac{\boldsymbol{q}^2 q_2 (q_1 - \tau_1 \Sigma q_2)}{\left[(q_1 - \tau_1 \Sigma q_2)^2 + q_2^2\right]\left[(q_1 - \tau_2 \Sigma q_2)^2 + q_2^2\right]} \\ &\times \tilde{W}(\tau_1, \boldsymbol{q}) \tilde{W}(\tau_2, \boldsymbol{q}) \chi(\tau_1 - \tau_2, q), \end{aligned} \tag{15}$$

where

$$\ln \tilde{W}(t, \boldsymbol{q}) = -\alpha\tau - \nu q^2 \tau + \nu \Sigma \tau^2 q_1 q_2 - \nu \Sigma^2 q_2^2 \tau^3 / 3. \tag{16}$$

The expression is obtained from Eq. (12) after the substitution $\Gamma(q) = \alpha + \nu q^2$, $\Sigma \to -\Sigma$.

The expression (15) can be further simplified. One can estimate $q_2(\tau_1 - \tau_2) \sim \min(1/U, \Omega_f k_f)$, where Ω_f is the spectral width of the pumping. In any case, $|\tau_1 - \tau_2| \ll \tau_1$, since $q_2 \tau_1 \sim k_f/\Sigma$, that is, $q_2 \tau_1 U \sim k_f R \gg 1$. Therefore one can substitute $\tau_1, \tau_2 \to \tau$ everywhere, excluding the argument of χ. Integrating then over the difference $\tau_1 - \tau_2$ we find

$$\begin{aligned} \Pi = &-\int \frac{d^2q}{(2\pi)^2} \boldsymbol{q}^2 F(q_2 U, q) \\ &\times \int_0^\infty d\tau \frac{q_2(q_1 - \tau\Sigma q_2)}{\left[(q_1 - \tau\Sigma q_2)^2 + q_2^2\right]^2} \tilde{W}^2(\tau, \boldsymbol{q}). \end{aligned} \tag{17}$$

Here the factor $\tilde{W}^2$ guarantees convergence of the integral.

The factor $F(q_2 U, q)$ in Eq. (17) has a transparent physical interpretation: energy is transferred from fluctuations to the coherent flow in a resonant manner. If the fluctuation has the wave vector $\boldsymbol{q}$ then its frequency should be equal to $\omega = \boldsymbol{q}\boldsymbol{U}$ to be in resonance with the coherent flow. This is the mechanism analogous to the well-known Landau antidamping in plasma, when nonequilibrium (but randomized) particles do not quench, but rock the Langmuir wave [32].

The expression (17) is analyzed in the Appendix A. It is demonstrated there that the stress Π can be represented as the sum of three terms

$$\Pi = \Pi_0 + \Pi_1 + \Pi_\nu. \tag{18}$$

The contributions Π_0, Π_1, Π_ν are defined as

$$\Pi_0 = \frac{1}{2\Sigma}\int \frac{d^2q}{(2\pi)^2} F(q_2U, q). \tag{19}$$

$$\Pi_\nu = -\frac{\nu}{\Sigma}\int \frac{d^2q}{(2\pi)^2}\int_0^\infty d\tau\, \boldsymbol{q}^2 F(q_2U, q)\tilde{W}^2. \tag{20}$$

$$\Pi_1 = -\frac{1}{2\Sigma}\int \frac{d^2q}{(2\pi)^2}\frac{\alpha}{\alpha+\nu q^2/3} F(0, q), \tag{21}$$

where the factor $\tilde{W}$ is determined by the expression (16). The contribution (19) to the stress is positive whereas the contributions (20) and (21) are negative.

In accordance with Eqs. (18) to (21) the stress is written as $\Pi = \mu/\Sigma$, where μ is a function of U. Substituting the expression into Eq. (1) and neglecting the viscous term (which is relevant solely in the vortex core) we find the equation

$$\alpha U = -(\partial_r + 2/r)(\mu/\Sigma).$$

Combining the equation with Eq. (2), one finds the flat velocity profile (U is independent of r). The value of U is determined by the relation $U^2 = 3\mu/\alpha$. To find U, one should know a dependence of μ on U. Any case, μ should be positive to support the coherent vortex.

The characteristic q in the integrals (19) to (21) can be estimated as k_f. Therefore the value of Π depends on the ratio $\nu k_f^2/\alpha$. In laboratory experiments $\nu k_f^2/\alpha \lesssim 1$ since α can be estimated as ν/h^2 (h is the thickness of the fluid film) and $k_f h \lesssim 1$ (otherwise one cannot treat the flow as two-dimensional one). Say, in the numerical simulations reported in Ref. [33], the inverse cascade appears at $k_f h < 3$. However, if one thinks about a numerical check of our analytical results then the ratio $\nu k_f^2/\alpha$ can be arbitrary. Note also that the ratio $\nu k_f^2/\alpha$ can be increased by using the two-layer experimental setup [17,31]. Therefore below we present an analysis for an arbitrary value of the ratio $\nu k_f^2/\alpha$.

Consider first the case $\nu k_f^2/\alpha \gg 1$ which can be realized in numerical simulations. Then the contribution Π_1 (21) is less than Π_0 (19). In the integral (20) $q_1 \sim q_2 \sim k_f$ and the integral over τ is determined by the third order in the τ term in the expression (16). Then $\tau \sim (\nu k_f^2 \Sigma^2)^{-1/3}$ and $\Sigma\tau \sim [\Sigma/(\mu k_f^2)]^{1/3} \gg 1$, which justifies the assertion. Therefore we find

$$\Pi_\nu \sim \nu k_f^2 \tau \Pi_0 \sim (\nu k_f^2/\Sigma)^{2/3}\Pi_0 \ll \Pi_0,$$

because of the inequality (5). Thus, in this case, the contribution (19) to Π is the leading one. Therefore $\mu > 0$ and in this case the coherent vortex can be realized (in the quasilinear regime).

Now we turn to the opposite case $\nu k_f^2/\alpha \ll 1$. For the model of the pumping short correlated in time where $F(\omega, q)$ is independent of ω, the expression (A3) cancels Π_0 (19) in the limit $\nu \to 0$, as it was noted in our previous work [20]. For finite (but small) $\nu k_f^2/\alpha$ in the same short-correlated case we find

$$\Pi \approx -\frac{1}{6\Sigma}\int \frac{d^2q}{(2\pi)^2}\frac{\nu q^2}{\alpha} F(q_2U, q).$$

Thus, Π is negative and is small in $\nu k_f^2/\alpha$. For a finite correlation time of the pumping the situation is even more dramatic. Then in the main approximation

$$\Pi \approx \frac{1}{2\Sigma}\int \frac{d^2q}{(2\pi)^2}[F(q_2U, q) - F(0, q)].$$

The quantity is negative and has no smallness in $\nu k_f^2/\alpha$. We conclude that $\mu < 0$ in the limit $\nu k_f^2/\alpha \ll 1$ and therefore the coherent vortex cannot exist in the quasilinear regime.

The above analysis shows that Π changes its sign at increasing $\nu k_f^2/\alpha$. Therefore the possibility to observe the coherent vortex (in the quasilinear regime) is realized at $\nu k_f^2/\alpha > C$, where C is of order unity. The fact imposes an essential restriction on the experimental setup where the coherent vortices can be observed. Thus the experimental efforts to diminish α (as in Refs. [17,31]) are relevant. In all likelihood, the effect of disappearing the coherent vortex at increasing α was reported in [34] where it was called the transition from a strong to a weak condensate.

Our theoretical scheme can be straightforwardly generalized for the case of hyperviscosity frequently used in numerical simulations. Then the expression $\Gamma = \alpha + \nu k^2$ for the fluctuation damping is substituted by $\Gamma = \alpha + \tilde{\nu} k^{2n}$ in Eq. (8). Then we have to compare α and $\tilde{\nu} k_f^{2n}$: in the limit $\tilde{\nu} k_f^{2n} \gg \alpha$ we obtain the stress tensor Π close to the expression (19), whereas in the opposite case we find a negative value of Π. Note that in Ref. [19], where the hyperviscosity was used for numerical simulations, the inequality $\tilde{\nu} k_f^{2n} \gg \alpha$ was satisfied. That was why the flat velocity profile was observed in the work.

There is another case where the flow is effectively two-dimensional. We imply the case of fast rotation where column vortices can appear [35–38]. The vortices can be described by practically the same equations that are examined in our paper excluding the bottom friction. Therefore all the conclusions based on the representation (19) are applicable to the case as well.

We used the model of isotropic pumping. However, experimental and numerical simulations show that even in the case of strongly anisotropic pumping (say, containing a single harmonic) the produced vortices are isotropic. That can be explained by an effective averaging caused by fast rotation. In this situation our scheme is directly applicable. One should only add averaging over angles (in the reference system attached to the vortex) to the time averaging.

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APPENDIX

Here we analyze the expression (17) for the stress Π obtained in the main body. One identically rewrites the expression (17) as

$$\Pi = -\frac{1}{2\Sigma}\int \frac{d^2q}{(2\pi)^2} q^2 F(q_2U, q)\int_0^\infty d\tau\, \exp(-2\alpha\tau) \times \partial_\tau \frac{\exp\left(-2\nu q^2\tau + 2\nu\Sigma\tau^2 q_1q_2 - 2\nu\Sigma^2 q_2^2\tau^3/3\right)}{(q_1-\tau\Sigma q_2)^2 + q_2^2} + \Pi_\nu, \quad (A1)$$

where Π_ν is given by Eq. (20). At deriving Eqs. (20) and (A1) we exploit the explicit expression (16). Taking the integral over τ in Eq. (A1) by parts one obtains Eq. (18) where

$$\Pi_1 = -\frac{\alpha}{\Sigma}\int \frac{d^2q}{(2\pi)^2} q^2 F(q_2U, q)\int_0^\infty d\tau\, \exp(-2\alpha\tau) \times \frac{\exp\left(-2\nu q^2\tau + 2\nu\Sigma\tau^2 q_1q_2 - 2\nu\Sigma^2 q_2^2\tau^3/3\right)}{(q_1-\tau\Sigma q_2)^2 + q_2^2}, \quad (A2)$$

and Π_0 is defined by Eq. (19).

Since $\nu k_f^2 + \alpha \ll \Sigma$, see Eq. (5), then $\Sigma\tau \sim \Sigma/(\nu k_f^2 + \alpha) \gg 1$ and the integral over q_2 in Eq. (A2) is gained at $q_2 \sim q_1/(\tau\Sigma) \ll q_1$. Therefore one can substitute

$$\frac{1}{(q_1-\tau\Sigma q_2)^2 + q_2^2} \to \frac{\pi}{|q_1|}\delta[q_2 - q_1/(\tau\Sigma)].$$

Producing the substitution and taking the integral over q_2 in Eq. (A2), one finds

$$\Pi_1 = -\frac{\alpha}{\Sigma}\int_{-\infty}^{+\infty}\frac{dq_1}{4\pi}|q_1|\int_0^\infty d\tau \times \exp\left(-2\alpha\tau - \nu q_1^2\tau/3\right) F\left(\frac{q_1U}{\tau\Sigma}, |q_1|\right). \quad (A3)$$

Since in our approximation $\tau\Sigma \gg 1$ then it is possible to substitute by zero the first argument in F in Eq. (A3), to obtain

$$\Pi_1 = -\frac{1}{\Sigma}\int_0^\infty \frac{dq_1}{4\pi} q_1 \frac{\alpha}{\alpha + \nu q_1^2/3} F(0, q_1). \quad (A4)$$

The expression can be identically rewritten as Eq. (21).

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