Relationship between the Hamiltonian and non-Hamiltonian forms of a fourth-order nonlinear Schrödinger equation

Yu. V. Sedletsky [*] and I. S. Gandzha [†]
Institute of Physics, National Academy of Sciences of Ukraine, Prosp. Nauky 46, Kyiv 03028, Ukraine

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We show the equivalence between the Hamiltonian and non-Hamiltonian forms of a fourth-order nonlinear Schrödinger equation for a particular example of the physical system described by the nonlinear Klein-Gordon equation with cubic nonlinearity.

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I. INTRODUCTION

The nonlinear Schrödinger equation (NLSE) describes the slowly modulated envelope of a rapidly oscillating nonlinear wave train, which is strongly dispersive, nearly monochromatic (has a narrow spectrum), and weakly nonlinear (of a small amplitude) [1]. The NLSE coefficients are determined as functions of the parameters of a particular physical medium by expanding (1) the equations of motion, (2) Lagrangian, or (3) Hamiltonian in terms of small amplitude A. These three approaches yield the same results in the case of standard cubic NLSE (NLSE3).

In the case of a fourth-order NLSE (the so-called NLSE4 model, where the total maximum power of amplitude A and its derivatives with respect to coordinate is equal to four), the non-Hamiltonian and Hamiltonian forms of this equation are nonequivalent. Non-Hamiltonian high-order NLSEs can be derived from the equations of motion (see, e.g., Refs. [2–7]) or from the associated Lagrangian [8], while Hamiltonian high-order NLSEs are derived from Hamilton's equations (see, e.g., Refs. [9–12]). An infinite hierarchy of integrable NLSEs was derived in Ref. [13], with the Hamiltonian being one of its invariants.

The Hamiltonian approach is generally considered as canonical in the nonlinear theory of waves [14]. On the other hand, the non-Hamiltonian form of NLSE4 includes the $A^2\overline{A}_x$ term (A_x being the partial derivative with respect to coordinate, and the bar designating the complex conjugate). This term is absent in the Hamiltonian counterpart of this equation written in terms of another complex amplitude u defined as a function of a complex symplectic coordinate z [10].

The purpose of this work is to show the complete equivalence between the Hamiltonian and non-Hamiltonian forms of NLSE4 for a particular example of the physical system described by the nonlinear Klein-Gordon equation with cubic nonlinearity. To this end, we employ the transformation of variables that unambiguously transforms the "noncanonical" form of NLSE4 for the amplitude A to the "canonical" form for the amplitude u.

Note that in this article we deal only with the so-called narrow-banded spectrum approximation to describing the propagation of nonlinear waves. There exist much more general models going beyond this approximation; see, e.g., Ref. [15] and references therein. In the Hamiltonian approach, such a more general model is given by the celebrated Zakharov integro-differential equation [16], which describes nonlinear four-wave interactions within a spectrum of surface waves (see also Ref. [17] for more general four- and five-wave forms of the Zakharov equation that preserve the Hamiltonian). NLSE3 and NLSE4 can generally be obtained as a narrow-band limit of the Zakharov equations [11]. Additional conformal mappings and canonical transformations allow the cancellation of certain nontrivial four-wave resonant interactions and produce the so-called compact [18] and supercompact [19] modifications of the Zakharov equations. These modified Zakharov equations result in more general envelope equations valid without usual narrow-banded spectrum approximation [20,21].

Nevertheless, for reasons of simplicity and brevity, the main focus of this work stays in the classical NLSE framework. We refer to a simple example of a physical system with known Hamiltonian and non-Hamiltonian representations. We prove these two representations to be fully equivalent and interdeducible by a simple transformation of variables. This result allows us to suppose the same equivalence in more complicated systems and examples, in particular in those outlined above.

II. NONLINEAR KLEIN-GORDON EQUATION AND NLSE4

A. Nonlinear Klein-Gordon equation

We use the nonlinear Klein-Gordon (nKG) equation with cubic nonlinearity as an example of the model equation that is governing the wave evolution:

$$\phi_{tt} - c^2\phi_{xx} + \alpha_1\phi + \alpha_3\phi^3 = 0. \tag{1}$$

Here the unknown real function ϕ is a characteristic of the wave field, t is time, x is coordinate, and c is the velocity parameter that deals with the speed of interaction propagation. The subscripts denote the partial derivatives. In particular, Eq. (1) describes the ϕ^4 model in the quantum field theory

[*] sedlets@iop.kiev.ua
[†] gandzha@iop.kiev.ua

[22] and represents the leading terms of the more general sine-Gordon equation, which has multiple physical applications [23] (see also Ref. [24] for more details). The real coefficients α_1 and α_3 describe the linear and nonlinear responses of the medium, respectively.

B. Non-Hamiltonian form of NLSE4

The unknown function ϕ is represented as the Fourier expansion over the basis of rapidly oscillating harmonic functions with slowly modulated amplitudes:

$$\phi = \tfrac{1}{2}[\varepsilon A(\chi,\tau)\exp(i\theta) + \varepsilon^3 A_3(\chi,\tau)\exp(3i\theta) + \text{c.c.}], \quad (2)$$

where $\theta(x_0, t_0) = k_0x_0 - \omega_0 t_0$ is the carrier phase (k_0 and ω_0 being the carrier wave number and frequency, respectively),

$$\omega(k) = \sqrt{c^2k^2 + \alpha_1} \quad (3)$$

is the liner dispersion relation, and c.c. denotes the complex conjugate terms. Note that expansion (2) misses the even and zeroth harmonics inasmuch as they are identically equal to zero when only the odd powers of the function ϕ are present in the nonlinear part of the nKG equation (1).

The complex amplitude $A(\chi,\tau)$ describes the slow envelope of the rapidly oscillating carrier with phase $\theta(x_0, t_0)$. The long coordinate χ and slow time τ are expressed in terms of short coordinate x_0 and fast time t_0 as (the so-called method of multiple scales),

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial x_0} + \varepsilon\frac{\partial}{\partial \chi}, \qquad \frac{\partial}{\partial t} = \frac{\partial}{\partial t_0} + \varepsilon\frac{\partial}{\partial \tau}, \quad (4)$$

and ε is a formal small parameter describing the smallness of the wave amplitude and the slowness of its modulations [7,8]. The complex amplitude A satisfies a high-order NLSE, which we write here up to the fourth order:

$$i\varepsilon^2(A_\tau + \omega_0' A_\chi) + \varepsilon^3\big(\tfrac{1}{2}\omega_0'' A_{\chi\chi} + q^{(3)}A^2\overline{A}\big) + i\varepsilon^4\big(-\tfrac{1}{6}\omega_0''' A_{\chi\chi\chi} + q_1^{(4)}A\overline{A}A_\chi + q_2^{(4)}A^2\overline{A}_\chi\big) = 0. \quad (5)$$

It was derived in a more general form (up to the sixth order) by the method of multiple scales [7], by the method of the averaged Lagrangian [8], and by the method of two-parameter expansions [5]. Noteworthy is that these three different approaches yielded the same results in all six orders of smallness.

The coefficients ω_0', ω_0'', and ω_0''' in the above equation are the derivatives of the linear dispersion relation $\omega(k)$ with respect to wave number k calculated at the point $k = k_0$. The coefficients $q^{(3)}$, $q_1^{(4)}$, and $q_2^{(4)}$ are expressed explicitly in terms of the parameters of the original nKG equation:

$$q^{(3)} = -\frac{3\alpha_3}{8\omega_0}, \quad (6)$$

$$q_1^{(4)} = 2r\,q^{(3)}, \qquad r \equiv \frac{\omega_0'}{\omega_0}, \quad (7)$$

$$q_2^{(4)} = \tfrac{1}{2}\,q_1^{(4)} = r\,q^{(3)}. \quad (8)$$

The non-Hamiltonian nature of Eq. (5) is manifested by the fact that $q_2^{(4)} \neq 0$, so that there exists no equivalent Hamiltonian.

C. Hamiltonian form of NLSE4

When the NLSE of form (5) is derived as Hamilton's equation for the Hamiltonian density

$$H = \tfrac{1}{2}\big(\phi_t^2 + c^2\phi_x^2 + \alpha_1\phi^2 + \tfrac{1}{2}\alpha_3\phi^4\big), \quad (9)$$

it is written in terms of the complex symplectic coordinate

$$z = \varepsilon u\,\exp(ik_0x) \quad (10)$$

as the gauge transformation of the corresponding complex envelope amplitude u, so that

$$iu_t = \frac{\delta H}{\delta \overline{u}},$$

with δ denoting the variational derivative [10]. (Such a gauge transformation was also considered in Ref. [25] in application to the problem of surface water waves.) The corresponding NLSE4 for the function u is written as

$$i\varepsilon^2(u_\tau + \omega_0' u_\chi) + \varepsilon^3\big(\tfrac{1}{2}\omega_0'' u_{\chi\chi} + Q^{(3)}u^2\overline{u}\big) + i\varepsilon^4\big(-\tfrac{1}{6}\omega_0''' u_{\chi\chi\chi} + Q_1^{(4)}u\overline{u}\,u_\chi + Q_2^{(4)}u^2\overline{u}_\chi\big) = 0, \quad (11)$$

where

$$Q^{(3)} = -\frac{3\alpha_3}{4\omega_0^2}, \qquad Q_1^{(4)} = 2r\,Q^{(3)},$$
$$Q_2^{(4)} \equiv 0. \quad (12)$$

Equation (11) has a Hamiltonian structure, inasmuch as the term proportional to $u^2\overline{u}_\chi$ vanishes.

III. EQUIVALENCE OF THE NON-HAMILTONIAN AND HAMILTONIAN FORMS OF NLSE4

The purpose of this section is to prove that Eqs. (5) and (11) are equivalent. First, we have to establish the relationship between the functions ϕ and z and between the amplitudes A and u.

A. Relationship between ϕ and z

In the Hamiltonian approach that was proposed by Craig *et al.* [10] (see also Refs. [12,26] for more details), the quadratic part of Hamiltonian (9),

$$H_2 = \tfrac{1}{2}\big(\phi_t^2 + c^2\phi_x^2 + \alpha_1\phi^2\big),$$

is represented in the following operator form:

$$H_2 = \tfrac{1}{2}\big[\phi_t^2 + (\widehat{\omega}\phi)^2\big].$$

Here $\widehat{\omega}$ is a pseudo-differential operator [27–29] (or the so-called Fourier multiplier operator) such that the wave number k in the linear dispersion relation (3) is replaced with the differential operator $(-i\partial_x)$:

$$\widehat{\omega} = \omega(-i\partial_x) = \sqrt{c^2(-i\partial_x)^2 + \alpha_1}. \quad (13)$$

Then we have

$$H_2 = \tfrac{1}{2}(i\phi_t + \widehat{\omega}\,\phi)(-i\phi_t + \widehat{\omega}\,\phi) \equiv \sqrt{\widehat{\omega}}z\,\sqrt{\widehat{\omega}}\overline{z},$$

where z is the so-called complex symplectic coordinate that was introduced instead of function ϕ,

$$z = \frac{1}{\sqrt{2}}\left(i\frac{1}{\sqrt{\widehat{\omega}}}\,\phi_t + \sqrt{\widehat{\omega}}\,\phi\right), \quad (14)$$

with the spatial derivative ϕ_x hidden in the operator $\widehat{\omega}$. Since

$$z + \bar{z} = \sqrt{2\widehat{\omega}}\,\phi, \tag{15}$$

the function ϕ is expressed in terms of z as follows:

$$\phi = \frac{1}{\sqrt{2\widehat{\omega}}}\,z + \text{c.c.} \tag{16}$$

The above equation means that Hamilton's equation in terms of the complex variable $\bar{z}$ is the complex conjugate to Hamilton's equation in terms of variable z.

B. Relationship between A and u

Next we use Theorem 1 from Ref. [10] (see also Theorem 4.1 from Ref. [30]) for the operator $\widehat{\omega}$ and some sufficiently smooth function $f(\chi)$, namely,

$$\widehat{\omega}(-i\partial_x)[\exp(ik_0x)\,f(\chi)] = \exp(ik_0x)\,\widehat{\omega}(k_0 - i\varepsilon\partial_\chi)f(\chi). \tag{17}$$

This formula basically means the operator expansion around the carrier wave number k_0 with the assumption of narrow spectrum and slow modulations. Then, by using formula (17) with Eq. (16), we have

$$\phi = \frac{1}{\sqrt{2\omega_0}}\left[z + i\varepsilon\,\frac{r}{2}\,z_\chi + O(\varepsilon^2)\right] + \text{c.c.}, \tag{18}$$

where we took into account that

$$\frac{1}{\sqrt{\widehat{\omega}(k_0 - i\varepsilon\partial_\chi)}} = \frac{1}{\sqrt{\omega_0}} + i\varepsilon\frac{\omega_0'}{2\omega_0\sqrt{\omega_0}}\,\partial_\chi + O(\varepsilon^2). \tag{19}$$

Finally, by substituting relations (2) and (10) in Eq. (18) and averaging over the fast phase, we get

$$\frac{1}{2}A = \frac{1}{\sqrt{2\omega_0}}\left(u + i\varepsilon\frac{r}{2}\,u_\chi\right). \tag{20}$$

Similarly, by using formula (17) with Eq. (15), one can show that

$$u = \sqrt{\frac{\omega_0}{2}}\left(A - i\varepsilon\frac{r}{2}A_\chi\right). \tag{21}$$

Thus, the complex amplitude u is a combination of the amplitude A and its small derivative A_χ, the latter describing the slow variation of the envelope amplitude. The same relationship was also obtained in the Fourier space in the framework of the traditional (Zakharov's) Hamiltonian formalism in the nonlinear theory of surface water waves [see Eq. (B3a) from Ref. [11]].

C. Proof

Now, the proof of equivalence of Eqs. (5) and (11) is straightforward. By substituting relation (21) in Eq. (11), we easily come to Eq. (5) with

$$q^{(3)} = \frac{\omega_0}{2}Q^{(3)} = -\frac{3\alpha_3}{8\omega_0},$$

$$q_1^{(4)} = \frac{\omega_0}{2}Q_1^{(4)} = 2r\,q^{(3)},$$

$$q_2^{(4)} = \frac{\omega_0}{2}\left(Q_2^{(4)} + r\,Q^{(3)}\right) = \frac{1}{2}\,q_1^{(4)}.$$

IV. CONCLUSIONS

Thus, we demonstrated that the coefficients of the Hamiltonian form of NLSE4 expressed in terms of variable u (which was derived as Hamilton's equations for the nKG equation [10]) could unambiguously be transformed into the same coefficients of the non-Hamiltonian form of NLSE4 expressed in terms of variable A (which was derived as variational equations for the averaged Lagrangian [8] or as the multiple-scale expansion of the nKG equation [7]). Therefore, the non-Hamiltonian and Hamiltonian forms of NLSE4 derived in Refs. [7,8,10] are equivalent, which proves the validity of these three approaches for the reduction of the nKG equation to a high-order NLSE.

We believe that the results presented here will be useful in facilitating the ongoing discussions on the equivalence between the Hamiltonian and non-Hamiltonian forms of high-order NLSEs.

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