Quantum Otto engine working with interacting spin systems: Finite power performance in stochastic thermodynamics

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A quantum Otto engine using two-interacting spins as its working medium is analyzed within framework of stochastic thermodynamics. The time-dependent power fluctuations and average power are explicitly derived for a complete cycle of engine operation. We find that the efficiency and power fluctuations are affected significantly by interparticle interactions, but both of them become interaction-independent under maximal power via optimizing the external control parameter. The behavior of the efficiency at maximum power is further explained by analyzing the optimal protocol of the engine.

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I. INTRODUCTION

A heat engine working between a hot and a cold bath with constant inverse temperatures of β_h and $\beta_c(>\beta_h)$, bounded by the Carnot efficiency $\eta_C = 1-\beta_h/\beta_c$ due to the second law of thermodynamics, is an energy converter composed of different consecutive nonequilibrium processes, allowing us to understand the thermodynamics of quantum systems. As a cyclic engine in practice should take finite time to implement the processes for completing a cycle, its working system should always stay in nonequilibrium states during these thermodynamic processes. Nonequilibrium thermodynamics and statistics in quantum systems, together with the experimental realization of open quantum systems such as trapped particles [1–5], quantum dots [6], and molecules [7,8], have ignited much effort to study performance of quantum heat engines under interparticle interactions [9,10], quantum effects [11–18], nonthermal baths [19–23], and fluctuations [21,24]. Although these effects may cause novel performance of quantum heat engines beyond their classical counterparts [12,21,23,25], there has been evidence that quantum and classical heat engines share the same universal behaviors in certain regimes. For instance, in the linear response regime, heat engines ranging from microscale to macroscale have the universality of efficiency at maximum power: $\eta^* = \eta_C + \eta_C^2/8 + O(\eta_C^3)$ [9,25–35], which is exactly the same as the second-order expansion of the so-called Curzon-Ahlborn (CA) efficiency [36] $\eta_{CA} = 1-\sqrt{\beta_h/\beta_c} = 1-\sqrt{1-\eta_C}$. The investigation on possible bounds of the efficiency at maximum power has recently seen increased interest triggered by the low-dissipation Carnot-like model [37], showing that the efficiency for engines under maximal power must be situated between the upper and lower limits [34,37].

Different theoretical frameworks of thermodynamics were used to investigate the performance of the models of (quantum) heat engines, such as finite-time (quantum) thermodynamics [26,38], irreversible thermodynamics [25,26,35,39,40], and stochastic thermodynamics [25,41]. The relationship between finite-time and irreversible thermodynamics was examined by studying the finite-power performance of heat engines with their working substance ranging from classical to quantum systems [26,35]. In most studies of quantum heat engines, the time evolution of the working system was examined to obtain the finite-time performance of quantum heat engines[13,14,26,30,38] within the framework of quantum thermodynamics. However, the dynamics for a quantum system weakly coupled to a heat reservoir could be stochastic and classical, but the energy levels of the systems are quantized and thus some quantum effects are maintained in these systems [42,43]. As far as we know, there has been no comprehensive discussion of cyclic quantum heat engines under interparticle interactions from the stochastic thermodynamic point of view, in which the fluctuating macroscopic quantities describing engine performance can be obtained.

In the present paper, we consider the performance in finite time of a quantum Otto engine which uses two interacting spins as its working substance. The time-duration-dependent expressions of efficiency and power as well as power fluctuations are derived analytically via a stochastic dynamical description of the system based on weak system-bath coupling. We find that the efficiency and power fluctuations can be increasing or decreasing by tuning the strength of interparticle interactions, but the efficiency at maximum power is shown to be independent of interparticle interaction and identical to that obtained [26,30] from quantum heat engines within a framework of quantum thermodynamics. The physical implication of the efficiency at maximum power is discussed via optimization on minimal irreversible entropy production.

The paper is organized as follows. In Sec. II we derive the expressions of power and efficiency as well as power fluctuations by using the stochastic master equation and cyclic (peri-

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odic) constraint, and we find that the interparticle interactions can enhance the efficiency at the price of increasing power fluctuations. We then determine the efficiency at maximum power in Sec. III. Next, using the strategy of optimization based on the Euler-Lagrange equation, we show in Sec. IV that the heat engine under the optimal protocol of minimal irreversible entropy production falls into the low-dissipation model, thereby confirming the universality of efficiency at maximum power in linear responses. Finally, a discussion and conclusions are given in Sec. V.

II. DYNAMICAL ANALYSIS OF A QUANTUM OTTO ENGINE BASED ON TWO INTERACTING SPINS

A. Motion equation of the system

The working substance of the heat engine under consideration consists of a system of two interacting spins [44], with its Hamiltonian being

$$\hat{H}(\omega) = \hat{H}^{\text{int}} + \hat{H}^{\text{ext}}(\omega). \tag{1}$$

$\hat{H}^{\text{int}}$ is the internal part of the Hamiltonian, which denotes the interaction between the two spins, and $\hat{H}^{\text{ext}}(\omega)$ represents the control Hamiltonian that depends on the external control field ω. The internal interaction Hamiltonian and the external Hamiltonian for the system of two coupled spins can be given by ($\hbar \equiv 1$)

$$\hat{H}^{\text{int}} = \frac{1}{2} j\left(\hat{\sigma}_x^1 \otimes \hat{\sigma}_x^2 - \hat{\sigma}_y^1 \otimes \hat{\sigma}_y^2\right) \equiv j\hat{B}_2, \tag{2}$$

$$\hat{H}^{\text{ext}}(\omega) = \frac{1}{2}\omega\left(\hat{\sigma}_z^1 \otimes \hat{I}^2 + \hat{I}^1 \otimes \hat{\sigma}_z^2\right) \equiv \omega\hat{B}_1, \tag{3}$$

where $\hat{\sigma}$ represents the spin-Pauli operator, and j scales the interaction strength between two spins. We assume j to be constant for a given external control filed with constant ω. Diagonalizing the Hamiltonian (1), we find that the set of energy eigenvalues reads

$$\varepsilon_{-1} = -\Omega, \quad \varepsilon_0 = \varepsilon_0 = 0, \quad \varepsilon_1 = \Omega, \tag{4}$$

where $\Omega \equiv \sqrt{\omega^2 + j^2}$. The dynamics of the occupations at these states can be described via a master equation

$$\dot{\mathbf{p}} = R\mathbf{p}, \tag{5}$$

where R is the stochastic transition matrix. It is well known that the systems under an external drive do not necessarily reach the thermal equilibrium even after an infinite long time. These may be an integrable system which is in contact with thermal baths [45], and an ergodic nonintegrable system [46,47] in which a subsystem is acted upon by the rest of the system playing the role of a heat bath. As the external drive is set to be frozen when the system is coupled to a heat reservoir of constant inverse temperature β with $\beta = 1/T\,(k_B \equiv 1)$, this system would be reach the thermal equilibrium after infinite long time. At thermal equilibrium, the occupation probabilities p_n must achieve their equilibrium values π_n, which are obtained by the steady-state solution of Eq. (5) and given by the Boltzmann distribution:

$$\pi_n(\beta) = \frac{e^{-\beta\varepsilon_n}}{Z_\beta}, \tag{6}$$

where $Z_\beta = \sum_n e^{-\beta\varepsilon_n}$ is the canonical partition function and the energy ε_n of state n was given by Eq. (4). The average population $\langle n\rangle^{\text{eq}}$ for the system at thermal equilibrium can be calculated as $\langle n\rangle^{\text{eq}} = \sum_n n\pi_n(\beta) = -\tanh(\beta\Omega/2)$. We assume that the elements R_{nm} which denote the transition rate from state m to n fulfill the detailed balance, $R_{nm}e^{-\beta\varepsilon_m} = R_{mn}e^{-\beta\varepsilon_n}$, such that the system can asymptotically achieve the thermal equilibrium in a specific manner, and R_{mn} are in the Arrhenius form of

$$R_{mn} = \begin{cases} \gamma e^{-\beta(B_{mn}-E_n)} & m \neq n \\ -\sum_{l\neq n} R_{ln} & n = m' \end{cases} \tag{7}$$

where $\gamma > 0$ is a constant rate for these transitions and $B_{mn} = B_{nm}$ is the energy barrier between states m an n. From Eq. (7), the stochastic transition matrix for the system under consideration is obtained as

$$R = \begin{pmatrix} -2k_\uparrow & k_\downarrow & k_\downarrow & 0 \\ k_\uparrow & -(k_\uparrow + k_\downarrow) & 0 & k_\downarrow \\ k_\uparrow & 0 & -(k_\uparrow + k_\downarrow) & k_\downarrow \\ 0 & k_\uparrow & k_\uparrow & -2k_\downarrow \end{pmatrix}, \tag{8}$$

where $k_\uparrow$ and $k_\downarrow$ are parameterized by $k_\uparrow = \frac{\gamma}{2}(1 + \langle n\rangle^{\text{eq}})$ and $k_\downarrow = \frac{\gamma}{2}(1 - \langle n\rangle^{\text{eq}})$.

B. Stochastic analysis of a quantum Otto cycle

Now we consider the time evolution of the probability for the quantum Otto engine per cycle in advance. Throughout the paper we call the engine a "quantum Otto engine" just because of the discrete energy spectrum (4), meaning that any genuine quantum is not considered in our approach. During a cycle, the interacting spin system as the working substance is alternatively coupled to two heat baths at inverse temperatures β_c and $\beta_h(< \beta_c)$, with the external control field $\omega(t)$ changing between ω_h and $\omega_c(< \omega_h)$. The interaction strength j is set to be $j = j_h$ and $j = j_c$ for the hot ($\omega = \omega_h$) and cold ($\omega = \omega_c$) isochoric processes, respectively. The four consecutive steps in a single cycle are described as follows.

(1) *Isochoric heating:* The frequency ω is fixed at constant value ω_h and thus no work is produced. For this step the system is coupled to the heat reservoir at temperature β_h in a period τ_h. We assume the initial time of the isochoric heating to be zero. From Eq. (5), the probabilities $\mathbf{p}(t)$ at any instant of the isochore ($0 \leqslant t \leqslant \tau_h$) can be obtained as

$$\mathbf{p}(t) = \exp(R_h t)\mathbf{p}(0), \tag{9}$$

where

$$R_h = \frac{\gamma_h}{2}\begin{pmatrix} -2\langle n\rangle_h^+ & -\langle n\rangle_h^- & -\langle n\rangle_h^- & 0 \\ \langle n\rangle_h^+ & -2 & 0 & -\langle n\rangle_h^- \\ \langle n\rangle_h^+ & 0 & -2 & -\langle n\rangle_h^- \\ 0 & \langle n\rangle_h^+ & \langle n\rangle_h^+ & 2\langle n\rangle_h^- \end{pmatrix}, \tag{10}$$

where $\langle n\rangle_h^{\text{eq}} = -\tanh(\beta_h\Omega_h/2)$, with $\Omega_h = \sqrt{\omega_h^2 + j_h^2}$. Here and hereafter we define $\langle n\rangle_\nu^+ \equiv \langle n\rangle_\nu^{\text{eq}} + 1$, and $\langle n\rangle_\nu^- \equiv \langle n\rangle_\nu^{\text{eq}} - 1$, with $\nu = c, h$.

(2) *Adiabatic expansion:* Along this isentropic branch, where the system is isolated from the hot reservoir, the frequency changes from ω_h to ω_c in a period τ_{hc}. Constancy of entropy leads to constant probabilities,

$$\mathbf{p}(t) = \mathbf{p}(\tau_h), \tag{11}$$

with $\tau_h \leqslant t \leqslant \tau_h + \tau_{hc}$. We should keep in mind that the forms [Eqs. (9) and (11)] of time evolution in the adiabatic and isochoric process are applicable only for the system with long-time scales under consideration. These forms should not hold in general, especially on short-time scales, since quantum interference of the system might suppress some transitions.

(3) *Isochoric cooling:* The system with $\omega = \omega_c$ is coupled to a cold reservoir at inverse temperature β_c in a time of τ_c, and its probabilities $\mathbf{p}(t)$ evolve as

$$\mathbf{p}(t) = \exp(R_c t)\mathbf{p}(\tau_h + \tau_{hc}), \tag{12}$$

where

$$R_c = \frac{\gamma_c}{2}\begin{pmatrix} -2\langle n\rangle_c^+ & -\langle n\rangle_c^- & -\langle n\rangle_c^- & 0 \\ \langle n\rangle_c^+ & -2 & 0 & -\langle n\rangle_c^- \\ \langle n\rangle_c^+ & 0 & -2 & -\langle n\rangle_c^- \\ 0 & \langle n\rangle_c^+ & \langle n\rangle_c^+ & 2\langle n\rangle_c^- \end{pmatrix}, \tag{13}$$

with $\langle n\rangle_c^{\rm eq} = -\tanh(\beta_c\Omega_c/2)$ and $\Omega_c = \sqrt{\omega_c^2 + j_c^2}$.

(4) *Adiabatic compression:* The frequency ω changes back (as in the adiabatic expansion) to its initial value, with corresponding time duration τ_{ch}. When the time t satisfies $\tau_h + \tau_{hc} + \tau_c \leqslant t \leqslant \tau_h + \tau_{hc} + \tau_c + \tau_{ch}$, we have

$$\mathbf{p}(t) = \mathbf{p}(\tau_h + \tau_{hc} + \tau_c). \tag{14}$$

When the time evolution is described by Eqs. (12) and (14), the system evolution along the adiabatic and isochoric processes should be within long-time scales. By combining Eqs. (9), (11), (12), and (14), we find that

$$\begin{pmatrix} p_{-1}(\tau_{\rm cyc}) \\ p_0(\tau_{\rm cyc}) \\ p_0(\tau_{\rm cyc}) \\ p_1(\tau_{\rm cyc}) \end{pmatrix} = \mathcal{M}\begin{pmatrix} p_{-1}(0) \\ p_0(0) \\ p_0(0) \\ p_1(0) \end{pmatrix}, \tag{15}$$

where $\mathcal{M} = \exp(R_c\tau_c)\exp(R_h\tau_h)$ is the transition matrix for the interacting system proceeding a cycle and $\tau_{\rm cyc} = \tau_h + \tau_{hc} + \tau_c + \tau_{ch}$ denotes the cycle time. Since the cyclic engine is a periodic steady state, namely, $\mathbf{p}(0) = \mathbf{p}(\tau_{\rm cyc})$, the probabilities $\mathbf{p}(0)$ at the initial instant in a cycle is obtained by using Eq. (15),

$$\begin{pmatrix} p_{-1}(0) \\ p_0(0) \\ p_0(0) \\ p_1(0) \end{pmatrix} = \frac{\mathcal{G}}{4}\begin{pmatrix} [x_h(\langle n\rangle_h^{\rm eq} - \langle n\rangle_c^{\rm eq} + x_c\langle n\rangle_c^-) - \langle n\rangle_h^-]^2 \\ [\langle n\rangle_h^- - x_h(\langle n\rangle_h^{\rm eq} + x_c\langle n\rangle_c^- - \langle n\rangle_c^{\rm eq})][x_h(\langle n\rangle_h^{\rm eq} - \langle n\rangle_c^{\rm eq} + x_c\langle n\rangle_c^+) - \langle n\rangle_h^+] \\ [\langle n\rangle_h^- - x_h(\langle n\rangle_h^{\rm eq} + x_c\langle n\rangle_c^- - \langle n\rangle_c^{\rm eq})][x_h(\langle n\rangle_h^{\rm eq} - \langle n\rangle_c^{\rm eq} + x_c\langle n\rangle_c^+) - \langle n\rangle_h^+] \\ [\langle n\rangle_h^+ - x_h(\langle n\rangle_h^{\rm eq} - \langle n\rangle_c^{\rm eq} + x_c\langle n\rangle_c^+)]^2 \end{pmatrix}, \tag{16}$$

where $\mathcal{G} = (x_c x_h - 1)^{-2}$ with $x_h \equiv e^{\gamma_h\tau_h}$ and $x_c \equiv e^{\gamma_c\tau_c}$. A combination of Eqs. (9) and (16) gives rise to the probabilities at time $t = \tau_h$, $\mathbf{p}(\tau_h)$, which take the form

$$\begin{pmatrix} p_{-1}(\tau_h) \\ p_0(\tau_h) \\ p_0(\tau_h) \\ p_1(\tau_h) \end{pmatrix} = \frac{\mathcal{G}}{4}\begin{pmatrix} [x_c(\langle n\rangle_c^{\rm eq} - \langle n\rangle_h^{\rm eq} + x_h\langle n\rangle_h^-) - \langle n\rangle_c^-]^2 \\ [\langle n\rangle_c^- - x_c(\langle n\rangle_c^{\rm eq} + x_h\langle n\rangle_h^- - \langle n\rangle_h^{\rm eq})][x_c(\langle n\rangle_c^{\rm eq} - \langle n\rangle_h^{\rm eq} + x_h\langle n\rangle_h^+) - \langle n\rangle_c^+] \\ [\langle n\rangle_c^- - x_c(\langle n\rangle_c^{\rm eq} + x_h\langle n\rangle_h^- - \langle n\rangle_h^{\rm eq})][x_c(\langle n\rangle_c^{\rm eq} - \langle n\rangle_h^{\rm eq} + x_h\langle n\rangle_h^+) - \langle n\rangle_c^+] \\ [\langle n\rangle_c^+ - x_c(\langle n\rangle_c^{\rm eq} - \langle n\rangle_h^{\rm eq} + x_h\langle n\rangle_h^+)]^2 \end{pmatrix}. \tag{17}$$

Since for each cycle the work is produced only in the two adiabatic processes, the stochastic work done by the system is the total work produced along the two adiabatic microscopic trajectories, implying that the stochastic work should be

$$w[|n(\tau_h)\rangle; |n(\tau_{\rm cyc} - \tau_{ch})\rangle] = (\Omega_h - \Omega_c)[n(\tau_h) - n(\tau_{\rm cyc} - \tau_{ch})], \tag{18}$$

where $n(t)(= -1, 0, 0, 1)$ is a quantum number indicating the state that the system is occupying at time t. The states $|n(\tau_h)\rangle$ and $|n(\tau_{\rm cyc} - \tau_{ch})\rangle$ can be assumed to be independent since the system would relax to the equilibrium in an isochoric process if the time duration is long enough. The probability density of the work w is then given by

$$p(w) = \sum_n p_n(\tau_h)p_n(\tau_{\rm cyc} - \tau_{ch})\delta\{w - w[|n(\tau_h)\rangle; \times |n(\tau_{\rm cyc} - \tau_{ch})\rangle]\}, \tag{19}$$

where $\delta(\cdots)$ is the Dirac's δ function. Using $\langle n(t)\rangle = \sum_n np_n(t)$, the average work output per cycle is obtained as

$$\mathcal{W} \equiv \langle w\rangle = \int wp(w)\,dw = (\Omega_h - \Omega_c)[\langle n(\tau_h)\rangle - \langle n(0)\rangle], \tag{20}$$

where we have used $\langle n(0)\rangle = \langle n(\tau_{\rm cyc} - \tau_{ch})\rangle$ holding in the adiabatic compression. In the isochoric process, the heat absorbed by the system leads to an increase in its internal energy because no work is done. That is, the heat input into the system along the hot isochoric process is given by

$$\mathcal{Q}_h = \langle \hat{H}(\tau_h)\rangle - \langle \hat{H}(0)\rangle = \Omega_h[\langle n(\tau_h)\rangle - \langle n(0)\rangle]. \tag{21}$$

The machine efficiency can be given by

$$\eta = \frac{\mathcal{W}}{\mathcal{Q}_h} = 1 - \frac{\Omega_c}{\Omega_h} = 1 - \frac{\omega_c\sqrt{1 + j_c^2/\omega_c^2}}{\omega_h\sqrt{1 + j_h^2/\omega_h^2}}, \tag{22}$$

which simplifies to $\eta^0 = 1 - \omega_c/\omega_h$ in the absence of interaction. The efficiency η can be increased or decreased by tuning the strength of interaction j_c and j_h, as shown in Fig. 1(a), where we set $j_h = 1$. For $j_c/j_h < \omega_c/\omega_h$, interparticle

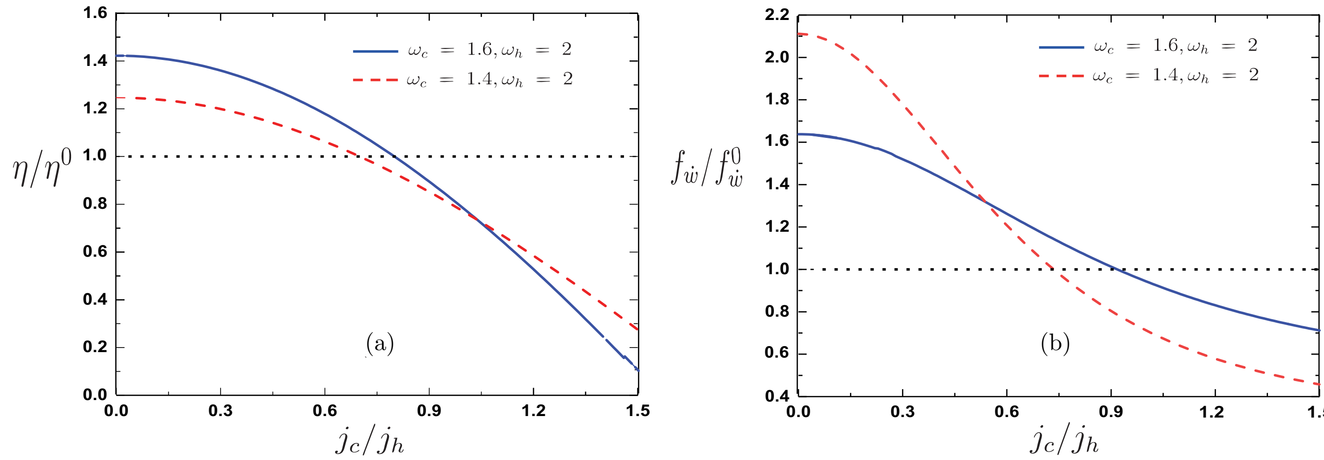

FIG. 1. Efficiency η (normalized to the noninteracting value η^0) versus the ratio of j_c to j_h (with $j_h = 1$ being adopted). For given $\omega_h = 2$, the values of ω_c are adopted, blue solid line: $\omega_c = 1.6$; and red dashed line 1.4, respectively.

interaction can significantly enhance the efficiency in magnitude as compared to noninteracting value η^0. We also note that the interaction reduces the efficiency for $j_c/j_h > \omega_c/\omega_h$ and has no effects on the efficiency when $j_c/j_h = \omega_c/\omega_h$.

The relation between the probabilities of initial and final states along an isochoric thermalization process can be identified for the cyclic engine. Combining Eqs. (9) and (16) [Eqs. (12) and (17)], we find that, for the hot (cold) isochore with $0 \leqslant t \leqslant \tau_h$ ($\tau_h + \tau_{hc} \leqslant t \leqslant \tau_{\rm cyc} - \tau_{ch}$), the instantaneous average population $\langle n(t)\rangle$ satisfies the relation

$$\langle n(t)\rangle = \langle n\rangle_\nu^{\rm eq} + \big[\langle n(t_i)\rangle - \langle n\rangle_\nu^{\rm eq}\big]e^{-\gamma_\nu t}. \tag{23}$$

This formula, obtained at steady state via use of periodic boundary conditions (15), has been derived previously using the Lindblad master equation [11,26]. While describing quantum dynamics of an open system a Lindblad approach must go beyond the classical treatment as presented in Eq. (5), both these approaches for the system weakly coupled to a heat reservoir are expected to give the same evolution of the average population when neglecting the quantum effects (like quantum coherence) which cause the system state to be nondiagonal.

Setting the derivatives of $\langle n(t)\rangle$ in Eq. (23) with respect to t, we find the solution at

$$\langle \dot n(t)\rangle = -\gamma(\langle n\rangle - \langle n\rangle^{\rm eq}), \tag{24}$$

where γ was defined in Eq. (7). Substituting $\langle n(\tau_h)\rangle = \langle n(\tau_h + \tau_{hc})\rangle$ and $\langle n(0)\rangle = \langle n(\tau_{\rm cyc} - \tau_{ch})\rangle$ required by quantum adiabatic condition into Eq. (23), we obtain

$$\langle n(\tau_h)\rangle - \langle n(0)\rangle = \Delta n^{\rm eq}\mathcal{F}(x_c, x_h), \tag{25}$$

where we defined the scaled time allocations $\mathcal{F}(x_c, x_h) = (x_c - 1)(x_h - 1)/(x_h x_c - 1)$ and $\Delta n^{\rm eq} = \langle n\rangle_h^{\rm eq} - \langle n\rangle_c^{\rm eq}$. Then the total work (20) done by the system after a complete cycle becomes

$$\mathcal{W} = \mathcal{F}(x_c, x_h)(\Omega_h - \Omega_c)\Delta n^{\rm eq}. \tag{26}$$

Using $\langle n^2\rangle = \sum n^2 p_n$ and $\delta w^2 = \langle w^2\rangle - \langle w\rangle^2$, the work fluctuations can be obtained as

$$\delta w^2 = (\Omega_h - \Omega_c)^2[\langle n^2(\tau_h)\rangle - \langle n(\tau_h)\rangle^2 + \langle n^2(\tau_c)\rangle - \langle n(\tau_c)\rangle^2]. \tag{27}$$

The work fluctuations are rewritten as a function of time allocations (τ_h and τ_c),

$$\delta w^2 = (\Omega_h - \Omega_c)^2\big\{\big[2 - \big(\langle n\rangle_h^{\rm eq}\big)^2 - \big(\langle n\rangle_c^{\rm eq}\big)^2\big]\mathcal{A} - \mathcal{B} - \mathcal{C}\big(\langle n\rangle_h^{\rm eq}\big)^2 - \mathcal{D}\big(\langle n\rangle_c^{\rm eq}\big)^2 - \mathcal{N}\langle n\rangle_h^{\rm eq}\langle n\rangle_c^{\rm eq}\big\}, \tag{28}$$

where $\mathcal{A} \equiv (x_c x_h)^2\mathcal{G}/2$, $\mathcal{B} \equiv \mathcal{G}(2x_c x_h - 1)$, $\mathcal{C} \equiv \mathcal{G}(1 - 2x_h + x_c^2 + x_h^2 - 2x_c^2 x_h)/2$, $\mathcal{D} \equiv \mathcal{G}(1 - 2x_c + x_c^2 + x_h^2 - 2x_h^2 x_c)/2$, and $\mathcal{N} \equiv \mathcal{G}(x_c + x_h - x_h^2 - x_c^2 - 2x_c x_h + x_h x_c^2 + x_c x_h^2)$, with $\mathcal{G}$ being defined in Eq. (16). Because the stochastic power output is $\dot w[|n(\tau_h)\rangle; |n(\tau_{\rm cyc} - \tau_{ch})\rangle] = w[|n(\tau_h)\rangle; |n(\tau_{\rm cyc} - \tau_{ch})\rangle]/\tau_{\rm cyc}$, with $\tau_{\rm cyc}$ being the total cycle time, the relative fluctuations of the power are equivalent to corresponding those of work. From Eqs. (26) and (28), one derives the relative power fluctuations as

$$f_{\dot w} = \frac{\big[2 - \big(\langle n\rangle_h^{\rm eq}\big)^2 - \big(\langle n\rangle_c^{\rm eq}\big)^2\big]\mathcal{A} - \mathcal{B} - \mathcal{C}\big(\langle n\rangle_h^{\rm eq}\big)^2 - \mathcal{D}\big(\langle n\rangle_c^{\rm eq}\big)^2 - \mathcal{N}\langle n\rangle_h^{\rm eq}\langle n\rangle_c^{\rm eq}}{\mathcal{F}^2(x_c, x_h)(\Delta n^{\rm eq})^2}. \tag{29}$$

We numerically calculate $f_{\dot w}/f_{\dot w}^0$ [with $f_{\dot w}^0 \equiv f_{\dot w}(j_c = j_h = 0)$] as a function of j_c/j_h along the finite-time cycle duration. The results are shown in Fig. 1(b). In the numerical calculation, we choose $x_h = 2$, $x_c = 1.2$, $\beta_h = 1$, $\beta_c = 2$, and set $j_h = 1$. As compared to noninteracting value $f_{\dot w}^0$, the relative power fluctuations can be increased or decreased by

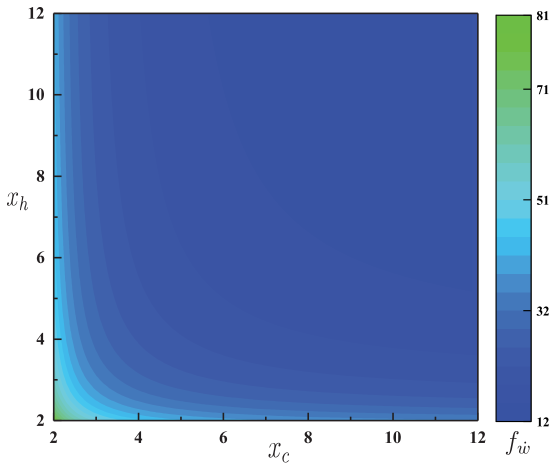

FIG. 2. Contour plot of the relative power fluctuations $f_{\dot{w}}$ in the effective time duration (x_h, x_c) plane for the quantum Otto engine model with $\beta_h = 0.4$, $\beta_c = 1$, $\Omega_h = 1$, and $\Omega_c = 0.8$.

tuning the strength of j_c and j_h. The relative power fluctuations $f_{\dot{w}}$ in Fig. 1(b) are shown to decrease monotonically with increasing j_c/j_h, and thus they behave in a similar manner to the machine efficiency η in Fig. 1(a). Because of the trade-off between efficiency and power fluctuations, the price for enhancing the efficiency by tuning the interaction strength is that the relative power fluctuations are increased.

To see the effects of finite time duration on the power fluctuations, a three-dimensional diagram $(x_c, x_h, f_{\dot{w}})$ for given parameters (β_h, β_c, Ω_c, and Ω_h) is plotted in Fig. 2, where $\beta_h = 0.4$, $\beta_c = 1$, $\Omega_h = 1$, and $\Omega_c = 0.8$. It shows that the relative fluctuations $f_{\dot{w}}$ are increasing with decreasing interaction interval (τ_h or τ_c) of the system-bath interaction and *vice versa*. Physically, this behavior follows from the fact that the system increasingly deviates form thermal equilibrium as τ_h (or τ_c) decreases and *vice versa*. The quasistatic limit of $x_c, x_h \to \infty$, leads to the facts that parameters $\mathcal{A} \to 1/2$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$, and $\mathcal{N}$ are vanishing, and that the work fluctuations become

$$\delta w^2 = (\Omega_h - \Omega_c)^2\left[1 - \frac{\left(\langle n\rangle_h^{\rm eq}\right)^2}{2} - \frac{\left(\langle n\rangle_c^{\rm eq}\right)^2}{2}\right]. \tag{30}$$

Unlike in the quantum engines working with Bose systems [21] or classical systems [48], where the work fluctuations may not be bounded from above due to the divergence of specific heat in the phase transition point, for quantum spin heat engines the work fluctuations are bounded from above by $(\delta w^2)^+ = (\Omega_h - \Omega_c)^2$. In the quasistatic limit when $x_{c,h} \to \infty$, these relative fluctuations approach their lower limit,

$$f_{\dot{w}}^- = \frac{2 - \left(\langle n\rangle_h^{\rm eq}\right)^2 - \left(\langle n\rangle_c^{\rm eq}\right)^2}{2\left(\langle n\rangle_h^{\rm eq} - \langle n\rangle_c^{\rm eq}\right)^2} < \frac{(\Omega_h - \Omega_c)^2}{(\Omega_h - \Omega_c)^2\left(\langle n\rangle_h^{\rm eq} - \langle n\rangle_c^{\rm eq}\right)^2}. \tag{31}$$

From Eq. (56) in the following Sec. III, we will prove for the cyclic engine that $(\Omega_h - \Omega_c)(\langle n\rangle_h^{\rm eq} - \langle n\rangle_c^{\rm eq}) \simeq (1/\beta_h - 1/\beta_c)\Delta S$, where ΔS is the protocol-independent entropy change. When the engine efficiency (22) is close to the Carnot value η_C, $\beta_h\Omega_h$ tends to be $\beta_c\Omega_c$. It follows, together with formula (31), that the relative power fluctuations for the engine working at any finite temperatures must satisfy the constraint:

$$\sqrt{f_{\dot{w}}^-} < \beta_h\frac{\Omega_h}{\Delta S}. \tag{32}$$

In contrast to the steady-state heat engines where the trade-off between power and efficiency is overcome by increasing power fluctuations [49,50], the quantum Otto engine based on interacting spin systems can operate in the state of efficiency η asymptotically closing to η_C at finite power, with small and even vanishing power fluctuations. This agrees with the result obtained previously from a cyclic heat engine based on either a classical simplified system [48] or a (noninteracting) harmonic system [21].

III. ANALYTIC EXPRESSION OF EFFICIENCY AT MAXIMUM POWER

Having obtained average heat and work per cycle, one can maximize the power output to determine the corresponding efficiency. With consideration of Eq. (26), the power output can be given by

$$\mathcal{P} = F(\tau_c, \tau_h)\Omega_h\eta\Delta n^{\rm eq}, \tag{33}$$

where we have used $F(\tau_h, \tau_c) = \mathcal{F}(x_c, x_h)/\tau_{\rm cyc}$. An exact analytical analysis on power optimization seems to be difficult at first sight, as power is a complicated function of the time-dependent protocols in the hot and cold isochoric branches. This optimization can, however, be present in two consecutive steps. First, we fix parameter values ω_h and ω_c to maximize the power with respect to the time durations of τ_h and τ_c. The second step is that we further maximize the power by tuning the remaining degrees of freedom ω_c and ω_h. From Eq. (33), we see that maximizing power $\mathcal{P}$ with respect to τ_h and τ_c is equivalent to maximizing F with respect to τ_h and τ_c.

In the sudden limit [11] where the time required for an adiabatic process is negligible, the extremal conditions of $\partial F/\partial\tau_c = 0$ and $\partial F/\partial\tau_h = 0$ yield the optimal allocation between the hot and cold isochoric branches as follows: $\gamma_h[\cosh(\gamma_c\tau_c - 1)] = \gamma_c[\cosh(\gamma_h\tau_h - 1)]$. The optimal time allocations on the isochores simplify to $\tau_h = \tau_c$, when and only when $\gamma_h = \gamma_c$. In the second step we maximize the power output by setting $\partial\mathcal{P}/\partial\omega_c = 0$ and $\partial\mathcal{P}/\partial\omega_h = 0$, leading to

$$\frac{\beta_c\chi_c(\Omega_h - \Omega_c)}{\chi_c + 1} = \frac{\chi_h - \chi_c}{\chi_h + 1}, \tag{34}$$

$$\frac{\beta_h\chi_h(\Omega_h - \Omega_c)}{\chi_h + 1} = \frac{\chi_h - \chi_c}{\chi_c + 1}, \tag{35}$$

where $\chi_c = e^{-\Omega_c\beta_c}$ and $\chi_h = e^{-\Omega_h\beta_h}$. Based on Eqs. (34) and (35), one can easily prove [26] that the efficiency at maximum power can be well approximated by

$$\eta^* = \frac{\eta_C^2}{\eta_C - (1 - \eta_C)\ln(1 - \eta_C)}, \tag{36}$$

which is identical to that obtained from various heat engine models [26,27,29] based on noninteracting systems. Expanding η^* up to the third term of η_C results into $\eta^* = \eta_C/2 + \eta_C^2/8 + 7\eta_C^3/96 + O(\eta_C^4)$, which agrees well with the expansion of the CA efficiency η_{CA}, with $\eta_{CA} = \eta_C/2 +$

$\eta_C^2/8 + 16\eta_C^3/96 + O(\eta_C^4)$. Consequently, in the linear response regime where the relative difference of temperatures is small, both η^* and η_{CA} have the same universality of $\eta_C/2 + \eta_C^2/8$.

IV. OPTIMAL PROTOCOL BY MINIMIZING IRREVERSIBLE ENTROPY PRODUCTION

According to stochastic thermodynamics, the irreversible entropy production rate [51] of the system and the heat reservoir can be expressed by

$$\dot{S}_\nu^{ir} = \sum_{m,n} R_{mn}^\nu p_n \ln \frac{R_{mn}^\nu p_n}{R_{nm}^\nu p_m} = \sum_{m<n} \left(R_{mn}^\nu p_n - R_{nm}^\nu p_m\right) \ln \frac{R_{mn}^\nu p_n}{R_{nm}^\nu p_m}, \tag{37}$$

where $\nu = h, c$ is used for the hot and cold isochoric processes, respectively. $\dot{S}_\nu^{ir}$ must be non-negative, and it becomes vanishing when and only when the thermal equilibrium is reached for the system in which the detailed balance is satisfied, $R_{mn}^\nu p_n = R_{nm}^\nu p_m$. The irreversible entropy production along the hot (cold) isochoric process with the initial time t_i and final time t_f is therefore given by

$$\Delta S_\nu^{ir} = \int_{t_i}^{t_f} \dot{S}_\nu^{ir}(t)\, dt. \tag{38}$$

Inserting Eq. (37) into Eq. (38) and considering Eq. (7), we can obtain the irreversible entropy production as

$$\Delta S_\nu^{ir} = -\beta_\nu \int_{t_i}^{t_f} \sum_n \dot{p}_n(t)\varepsilon_n\, dt + \Delta S_\nu. \tag{39}$$

Here we define the entropy change $\Delta S_\nu \equiv \Delta S(t_i, \tau_\nu) = S(t_f) - S(t_i)$, where $\tau_\nu = t_f - t_i$ is the time duration for the hot ($\nu = h$) or cold ($\nu = c$) isochoric process, and the von Neumann entropy $S(t) = -\sum_n p_n(t) \ln p_n(t)$. $S(t)$ is thus a state variable and

$$S(t) = -[(1 - \langle n\rangle)\ln(1 - \langle n\rangle) + (1 + \langle n\rangle)\ln(1 + \langle n\rangle)] + 2\ln 2 \tag{40}$$

for the system under consideration. It shows from Eq. (39) that minimizing the irreversible entropy production is equivalent to maximizing the heat absorbed by the system along a process with duration τ_ν. Since the change in the probabilities of p_n ($n = -1, 0, 0, 1$) accounts for the heat exchanged, $đ\mathcal{Q} = \sum_n \dot{p}_n \varepsilon_n$, one finds

$$\beta_\nu \mathcal{Q}_\nu = \beta_\nu \int_{t_i}^{t_f} \Omega(t)\langle \dot{n}(t)\rangle\, dt. \tag{41}$$

We can determine the optimal protocol minimizing irreversible entropy production, using the Euler-Lagrange approach [30,34], to search for the optimal schedule $n(t)$ and $\dot{n}(t)$, both of which are functionals of $\Omega(t)$. From Eqs. (24) and (41), we can obtain

$$\beta_\nu \mathcal{Q}_\nu = \int_{t_i}^{t_f} \mathcal{L}(\langle n\rangle, \langle \dot{n}\rangle)\, dt, \tag{42}$$

where

$$\mathcal{L} = \langle \dot{n}\rangle \ln\left(\frac{\gamma_\nu - \gamma_\nu\langle n\rangle - \langle \dot{n}\rangle}{\gamma_\nu + \gamma_\nu\langle n\rangle + \langle \dot{n}\rangle}\right). \tag{43}$$

By integrating the Euler-Lagrange equation, we have $\mathcal{L} - \langle \dot{n}\rangle \partial\mathcal{L}/\partial\langle \dot{n}\rangle = \mathcal{K}_\nu$, with the constant $\mathcal{K}_\nu$ of integration, we obtain

$$\frac{2\gamma_\nu \langle \dot{n}\rangle^2}{(\gamma_\nu - \gamma_\nu\langle n\rangle - \langle \dot{n}\rangle)(\gamma_\nu + \gamma_\nu\langle n\rangle + \langle \dot{n}\rangle)} = \mathcal{K}_\nu. \tag{44}$$

Its solution for $\dot{n}(t)$ can be derived as

$$\frac{\langle \dot{n}\rangle}{\gamma_\nu} = \frac{-\mathcal{K}_\nu\langle n\rangle \mp \sqrt{2\gamma_\nu\mathcal{K}_\nu + \mathcal{K}_\nu^2 - 2\gamma_\nu\mathcal{K}_\nu\langle n\rangle^2}}{2\gamma_\nu + \mathcal{K}_\nu}, \tag{45}$$

where the plus sign (+) refers to the upward process with rising quantum level, and the minus sign (−) to the downward process. With consideration of Eqs. (45) and (24), the explicit expression for instantaneous mean populations can be obtained as

$$\langle n(t)\rangle = \langle n\rangle^{\text{eq}} \pm \sqrt{\frac{\mathcal{K}_\nu}{2\gamma_\nu}[1 - (\langle n\rangle^{\text{eq}})^2]}. \tag{46}$$

When $\mathcal{K}_\nu = 0$, the system achieves the thermal equilibrium state and $\langle n(t)\rangle$ tends to be $\langle n\rangle^{\text{eq}} = \langle n(t \to \infty)\rangle$, thereby implying that $\mathcal{K}_\nu = 0$ represents the quasistatic limit. When $\mathcal{K}_\nu \neq 0$, the system evolves in finite time and it deviates from the thermal equilibrium. For the hot (cold) isochoric process where $\langle n\rangle < \langle n\rangle^{\text{eq}}$ ($\langle n\rangle > \langle n\rangle^{\text{eq}}$), the constant $\mathcal{K}_\nu$ indicates how far the thermodynamic process is away from the quasistatic limit.

We can solve Eq. (45) by separating the variables $\langle n\rangle$ and t to obtain

$$\gamma_\nu(t - t_i) = \mathcal{G}[\langle n(t)\rangle; \mathcal{K}_\nu] - \mathcal{G}[\langle n(t_i)\rangle; \mathcal{K}_\nu], \tag{47}$$

where

$$\mathcal{G}[\langle n\rangle; \mathcal{K}_\nu] = -\ln(1 - \langle n\rangle) + \sqrt{\frac{2\gamma_\nu}{\mathcal{K}_\nu}} \arcsin\left(\sqrt{\frac{2\gamma_\nu}{2\gamma_\nu + \mathcal{K}_\nu}}\langle n\rangle\right) + \frac{1}{2}\ln\left[\frac{\mathcal{K}_\nu + 2\gamma_\nu(1 - \langle n\rangle) + \sqrt{\mathcal{K}_\nu(2\gamma_\nu + \mathcal{K}_\nu - 2\gamma_\nu\langle n\rangle^2)}}{\mathcal{K}_\nu + 2\gamma_\nu(1 + \langle n\rangle) + \sqrt{\mathcal{K}_\nu(2\gamma_\nu + \mathcal{K}_\nu - 2\gamma_\nu\langle n\rangle^2)}}\right]. \tag{48}$$

When we restrict our analysis to a long (but not infinite) time duration of the system-bath interaction, we can use a perturbation method by assuming very small $\mathcal{K}_\nu$. The first-order term of the Taylor expansion of $\mathcal{G}[\langle n\rangle; \mathcal{K}_\nu]$ with respect to $\sqrt{\mathcal{K}_\nu}$ is obtained as

$$\mathcal{G}[\langle n\rangle;\mathcal{K}_\nu]=\sqrt{\frac{2\gamma_\nu}{\mathcal{K}_\nu}}\arcsin\langle n\rangle. \tag{49}$$

It is therefore indicated that, for the process of duration τ_ν, Eq. (47) can be approximated by $\gamma_\nu\tau_\nu=\varphi_\nu/\sqrt{\mathcal{K}_\nu}$, where $\varphi_\nu\equiv\sqrt{2}[\arcsin\langle n(t_f)\rangle-\arcsin\langle n(t_i)\rangle]$, and $\varphi_c=-\varphi_h$ due to $\langle n(\tau_h)\rangle=\langle n(\tau_h+\tau_{hc})\rangle$ and $\langle n(0)\rangle=\langle n(\tau_{\rm cyc}-\tau_{ch})\rangle$. Assuming that the heat engine works in the linear response regime where the difference between the temperatures of the hot and cold reservoirs is small, the difference of the equilibrium populations, $\Delta n^{\rm eq}$, must be very small. With consideration of Eq. (25), we can obtain the quadratic approximation to the expression of $\sqrt{2}[\arcsin\langle n(t_f)\rangle-\arcsin\langle n(t_i)\rangle]$ about $\langle n\rangle_c^{\rm eq}$ by making Taylor series expansion in isochoric heating process, leading to $\varphi_h=\Delta n^{\rm eq}\mathcal{F}(x_c,x_h)/[1-(\langle n\rangle_h^{\rm eq})^2]^{\frac{1}{2}}-\langle n\rangle_h^{\rm eq}(\Delta n^{\rm eq})^2\mathcal{F}^2(x_c,x_h)/2[1-(\langle n\rangle_h^{\rm eq})^2]^{\frac{3}{2}}+O[(\Delta n^{\rm eq})^3]$. Since function $\mathcal{F}(x_c,x_h)$ decreases exponentially to approach its maximum value 1 as time duration τ_h or τ_c increases, for the long duration of system-bath interaction, we have the approximation $-\varphi_c=\varphi_h\simeq\Delta n^{\rm eq}/[1-(\langle n\rangle_h^{\rm eq})^2]^{\frac{1}{2}}-(\Delta n^{\rm eq})^2\langle n\rangle_h^{\rm eq}/2[1-(\langle n\rangle_h^{\rm eq})^2]^{\frac{3}{2}}$. It follows, using Eq. (47), that for the isochoric process of time duration τ_ν,

$$\sqrt{\gamma_\nu}\tau_\nu=\frac{\varphi_\nu}{\sqrt{\mathcal{K}_\nu}}. \tag{50}$$

As emphasized, in deriving formula (50) we also used the condition that the heat engine runs in the linear response regime.

Using Eq. (46) and $\beta_\nu\mathcal{Q}_\nu=\int_{\langle n(t_i)\rangle}^{\langle n(t_f)\rangle}\beta_\nu\Omega\,d\langle n\rangle$, the entropy change due to heat exchange can be rewritten as

$$\beta_\nu\mathcal{Q}_\nu=\tilde{\mathcal{S}}[\langle n(t_f)\rangle;\mathcal{K}_\nu]-\tilde{\mathcal{S}}[\langle n(t_i)\rangle;\mathcal{K}_\nu], \tag{51}$$

where

$$\tilde{\mathcal{S}}[\langle n\rangle;\mathcal{K}_\nu]=-2\ln(1-\langle n\rangle)+\ln\left[\frac{\mathcal{K}_\nu+2\gamma_\nu(1-\langle n\rangle)+\sqrt{\mathcal{K}_\nu(2\gamma_\nu+\mathcal{K}_\nu-2\gamma_\nu\langle n\rangle^2)}}{\mathcal{K}_\nu+2\gamma_\nu(1+\langle n\rangle)+\sqrt{\mathcal{K}_\nu(2\gamma_\nu+\mathcal{K}_\nu-2\gamma_\nu\langle n\rangle^2)}}\right]$$
$$+\langle n\rangle\ln\left[\frac{\gamma_\nu+\mathcal{K}_\nu-\gamma_\nu\langle n\rangle^2-\sqrt{\mathcal{K}_\nu(2\gamma_\nu+\mathcal{K}_\nu-2\gamma_\nu\langle n\rangle^2)}}{\gamma_\nu(1+\langle n\rangle)^2}\right]-\sqrt{\frac{2\mathcal{K}_\nu}{\gamma_\nu}}\arcsin\left(\sqrt{\frac{2\gamma_\nu}{2\gamma_\nu+\mathcal{K}_\nu}}\langle n\rangle\right). \tag{52}$$

Making the first-order Taylor expansion of $\tilde{\mathcal{S}}[\langle n\rangle;\mathcal{K}_\nu]$ with respect to $\sqrt{\mathcal{K}_\nu}$ gives rise to $\tilde{\mathcal{S}}[\langle n\rangle;\mathcal{K}_\nu]=-[(1-\langle n\rangle)\ln(1-\langle n\rangle)+(1+\langle n\rangle)\ln(1+\langle n\rangle)]-\sqrt{2\mathcal{K}_\nu/\gamma_\nu}\arcsin(\langle n\rangle)$, which, together with Eq. (40), leads to

$$\Delta\tilde{S}_\nu=\Delta S_\nu-\Delta S_\nu^{ir}, \tag{53}$$

where

$$\Delta S_\nu^{ir}=\sqrt{\frac{2\mathcal{K}_\nu}{\gamma_\nu}}[\arcsin\langle n(t_f)\rangle-\arcsin\langle n(t_i)\rangle] \tag{54}$$

represents the irreversible entropy production. While ΔS_ν is a state variable depending on only the initial and final states of the process, ΔS_ν^{ir} is a protocol-dependent quantity indicating the deviation from the quasistatic limit. From Eqs. (37) and (53), we find that, under the optimized protocol above, the irreversible entropy production in stochastic thermodynamics can be approximated by

$$\Delta S_\nu^{ir}=\int_{t_i}^{t_f}\sum_{m,n}R_{mn}^\nu p_n\ln\frac{R_{mn}^\nu p_n}{R_{nm}^\nu p_m}\,dt=\frac{\varphi^2}{\gamma_\nu\tau_\nu}, \tag{55}$$

where we have used $\varphi=\varphi_h=-\varphi_c$, thereby confirming the low dissipation assumption [30,34,37]. For the quantum Otto engine, the two individual isochoric processes are connected by the two adiabatic, isentropic processes, meaning that $\Delta S\equiv\Delta S_h=-\Delta S_c$. Therefore, the heats transferred along the hot and cold isothermal contact of the engine cycle are given by

$$\mathcal{Q}_h=\frac{\Delta S}{\beta_h}-\frac{\varphi^2}{\gamma_h\beta_h\tau_h},\quad \mathcal{Q}_c=\frac{-\Delta S}{\beta_c}-\frac{\varphi^2}{\gamma_c\beta_c\tau_c}. \tag{56}$$

Maximizing power $\mathcal{P}=(\mathcal{Q}_h+\mathcal{Q}_c)/(\tau_c+\tau_h)$ with respect to τ_h and τ_c, we obtain the efficiency at maximum power as

$$\eta^*=\frac{\eta_C}{2-r\eta_C}, \tag{57}$$

where we use $r=(1+\sqrt{\gamma_h\beta_h/\gamma_c\beta_c})^{-1}$. This optimal efficiency η^* is thus situated between $\eta_C/2\leqslant\eta^*\leqslant\eta_C/(2-\eta_C)$ [37]. The upper and lower bounds are achieved in the asymmetric limits $\gamma_h/\gamma_c\to 0$ and $\gamma_h/\gamma_c\to\infty$, respectively. The CA efficiency $\eta_{CA}=1-\sqrt{\beta_h/\beta_c}$ is recovered in the symmetric case of $\gamma_h=\gamma_c$. When $\gamma_h\beta_h=\gamma_c\beta_c$ leads to $r=1/2$, we can obtain $\eta^*=\eta_C/2+\eta_C^2/8+O(\eta_C^3)$, recovering the universal behavior in the linear response regime.

V. DISCUSSION AND CONCLUSIONS

Consider an engine with working substance consists of N spin$-1/2$ particles with cluster interactions. If such a system falls into the one-dimensional spin-1/2 XX model [52], its Hamiltonian can be given by

$$\hat{H}=\Omega\sum_{i=1}^N\left(\hat{\sigma}_i^x\hat{\sigma}_{i+1}^x+\hat{\sigma}_i^y\hat{\sigma}_{i+1}^y\right)$$
$$+J\sum_{i=1}^N\left(\hat{\sigma}_i^x\hat{\sigma}_{i+1}^z\hat{\sigma}_{i+2}^x+\hat{\sigma}_i^y\hat{\sigma}_{i+1}^z\hat{\sigma}_{i+2}^y\right), \tag{58}$$

where $\Omega>0$ denotes the antiferromagnetic exchange coupling which depends on the external control parameter ω and j interaction between two spins, and $J>0$ presents the strength of the cluster interaction. As a specific example, for a system of three spins (i.e., $N=3$) the eigenvalues are in

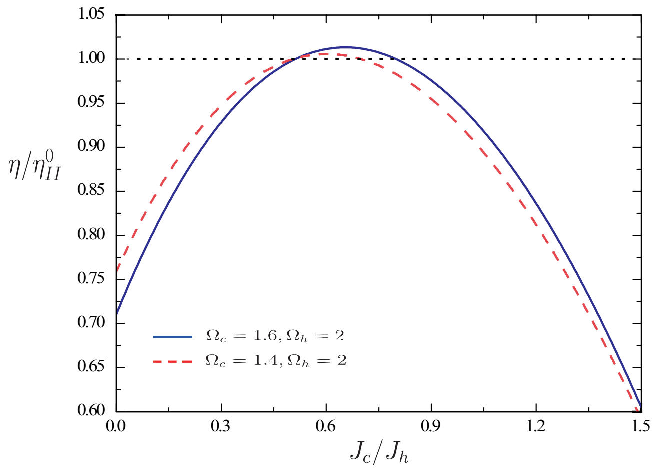

FIG. 3. Efficiency η (normalized to the efficiency η_{II}^0 without three-spin interaction) versus the ratio of J_c to J_h for $J_h = 1$, $\beta_h = 1$, $\beta_c = 2$, and $\Omega_h = 2$. The values of Ω_c are adopted: blue solid line: $\Omega_c = 1.6$, and red dashed line 1.4, respectively.

the form of $\varepsilon_1 = \varepsilon_2 = 0$, $\varepsilon_3 = -\varepsilon_4 = \frac{J}{4}$, $\varepsilon_5 = -\varepsilon_6 = \frac{1}{8}(J - \Delta)$, $\varepsilon_7 = -\varepsilon_8 = \frac{1}{8}(J + \Delta)$, with $\Delta \equiv \sqrt{32\Omega^2 + J^2}$. The occupation probabilities for this system at thermal equilibrium are then $\pi_n = \frac{1}{Z_\beta} e^{-\beta\varepsilon_n}$, with the partition function $Z_\beta = 4\cosh(\frac{\beta J}{8})[\cosh(\frac{\beta J}{8}) + \cosh(\frac{\beta\Delta}{8})]$. In the hot (cold) isochoric process, $\Omega = \Omega_h$ and $J = J_h$ ($\Omega = \Omega_c$ and $J = J_c$).

Since an analytical analysis of the finite-power engine with three interacting spins ($N = 3$) becomes a formidable task, we assume that the system with constant Ω and J equilibrates with the hot and the cold reservoir, respectively. Under this assumption, the heat injection and heat rejection along the hot and the cold isochoric process is $\mathcal{Q}_h = \sum_n \varepsilon_n(\Omega_h, J_h)[\pi_n(\beta_h, \Omega_h, J_h) - \pi_n(\beta_c, \Omega_c, J_c)]$ and $\mathcal{Q}_c = \sum_n \varepsilon_n(\Omega_c, J_c)[\pi_n(\beta_h, \Omega_h, J_h) - \pi_n(\beta_c, \Omega_c, J_c)]$, from which we can numerically determine the machine efficiency, $\eta = 1 - \mathcal{Q}_c/\mathcal{Q}_h$. It can simplify to $\eta = \eta_{II}^0 \equiv 1 - \Omega_c/\Omega_h$ if only two-spin interaction is considered ($J_h = J_c = 0$). The effects of three-spin interaction on the efficiency are shown in Fig. 3, where $\beta_h = 1$, $\beta_c = 2$, $J_h = 1$, and $\Omega_h = 2$. Figure 3 shows that, only except for $0.51 \leqslant J_c/J_h \leqslant 0.80$ ($0.51 \leqslant J_c/J_h \leqslant 0.70$) at $\Omega_c = 1.6 (\Omega_c = 1.4)$, the normalized efficiency η/η_{II}^0 is smaller than 1. In contrast to the interacting system with particle number $N = 2$, the interaction among a many-body system ($N \geqslant 3$) would bring down the efficiency. Physically, for large-scale systems interactions generically result in quantum chaos, and uncontrolled increase of entropy is inevitable, thereby leading to a decrease in the efficiency. For the heat engine operating with many-body interacting system ($N \geqslant 3$), the Floquet techniques in suppressing quantum chaos and entropy growth [46,47] may be used in order for the engine to run efficiently.

When the time allocations to the four processes of the cycle are given, the maximal power output can be obtained by differentiating work with respect to the frequencies Ω_h and Ω_c. The efficiency at maximum power for three-particle system, which is not plotted in this article, is numerically shown to be weakly dependent on the interaction strength J. While for the two-particle system the expressions of P and η in the interacting case ($j \neq 0$) take the same forms as the corresponding those of noninteracting spins ($j = 0$) as long as replacing Ω with ω, for the many-body system ($N \geqslant 3$) this is not the case, and thus the efficiency at maximum power depends on the cluster interaction strength J [though it is still situated between the lower bound $\eta_C/2$ and upper bound $\eta_C/(2 - \eta_C)$].

As a final remark, our approach can be directly used to analyze how entanglement of the interacting system behaves in the engine under finite power. We use concurrence directly as the measurement of entanglement, since the concurrence is a monotone of the entanglement of formation. The entanglement of the interacting spin system at the final states of hot and cold isochoric processes is represented by $\mathbb{C}_1[\equiv \mathbb{C}(\tau_h)]$ and $\mathbb{C}_2[\equiv \mathbb{C}(\tau_{\rm cyc} - \tau_{ch})]$, respectively. It was shown in Refs. [53,54] that the concurrence can be determined according to

$$\mathbb{C}_1 = \begin{cases} \frac{\sinh[\beta(\tau_h)\Omega_h]-1}{\cosh[\beta(\tau_h)\Omega_h]+1} & \beta(\tau_h)\Omega_h > \text{arcsinh}(1) \\ 0 & \beta(\tau_h)\Omega_h \leqslant \text{arcsinh}(1) \end{cases} \tag{59}$$

and

$$\mathbb{C}_2 = \begin{cases} \frac{\sinh[\beta(\tau_{\rm cyc}-\tau_{ch})\Omega_c]-1}{\cosh[\beta(\tau_{\rm cyc}-\tau_{ch})\Omega_c]+1} & \beta(\tau_{\rm cyc} - \tau_{ch})\Omega_c > \text{arcsinh}(1) \\ 0 & \beta(\tau_{\rm cyc} - \tau_{ch})\Omega_c \leqslant \text{arcsinh}(1) \end{cases}. \tag{60}$$

The power output (33) and efficiency (22) can be rewritten in terms of the entanglement, $\mathcal{P} = \sqrt{2}(\sqrt{1+\mathbb{C}_c} - \sqrt{1+\mathbb{C}_h})[\frac{1}{\beta_c}\ln(-1 + \sqrt{\frac{2}{1+\mathbb{C}_c}}) - \frac{1}{\beta_h}\ln(-1 + \sqrt{\frac{2}{1+\mathbb{C}_h}})]$ $F(\tau_c, \tau_h)$. Maximizing power with respect to $\mathbb{C}_h$ and $\mathbb{C}_c$ can therefore reproduce the optimal efficiency (36) at the interaction-independent value of $\mathbb{C}_h/\mathbb{C}_c = \frac{\{1+\cosh[1-\frac{\ln(1-\eta_C)}{\eta_C}]\}\{-1+\sinh[1-\frac{(1-\eta_C)\ln(1-\eta_C)}{\eta_C}]\}}{\{-1+\sinh[1-\frac{\ln(1-\eta_C)}{\eta_C}]\}\{1+\cosh[1-\frac{(1-\eta_C)\ln(1-\eta_C)}{\eta_C}]\}}$, where $\mathbb{C}_h = \mathbb{C}_1|_{\beta(\tau_h\to\infty)=\beta_h}$, and $\mathbb{C}_c = \mathbb{C}_2|_{\beta(\tau_c\to\infty)=\beta_c}$.

We have examined the finite-power thermodynamics of a quantum Otto engine using two interacting spins as its working substance. From the stochastic master equation, explicit expressions were derived analytically for the power and (relative) power fluctuations as functions of the time durations and control variable. We found that, for the engine under finite (but not maximal) power, the interactions can enhance the machine efficiency via tuning strength of interactions between particles, but with the sacrifice of increasing power fluctuations. We showed that the relative power fluctuations were bounded by the upper limit and the engine can be close to the Carnot efficiency at finite and even vanishing relative power fluctuations. The efficiency at maximum power for the engine was derived analytically, and it has the same universality as η_{CA} in a linear response regime. The physical implication of the efficiency at maximum power was given by minimizing irreversible entropy production (subject to finite time cycle duration) based on the Euler-Lagrange approach, which confirms the universality of efficiency at maximum power in linear responses.

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[1] J. P. S. Peterson, T. B. Batalhão, M. Herrera, A. M. Souza, R. S. Sarthour, I. S. Oliveira, and R. M. Serra, Experimental Characterization of a Spin Quantum Heat Engine, Phys. Rev. Lett. **123**, 240601 (2019).

[2] G. Maslennikov, S. Ding, R. Hablutzel, J. Gan, A. Roulet, S. Nimmrichter, J. Dai, V. Scarani, and D. Matsukevich, Quantum absorption refrigerator with trapped ions, Nat. Commun. **10**, 202 (2019).

[3] J. Rofinagel, S. T. Dawkins, K. N. Tolazzi, O. Abah, E. Lutz, F. Schmidt-Kaler, and K. Singer, A single-atom heat engine, Science **352**, 325 (2016).

[4] J. Klatzow, J. N. Becker, P. M. Ledingham, C. Weinzetl, K. T. Kaczmarek, D. J. Saunders, J. Nunn, I. A. Walmsley, R. Uzdin, and E. Poem, Experimental Demonstration of Quantum Effects in the Operation of Microscopic Heat Engines, Phys. Rev. Lett. **122**, 110601 (2019).

[5] D. Gangloff, A. Bylinskii, I. Counts, W. Jhe, and V. Vuletiá, Velocity tuning of friction with two trapped atoms, Nat. Phys. **11**, 915 (2015).

[6] B. Sothmann, R. Sánchez, and A. N. Jordan, Thermoelectric energy harvesting with quantum dots, Nanotechnolgy **26**, 032001 (2015).

[7] C. Lotze, M. Corso, K. J. Franke, F. von Oppen, and J. I. Pascual, Driving a macroscopic oscillator with the stochastic motion of a hydrogen molecule, Science **338**, 779 (2012).

[8] V. Blickle and C. Bechinger, Realization of a micrometre-sized stochastic heat engine, Nat. Phys. **8**, 143 (2012).

[9] J. H. Wang, Z. L. Ye, Y. M. Lai, W. S. Li, and J. Z. He, Efficiency at maximum power of a quantum heat engine based on two coupled oscillators, Phys. Rev. E **91**, 062134 (2015).

[10] Y. Zhang, F. Bariani, and P. Meystre, Quantum Optomechanical Heat Engine, Phys. Rev. Lett. **112**, 150602 (2014).

[11] R. Kosloff and Y. Rezek, Irreversible performance of a quantum harmonic heat engine, New J. Phys. **8**, 83 (2006).

[12] W. Niedenzu, V. Mukherjee, A. Ghosh, A. G. Kofman, and G. Kurizki, Quantum engine efficiency bound beyond the second law of thermodynamics, Nat. Commun. **9**, 165 (2018).

[13] R. Kosloff, Quantum thermodynamics: A dynamical view point, Entropy **15**, 2100 (2013).

[14] R. Kosloff and A. Levy, Quantum heat engines and refrigerators: Continuous devices, Annu. Rev. Phys. Chem. **65**, 365 (2014).

[15] R. Uzdin, A. Levy, and R. Kosloff, Equivalence of Quantum Heat Machines, and Quantum-Thermodynamic Signatures, Phys. Rev. X **5**, 031044 (2015).

[16] S. H. Su, X. Q. Luo, J. C. Chen, and C. P. Sun, Angle-dependent quantum Otto heat engine based on coherent dipole-dipole coupling, Europhys. Lett. **115**, 30002 (2016).

[17] X. L. Huang, T. Wang, and X. X. Yi, Effects of reservoir squeezing on quantum systems and work extraction, Phys. Rev. E **86**, 051105 (2012).

[18] L. A. Correa, J. P. Palao, D. Alonso, and G. Adesso, Quantum-enhanced absorption refrigerators, Sci. Rep. **4**, 3949 (2015).

[19] J. Rofinagel, O. Abah, F. Schmidt-Kaler, K. Singer, and E. Lutz, Nanoscale Heat Engine Beyond the Carnot Limit, Phys. Rev. Lett. **112**, 030602 (2014).

[20] J. Klaers, S. Faelt, A. Imamoglu, and E. Togan, Squeezed Thermal Reservoirs as a Resource for a Nanomechanical Engine Beyond the Carnot Limit, Phys. Rev. X **7**, 031044 (2017).

[21] J. H. Wang, J. Z. He, and Y. L. Ma, Finite-time performance of a quantum heat engine with a squeezed thermal bath, Phys. Rev. E **100**, 052126 (2019).

[22] W. Niedenzu, D. Gelbwaser-Klimovsky, A. G. Kofman, and G. Kurizki, On the operation of machines powered by quantum non-thermal baths, New J. Phys. **18**, 083012 (2016).

[23] B. K. Agarwalla, J.-H. Jiang, and D. Segal, Quantum efficiency bound for continuous heat engines coupled to noncanonical reservoirs, Phys. Rev. B **96**, 104304 (2017).

[24] M. Campisi, J. Pekola, and R. Fazio, Nonequilibrium fluctuations in quantum heat engines: Theory, example, and possible solid state experiments, New J. Phys. **17**, 035012 (2015).

[25] K. Brandner, K. Saito, and U. Seifert, Thermodynamics of Micro- and Nano-Systems Driven by Periodic Temperature Variations, Phys. Rev. X **5**, 031019 (2015).

[26] F. L. Wu, J. Z. He, Y. L. Ma, and J. H. Wang, Efficiency at maximum power of a quantum Otto cycle within finite-time or irreversible thermodynamics, Phys. Rev. E **90**, 062134 (2014).

[27] C. Van den Broeck and K. Lindenberg, Efficiency at maximum power for classical particle transport, Phys. Rev. E **86**, 041144 (2012).

[28] J. Guo, J. Wang, Y. Wang, and J. Chen, Universal efficiency bounds of weak-dissipative thermodynamic cycles at the maximum power output, Phys. Rev. E **87**, 012133 (2013).

[29] Z. C. Tu, Efficiency at maximum power of Feynman's ratchet as a heat engine, J. Phys. A **41**, 312003 (2008).

[30] Q. Liu, J. Z. He, Y. L. Ma, and J. H. Wang, Finite-power performance of quantum heat engines in linear response, Phys. Rev. E **100**, 012105 (2019).

[31] J. Gonzalez-Ayala, M. Santillán, I. Reyes-Ramírez, and A. Calvo-Hernández, Link between optimization and local stability of a low-dissipation heat engine: Dynamic and energetic behaviors, Phys. Rev. E **98**, 032142 (2018).

[32] V. Singh and R. S. Johal, Low-dissipation Carnot-like heat engines at maximum efficient power, Phys. Rev. E **98**, 062132 (2018).

[33] A. Calvo Hernández, A. Medina, and J. M. M. Roco, Time, entropy generation, and optimization in low-dissipation heat devices, New J. Phys. **17**, 075011 (2015).

[34] M. Esposito, R. Kawai, K. Lindenberg, and C. Van den Broeck, Quantum-dot Carnot engine at maximum power, Phys. Rev. E **81**, 041106 (2010).

[35] C. Van den Broeck, Thermodynamic Efficiency at Maximum Power, Phys. Rev. Lett. **95**, 190602 (2005).

[36] F. L. Curzon and B. Ahlborn, Efficiency of a Carnot engine at maximum power output, Am. J. Phys. **43**, 22 (1975).

[37] M. Esposito, R. Kawai, K. Lindenberg, and C. Van den Broeck, Efficiency at Maximum Power of Low-Dissipation Carnot Engines, Phys. Rev. Lett. **105**, 150603 (2010).
[38] R. Alicki and R. Kosloff, Introduction to quantum thermodynamics: History and prospects, in *Thermodynamics in the Quantum Regime*, edited by F. Binder, L. Correa, C. Gogolin, J. Anders, G. Adesso, Fundamental Theories of Physics Vol. 195 (Springer Nature, Cham, 2019), Chap. 1, pp. 1–33.
[39] Y. Apertet, H. Ouerdane, C. Goupil, and Ph. Lecoeur, True nature of the Curzon-Ahlborn efficiency, Phys. Rev. E **96**, 022119 (2017).
[40] Y. Izumida, K. Okuda, J. M. M. Roco, and A. Calvo Hernández, Heat devices in nonlinear irreversible thermodynamics, Phys. Rev. E **91**, 052140 (2015).
[41] V. Holubec, An exactly solvable model of a stochastic heat engine: Optimization of power, power fluctuations and efficiency, J. Stat. Mech. (2014) P05022.
[42] M. Campisi and R. Fazio, The power of a critical heat engine, Nat. Commun. **7**, 11895 (2016).
[43] D. Gelbwaser-Klimovsky, A. Bylinskii, D. Gangloff, R. Islam, A. Aspuru-Guzik, and V. Vuletic, Single-Atom Heat Machines Enabled by Energy Quantization, Phys. Rev. Lett. **120**, 170601 (2018).
[44] F. Boldt, K. H. Hoffmann, P. Salamon, and R. Kosloff, Time-optimal processes for interacting spin systems, Eur. Phys. Lett. **99**, 40002 (2012).
[45] T. Prosen, and E. Ilievski, Nonequilibrium Phase Transition in a Periodically Driven XY Spin Chain, Phys. Rev. Lett. **107**, 060403 (2011).
[46] A. Haldar, R. Moessner, and A. Das, Onset of Floquet thermalization, Phys. Rev. B **97**, 245122 (2018).
[47] A. Haldar, D. Sen, R. Moessner, and A. Das, Scars in strongly driven Floquet matter: Resonance vs emergent conservation laws, arXiv:1909.04064.
[48] V. Holubec and A. Ryabov, Cycling Tames Power Fluctuations Near Optimum Efficiency, Phys. Rev. Lett. **121**, 120601 (2018).
[49] P. Pietzonka and U. Seifert, Universal Trade-Off Between Power, Efficiency, and Constancy in Steady-State Heat Engines, Phys. Rev. Lett. **120**, 190602 (2018).
[50] N. Shiraishi, K. Saito, and H. Tasaki, Universal Trade-Off Relation Between Power and Efficiency for Heat Engines, Phys. Rev. Lett. **117**, 190601 (2016).
[51] M. Esposito and C. Van den Broeck, Three faces of the second law. I. Master equation formulation, Phys. Rev. E **82**, 011143 (2010).
[52] I. Titvinidze and G. I. Japaridze, Phase diagram of the spin extended model, Eur. Phys. J. B **32**, 383 (2003).
[53] G.-F. Zhang, Entangled quantum heat engines based on two two-spin systems with Dzyaloshinski-Moriya anisotropic antisymmetric interaction, Eur. Phys. J. D **49**, 123 (2008).
[54] X. G. Wang, Effects of anisotropy on thermal entanglement, Phys. Lett. A **281**, 101 (2001).