Particle-number distribution in large fluctuations at the tip of branching random walks

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We investigate properties of the particle distribution near the tip of one-dimensional branching random walks at large times t, focusing on unusual realizations in which the rightmost lead particle is very far ahead of its expected position, but still within a distance smaller than the diffusion radius $\sim\sqrt{t}$. Our approach consists in a study of the generating function $G_{\Delta x}(\lambda) = \sum_n \lambda^n p_n(\Delta x)$ for the probabilities $p_n(\Delta x)$ of observing n particles in an interval of given size Δx from the lead particle to its left, fixing the position of the latter. This generating function can be expressed with the help of functions solving the Fisher-Kolmogorov-Petrovsky-Piscounov (FKPP) equation with suitable initial conditions. In the infinite-time and large-Δx limits, we find that the mean number of particles in the interval grows exponentially with Δx, and that the generating function obeys a nontrivial scaling law, depending on Δx and λ through the combined variable $[\Delta x - f(\lambda)]^3/\Delta x^2$, where $f(\lambda) \equiv -\ln(1-\lambda) - \ln[-\ln(1-\lambda)]$. From this property, one may conjecture that the growth of the typical particle number with the size of the interval is slower than exponential, but, surprisingly enough, only by a subleading factor at large Δx. The scaling we argue is consistent with results from a numerical integration of the FKPP equation.

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I. INTRODUCTION

Branching random walks are stochastic processes which rule the time evolution of a set of particles through a random walk combined with a branching process. Each particle is characterized by its position in some space and may evolve in time by splitting into several descendants, and/or by randomly changing its position, independently of the other particles present at the same time in the set. Branching random walk (BRW) processes turn out to be relevant in many physical and biological context. Hence, studying the universal properties of such processes is of wide interest. In this paper, we will address BRWs in one space dimension, starting with one single particle, and we will focus our discussion essentially on the branching Brownian motion (BBM). In this context, in each realization at any given time there are two lead particles, and the set of the positions of the latter indexed by the time form two boundaries in the two-dimensional plane. The distribution of the positions of these boundaries is given by solutions to the well known Fisher-Kolmogorov-Petrovsky-Piscounov (FKPP) [1,2] equation (For a review, see Ref. [3]).

Our motivation for studying BRWs initially comes from particle physics [4]. In the context of the scattering of hadrons at very high energies, the state of the hadrons, which determines observables such as the interaction cross section, is essentially a set of many gluons, which turn out to be generated by a peculiar BRW. This picture emerges from the theory of the strong interactions, quantum chromodynamics, in the semiclassical limit, relevant when the energy at which the scattering occurs is very large compared to any other energy scales, such as the hadron masses or the momenta transfered to the particles that go to the final state. An example of direct application of some detailed properties of BRWs in this physical context was found recently. We showed indeed that a subset of the events observed in the data collected at high-energy colliders, called "diffractive events" due to the characteristic angular distribution of the particles measured in the final state, can be understood as a consequence of the fluctuations of the position of a lead particle in a BRW [5,6]. The corresponding statistical physics problem is actually the genealogy of the particles generated by a BRW that have a position larger than some number, which, in the context of particle physics, is determined by some intrinsic properties of the interacting particles [7].

Fluctuations in a BRW mainly occur in two places: in the beginning of the evolution, when the system consists in few particles, and throughout the evolution in the vicinity of the two lead particles. Based on this observation and modeling these two kinds of fluctuations in a very simple way [8], it was possible to arrive at different quantitative results for general BRWs, such as analytical expressions for the moments of random variables defined on the set of particles present in the system at a given time. Such a phenomenological model has helped to formulate a mathematical conjecture, that by now has been proven and generalized [9].

The fluctuations occurring in the beginning of the time evolution have an effect on the particle distribution that persists throughout: They mainly result in an effective global shift of the mean positions of the particles in the vicinity of the boundary of the BRW with respect to the expectation values of their positions in typical events, by a distance which tends to a realization-dependent finite constant at large times. We

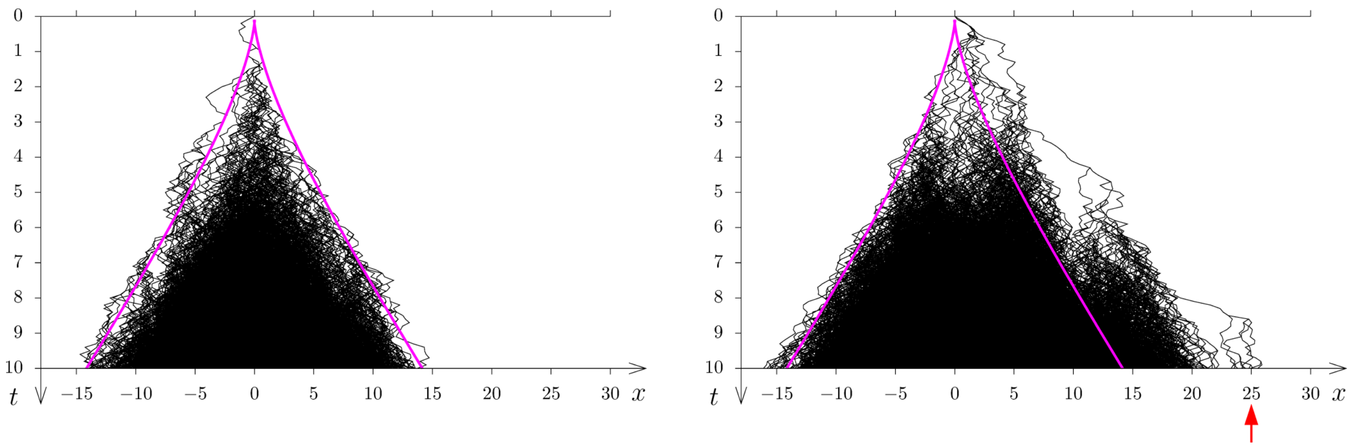

FIG. 1. Typical (left) and unusual (right) realizations of the branching Brownian motion. The continuous lines represent the expected values $x = \pm m_t$ of the positions of the lead particles, extracted from a numerical solution to the FKPP equation. The realization in the right plot is obtained by conditioning the rightmost particle at final time $t = 10$ to be to the right of the position $x = 25$ indicated by the arrow, that is to say, way ahead of its expected position. (The probability of such an event is about 10^{-4}). We see that while in the typical realization the lead particles are quite close to their expected values throughout, in the unusual realization instead, the particles on the right of $x = 25$ are brought there by a combination of a large front fluctuation, namely which develops at relatively early times, and a large tip fluctuation, namely which occurs at late times. In the latter event, the density of particles in an interval of size say $\Delta x \sim 5$ from the tip is visibly quite low, much lower than in the same interval in a typical realization.

call them "front fluctuations"; they had been formalized in mathematics in the earlier work of Lalley and Sellke [10]. Instead, late-time fluctuations originating from particles near the boundaries of the BRW have a small effect on the position of the latter, that vanishes as they occur at larger times. We call them "tip fluctuations". In the phenomenological model, in a given realization, both fluctuations are needed to send the lead particle significantly ahead of its expected position.

The properties of the tip of BRWs turn out to provide insight into the free energy of spin glasses [11], and have therefore been an active research topic for some time. Several recent works have focused on the properties of lead particles in the BBM. For example, the large deviations of a lead particle have recently been discussed [12], as well as the correlations between the positions of the two lead particles [13]. The systematic study of the properties of particle distributions near the tip of BRWs was pioneered in the physics literature in Refs. [14,15]; See also Refs. [16–18] for parallel mathematical developments. These works were essentially focused on the limiting distribution of extremal points in the BBM, namely, as seen from the lead particle. The object of the present paper is to propose an approach to the study of the statistical properties of the particles in large tip fluctuations, that is, near the tip of a BRW *conditioned to have its lead particle at a position very far from its expected position.* In the phenomenological model described above, the main assumption on tip fluctuations is actually that they consist of low-density sets of particles sent far away from the expected position of the lead particles, in the region which is void of particles in a typical realization. The main question we want to address is whether this crucial assumption[1] is indeed verified in an actual BRW.

To this aim, an appropriate quantity to compute is the distribution $p_n(\Delta x)$ of the number n of particles in the finite interval Δx from the position of the lead particle. (See Fig. 1 for an illustration.) Generally speaking, quantities like $p_n(\Delta x)$ were addressed in the literature in the case of sets of particles characterized by a single number chosen in various ways: For example, the sets of particles drawn from an inhomogeneous Poisson point process on a line or resulting from a Ruelle cascade [15,19] were considered; or simply sets of independent and identically distributed random variables [20]; or, also, sets of eigenvalues of random matrices [21]. In the case in which the generating process is the BBM (or more generally BRW), Brunet and Derrida have shown [15] that all properties of the particle distribution may be deduced from solutions to the FKPP equation with peculiar initial conditions, which is obeyed by generating functions of the particle-number probability distributions. In this paper, we shall discuss properties of these generating functions. While at this point exact solutions seem out of reach, we are able to find an interesting scaling form, from which one may conjecture qualitative features of the particle distribution near the tip in typical realizations of rare tip fluctuations.

In Sec. II, we formulate the calculation of the statistics of the number of particles in the tip of a branching random walk using a generating function; In Sec. III, we present our analytical insight on the solutions of the equations governing the time evolution of the generating function; In Sec. IV, we derive a few consequences for the mean of the particle

[1]Note that these tip fluctuations also drive the fluctuations in the position of stochastic fronts, namely determine the statistics of the position of the bulk of systems of particles generated by branching random walks supplemented with a selection process (see Ref. [22]), as well as the statistics of the genealogical trees (see Ref. [23]). Recently, evidence was given that the genealogies of extreme particles in BRWs without selection can be explained by the statistics of what we call tip fluctuations occurring during the evolution [24].

numbers in the tip, and we propose a conjecture for the parametric Δx dependence of their typical value.

II. GENERAL FRAMEWORK

As stated in the Introduction, we will essentially discuss BBM in one dimension. The evolution variable, namely the time, will be labeled by t, and the line on which the particles move by x. We shall always start at $t = 0$ with a single particle at the origin $x = 0$, which evolves in time through two independent stochastic processes: Diffusion (we shall set the diffusion coefficient D to 1), and splitting to two particles (at rate unity), which further evolve independently according to the very same rules.

We expect the results we will obtain to be universal, namely they should apply to a larger class of branching random walk models (such as the ones discussed in Ref. [15]), up to the replacement of the small set of constants which depend on the detailed definition of the particular process we may consider.

Our aim is to understand the particle density in a tip fluctuation. It is natural to select realizations of the BBM in which the rightmost particle, at some given large time t, is significantly ahead of its expected position, and then, to count the particles within an interval of fixed size Δx from the position of that lead particle. In this section, we shall introduce the formalism useful for our analytical work, as well as the numerical tools.

A. Generating functions of particle-number probabilities and their evolution

We start by introducing the joint probability $Q_n(x, x - \Delta x, t)$ that, at time t (which we will eventually take infinite), there is no particle to the right of x, and exactly n particles to the right of $y \equiv x - \Delta x$ ($\Delta x > 0$). Obviously, at the initial time, when the system consists of one single particle at the origin $x = 0$,

$$\begin{aligned} &Q_n(x, x - \Delta x, t = 0) \\ &\quad = \delta_{n,0}\Theta(x - \Delta x) + \delta_{n,1}\Theta(x)\Theta(\Delta x - x), \end{aligned} \tag{1}$$

where $z \mapsto \Theta(z)$ is the Heaviside function. Following Brunet and Derrida [15], who built on Ref. [25], we introduce the generating function[2]

$$\psi_\lambda(x, x - \Delta x, t) = \sum_{n=0}^{\infty} \lambda^n Q_n(x, x - \Delta x, t). \tag{2}$$

Then, defining

$$\phi(x) \equiv \lambda\, \Theta(x) + (1 - \lambda)\, \Theta(x - \Delta x), \tag{3}$$

it is a straightforward calculation to establish that the two-variable function

$$H_\phi(x, t) \equiv \psi_\lambda(x, x - \Delta x, t) \tag{4}$$

obeys the FKPP equation[3]

$$\partial_t H_\phi(x, t) = \partial_x^2 H_\phi(x, t) + H_\phi^2(x, t) - H_\phi(x, t) \tag{5}$$

with the initial condition

$$H_\phi(x, t = 0) = \phi(x). \tag{6}$$

It will prove useful to decompose $\phi(x)$ as the sum

$$\phi(x) = \phi_A(x) + \phi_B(x) - 1, \tag{7}$$

where

$$\begin{cases} \phi_A(x) \equiv 1 + (1 - \lambda)[\Theta(x - \Delta x) - \Theta(x)] \\ \phi_B(x) \equiv \Theta(x). \end{cases} \tag{8}$$

In these equations, Δx and λ are parameters, the value of which is fixed as far as the t evolution is concerned.

We then introduce the probability $Q(x, t)$ that there is no particle to the right of x. It is known to also obey the FKPP equation with the initial condition

$$\begin{aligned} &Q(x, t = 0) = \Theta(x) = \phi_B(x), \quad \text{and therefore,} \\ &Q(x, t) = H_{\phi_B}(x, t). \end{aligned} \tag{9}$$

Let us now write the generating functions for the particle-number probabilities within the interval Δx from the lead particle, considering two cases:

(1) First, the case in which the position of the lead particle is unconstrained, in order to make contact with Ref. [15]: If we call $p_n^{(0)}(\Delta x)$ the probability to find n particles in the interval of size Δx from the lead particle, then we write the corresponding generating function as

$$G_{\Delta x}^{(0)}(\lambda) = \sum_{n=1}^{\infty} \lambda^n p_n^{(0)}(\Delta x). \tag{10}$$

(In addition to λ and Δx, $G_{\Delta x}^{(0)}(\lambda)$ also depends on the time t, which is not explicitly mentioned since this dependency goes away in the limit $t \to +\infty$ of interest here).

(2) Second, for our purpose of understanding the tip fluctuations in rare realizations in which the lead particle sits very far ahead of its expected position, we need to fix its position x. Thus the quantity of interest for us is the generating function of the probability $p_n(\Delta x)$ that there be a number n of particles in the interval $[y \equiv x - \Delta x, x]$ *given* that the lead, rightmost, particle is at the position x:

$$G_{\Delta x}(\lambda) = \sum_{n=1}^{\infty} \lambda^n p_n(\Delta x). \tag{11}$$

(Again, $G_{\Delta x}(\lambda)$ also depends on the time t and in addition, on x, dependences which are understood since they are only relevant for finite t, a case that we shall not investigate here.)

The generating functions $G_{\Delta x}^{(0)}(\lambda)$ and $G_{\Delta x}(\lambda)$ do not directly obey the FKPP equation, unlike ψ_λ, but they can easily

[2]We slightly simplify the notations in Ref. [15]: Our ψ_λ would read $\psi_{0\lambda}$ in there.

[3]On terminology: In the literature, by FKPP equation is usually meant the nonlinear partial differential equation $\partial_t u = \partial_x^2 u + u(1 - u)$, up to dimensionful positive coefficients. We shall also use this name for the equation solved by $H_\Phi \equiv 1 - u$.

be deduced from ψ_λ and Q. It is clear that $G^{(0)}_{\Delta x}(\lambda)$ is related to ψ_λ through the following formula:[4]

$$G^{(0)}_{\Delta x}(\lambda) = \int_{-\infty}^{+\infty} dx\, \frac{\partial\psi_\lambda(x',y,t)}{\partial x'}\bigg|_{x'=x,\, y=x-\Delta x}. \tag{12}$$

It is equally straightforward to see that

$$G_{\Delta x}(\lambda) = \frac{\partial\psi_\lambda(x',y,t)/\partial x'\big|_{x'=x,\, y=x-\Delta x}}{\partial Q(x,t)/\partial x}. \tag{13}$$

Note that $G^{(0)}_{\Delta x}(\lambda)$ and $G_{\Delta x}(\lambda)$ can be written as a series of powers of $1-\lambda$ with the help of the factorial moments $n^{(k)}(\Delta x)$ of the particle number n in the interval $[x-\Delta x, x]$. For example,

$$G_{\Delta x}(\lambda) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!}(1-\lambda)^k n^{(k)}(\Delta x)$$
$$\text{with } n^{(k)}(\Delta x) \equiv \langle n(n-1)\cdots(n-k+1)\rangle, \tag{14}$$

where the expectation is taken over realizations which have the rightmost lead particle at the final time at position x.

B. Infinite-time asymptotics of the generating functions

1. Traveling wave solutions to the FKPP equation

When going to large times, the solutions to the FKPP equation Q and H_Φ, with $\Phi \equiv \phi$, ϕ_A or ϕ_B, all converge to *traveling waves* the shape of which is stable through time evolution and universal, i.e., largely independent of the details of the initial condition [26]. We shall call F the function of the space variable that encodes this shape. In this large-t limit, the evolution is essentially a translation of the traveling wave along the x axis. We shall denote by m_t the position of the wave front of Q at time t. The leading t dependence of dm_t/dt is also universal at large t. In practice, m_t can be computed by taking a particular integral of $Q(x,t)$ over the x variable, in which case it coincides with the expectation value of the position of the lead particle. More precisely, if we choose the definition

$$m_t \equiv \int_{-\infty}^{+\infty} dx\, x\, \frac{\partial Q(x,t)}{\partial x}, \tag{15}$$

then $m_t = \langle x_{\text{lead particle}}\rangle$, by definition of Q. It is then convenient to introduce a position variable in the frame of the traveling-wave front:

$$\xi \equiv x - m_t. \tag{16}$$

Equipped with these notations, we can rewrite the asymptotics of Q as

$$Q(\xi + m_t, t) \underset{t\to+\infty}{\simeq} F(\xi). \tag{17}$$

Because of the universality of the shape encoded in the function F and of the leading-t dependence of m_t at large t, the solutions for Q and H_ϕ can only differ by a shift $g(\lambda, \Delta x)$ in the argument of F. Hence

$$H_\phi(\xi + m_t, t) \underset{t\to+\infty}{\simeq} F[\xi - g(\lambda, \Delta x)]. \tag{18}$$

The shape function $F(z)$ connects 0 at $z=-\infty$ to 1 at $z=+\infty$. Its asymptotics read

$$F(z) \simeq \begin{cases} 1 - C \times z\, e^{-z} & \text{for large positive } z \\ C' \times e^{(\sqrt{2}-1)z} & \text{for large negative } z, \end{cases} \tag{19}$$

where C and C' are constants of order 1, which are unambiguous (although not calculable analytically) once m_t is properly defined. The t-dependent real number m_t has the following time dependence for large t:

$$m_t \underset{t\gg1}{\simeq} 2t - \frac{3}{2}\ln t, \tag{20}$$

up to an additive constant, independent of λ and of Δx but dependent on the very definition of the position of the wave front, and up to terms which vanish when t is large. With the definition $m_t \equiv \langle x_{\text{lead particle}}\rangle$, one has $m_{t=0} = 0$; But no analytical expression for m_t is known when t is small (i.e., of order unity).

Although we will eventually be interested only in the infinite-time limit, we will need the shape $F_t(z)$ of the FKPP traveling wave at finite (but large) time t:

$$F_t(z) \simeq 1 - C \times z\, e^{-z} e^{-z^2/4t} \quad \text{for large positive } z. \tag{21}$$

We note that $F_t(z) \simeq F(z)$ for $z < 2\sqrt{t}$, and that $1 - F_t(z)$ goes to 0 very quickly as soon as $z > 2\sqrt{t}$, due to the Gaussian damping factor.

2. Generating functions expressed in terms of the shift

In Ref. [14], Brunet and Derrida noticed that some quantitative properties of the tip of a branching random walk can be deduced from the only knowledge of the difference in the positions of the traveling waves resulting from the evolution of different initial conditions to very large times. In the context of the present paper, it turns out that a similar statement can be made: Generating functions of the particle-number probabilities in an interval of size Δx from the lead particle of a BRW are related to the differences of the positions of the asymptotic traveling waves H_ϕ and H_{ϕ_B}, encoded in the shift function g defined in Eq. (18).

Let us start with Eqs. (12) and (13). The derivative of ψ_λ in these formulas is easily expressed in terms of derivatives of the shape function F of the traveling waves and of the shift function g. From the relation between ψ_λ and H_ϕ, Eq. (4), and from the asymptotic behavior of the latter at infinite times, see Eq. (18), we deduce that the generating function $G^{(0)}_{\Delta x}(\lambda)$ is a simple function of the shift $g(\lambda, \Delta x)$:

$$G^{(0)}_{\Delta x}(\lambda) \simeq 1 - \frac{\partial g(\lambda, \Delta x)}{\partial \Delta x}. \tag{22}$$

In the same way, starting this time from Eq. (13), we find the following expression for the large-time asymptotics of the

[4]Note that the derivative $\partial\psi_\lambda(x',y,t)/\partial x'|_{x'=x,\, y=x-\Delta x}$ that appears in Eqs. (12) and (13) is with respect to the position of the lead particle, *keeping fixed the position of the left bound of the interval* in which one counts the particles. Therefore, this derivative does not coincide with a derivative of the two-variable function $H_\phi(x,t)$ that obeys the FKPP equation.

generating function $G_{\Delta x}(\lambda)$:

$$G_{\Delta x}(\lambda) \simeq \left[1 - \frac{\partial g(\lambda, \Delta x)}{\partial \Delta x}\right] \frac{F'[\xi - g(\lambda, \Delta x)]}{F'(\xi)}. \tag{23}$$

In order to simplify further this formula, let us pick a value of x such that $|\xi| \gg 1$ and $|g(\lambda, \Delta x)| \ll |\xi|$. Then, using Eq. (19), we see that $G_{\Delta x}(\lambda)$ boils down to a function of the shift only:

$$G_{\Delta x}(\lambda) \simeq \left[1 - \frac{\partial g(\lambda, \Delta x)}{\partial \Delta x}\right] \times \begin{cases} e^{g(\lambda, \Delta x)} & x > m_t \\ e^{-(\sqrt{2}-1) g(\lambda, \Delta x)} & x < m_t. \end{cases} \tag{24}$$

It is clear from these formulas that it is enough to compute $g(\lambda, \Delta x)$ in order to get the infinite-time asymptotics of $G_{\Delta x}(\lambda)$.

C. Particle-number probabilities from the generating function

The particle-number probabilities $p_n(\Delta x)$ (and, similarly, $p_n^{(0)}(\Delta x)$) can obviously be obtained from the generating function through an integration in the complex plane:

$$p_n(\Delta x) = \oint \frac{d\lambda}{2i\pi} \lambda^{-n-1} G_{\Delta x}(\lambda), \tag{25}$$

where the integration contour goes around the origin $\lambda = 0$, without encircling any singularity of $G_{\Delta x}(\lambda)$. However, this formula requires a precise knowledge of $G_{\Delta x}(\lambda)$, which we will not be able to arrive at here. But the behavior of the generating function as a function of λ and of the parameter Δx will enable us to conjecture properties of the particle number distribution.

It is clear that when $\lambda \to 1$ for fixed Δx, the generating function $G_{\Delta x}(\lambda)$ gets close to the unweighted sum over n of $p_n(\Delta x)$, and thus by unitarity, tends to 1. On the other hand, since $G_{\Delta x}(\lambda) = p_1(\Delta x)\lambda + \cdots$, it obviously tends to 0 when $\lambda \to 0$. From the values of λ at which the transition between 0 and 1 occurs, one may deduce the values of the particle numbers which are the most probable. Indeed, in the limit in which the typical particle numbers are large (which is necessarily relevant when Δx is very large),

$$G_{\Delta x}(\lambda) \underset{\Delta x \gg 1}{\simeq} \int_0^\infty dn\, e^{-n(1-\lambda)}\, p_n(\Delta x), \tag{26}$$

which shows that, under reasonable assumptions for the probabilities $p_n(\Delta x)$, only values of the particle numbers n less than $1/(1-\lambda)$ may contribute significantly to the integral.[5]

D. Numerical evaluation of the generating functions

Besides the analytical analysis, we shall also solve numerically the FKPP equation obeyed by the functions Q and H_ϕ: Once these functions are tabulated, the generating functions $G_{\Delta x}(\lambda)$ can be calculated.

We use an implementation proposed by Ebert and Van Saarloos [27]: The FKPP equation is discretized in space and time (with respective steps δx and δt) in a semi-implicit form, in such a way that the scheme leads to a close approximation of the continuous FKPP evolution, and is numerically stable even for very large times.

Concretely, let us call $u(x, t)$ a generic function which obeys the FKPP equation

$$\partial_t u = \partial_x^2 u + u - u^2, \tag{27}$$

and discretize it in x and t. We label the N_x spatial sites by the integer $j \in [1, N_x]$. For $j \in [2, N_x - 1]$, we replace the derivatives by finite differences as follows:

$$\begin{aligned} \frac{u_j(t+\delta t) - u_j(t)}{\delta t} &= \frac{1}{2}\frac{u_{j+1}(t) - 2u_j(t) + u_{j-1}(t)}{\delta x^2} \\ &\quad + \frac{1}{2}\frac{u_{j+1}(t+\delta t) - 2u_j(t+\delta t) + u_{j-1}(t+\delta t)}{\delta x^2} \\ &\quad + \frac{1}{2}\big[u_j(t+\delta t) + u_j(t)\big] - u_j(t+\delta t)u_j(t). \end{aligned} \tag{28}$$

Reshuffling the terms of this equation, we see that the evolution of u_j from times t to $t + \delta t$ is the solution to the linear set of equations

$$\begin{aligned} &-\frac{1}{2}u_{j+1}(t+\delta t) + u_j(t+\delta t)\left\{1 + \delta x^2\left[\frac{1}{\delta t} - \frac{1}{2} + u_j(t)\right]\right\} \\ &-\frac{1}{2}u_{j-1}(t+\delta t) \\ &= \frac{1}{2}u_{j+1}(t) - u_j(t)\left[1 - \delta x^2\left(\frac{1}{2} + \frac{1}{\delta t}\right)\right] + \frac{1}{2}u_{j-1}(t). \end{aligned} \tag{29}$$

We need a rule for the leftmost and rightmost x-lattice sites. We just extend the previous set of equations to the sites $j = 1$ and $j = N_x$ by assuming the following: $u_0 = 1$ and $u_{N_x+1} = 0$. Solving this equation then amounts to inverting a tridiagonal matrix at each timestep, of size the number N_x of lattice sites, for which there exist very efficient numerical methods [28].

In addition, in order to minimize the effects of the finite size of our spatial lattice, we shift back the front every $\mathcal{O}(1)$ time step, in such a way that its position on the lattice be approximately unchanged. In practice, at every integer time t, we replace $u_j(t)$ by $u_{j+k_t}(t)$ for all $j \in [1, N_x - k_t]$, and set $u_j(t)$ to 0 for $j \in [N_x - k_t + 1, N_x]$, where k_t is the integer part of the measured difference in the position of the front between times $t - 1$ and t.

The formulas we shall find below for the branching Brownian motion will still be valid, up to a set of constants appearing in the analytical expressions which will need to be slightly modified, since the effective diffusion constant and branching rate of the discretized branching Brownian motion are no longer strictly unity. The main parameters are the exponential rate γ_0 at which the traveling wave goes to zero at large x, its asymptotic velocity v_0 and a diffusion constant D, which are, respectively, 1, 2 and 1 in the case of the branching Brownian motion introduced above. The main changes are in the function $F_t(z)$ in Eq. (21), and in m_t in Eq. (20) which become

$$F_t(z) = 1 - C \times z\, e^{-\gamma_0 z} e^{-z^2/4Dt} \quad \text{and} \quad m_t = v_0 t - \frac{3}{2\gamma_0} \ln t. \tag{30}$$

[5]The same relations as (25),(26) exist of course between $p_n^{(0)}(\Delta x)$ and $G_{\Delta x}^{(0)}(\lambda)$.

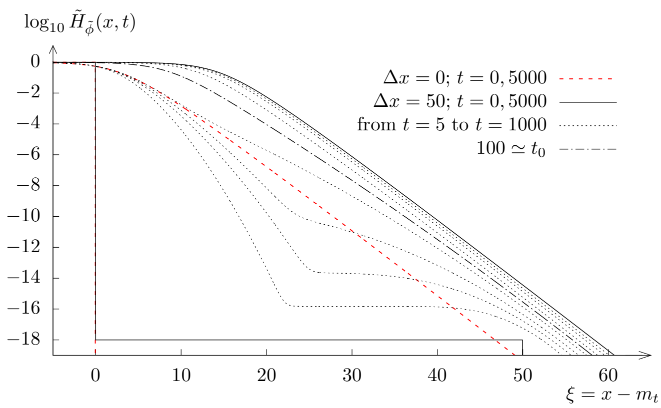

FIG. 2. Numerical evaluation of $\tilde{H}_{\tilde{\phi}}(x,t)$ as a function of $\xi = x - m_t$, that is, in the frame of the traveling wave $\tilde{H}_{\tilde{\phi}_B}$, at different times (logarithmic scale on the vertical axis). The parameters that characterize the initial condition are set to $\Delta x = 50$ and $1-\lambda = 10^{-18}$. *Solid line:* Initial condition $\log_{10}\tilde{\phi}(x)$ and $\log_{10}\tilde{H}_{\tilde{\phi}}(x, t = 5000)$, namely at a time at which the traveling wave has reached its asymptotic shape. The *dotted lines* represent $\log_{10}\tilde{H}_{\tilde{\phi}}(x,t)$ for $t = 5, 10, 20, 50, 200, 500, 1000$ to show the evolution between the initial condition and the universal traveling wave. The time $t = 100$, which is close to the special time τ computed from the formula (64) (see the discussion in Sec. III C 1 below) is singled out with the help of a *dash-dotted line*. *Dashed lines:* $\log_{10}\tilde{\phi}_B(x)$ and $\log_{10}\tilde{H}_{\tilde{\phi}_B}(x, t = 5000)$, for comparison.

The values of γ_0, v_0 and D are found by looking for solutions of the form $u_j(t) = e^{-\gamma[j\delta x - v(\gamma)t]}$ to the equation obtained by linearization of Eq. (28) when u is small. γ_0 minimizes $v(\gamma)$, i.e., solves $v'(\gamma_0) = 0$, and $v_0 = v(\gamma_0)$, $D = \frac{1}{2}\gamma_0 v''(\gamma_0)$.

In practice, we set the number of lattice sites to $N_x = 10^4$, the lattice spacing in x to $\delta x = 0.05$, and the lattice spacing in the t variable to $\delta t = 0.01$. With these latter two choices,

$$\begin{aligned} v_0 &\simeq 2 + \frac{2}{3}\delta t^2 + \frac{1}{12}\delta x^2 + \cdots = 2.000275\cdots, \\ \gamma_0 &\simeq 1 - \frac{2}{3}\delta t^2 - \frac{1}{8}\delta x^2 + \cdots = 0.9996208\cdots, \\ D &\simeq 1 + 3\delta t^2 + \frac{1}{2}\delta x^2 + \cdots = 1.00155\cdots. \end{aligned} \tag{31}$$

So the differences with the constants characteristic of the BBM are small. For completeness, let us mention that $F(z)$ in Eq. (19) is also modified for negative z: It becomes $F(z) \simeq C' \times e^{rz}$, with

$$\begin{aligned} r = \sqrt{2} - 1 + \left(\frac{4}{3} - \sqrt{2}\right)\delta t^2 + \frac{1}{3}\left(\frac{11}{8} - \sqrt{2}\right)\delta x^2 \\ + \cdots \simeq \left(\sqrt{2} - 1\right) - 0.0000407\cdots. \end{aligned} \tag{32}$$

The evolution of $\tilde{H}_{\tilde{\phi}}$ from an initial condition characterized by $\Delta x = 50$ and $1-\lambda = 10^{-18}$ to the large time $t = 5000$ is displayed in Fig. 2. At the final time, the comparison to $\tilde{H}_{\tilde{\phi}_B}$ is shown and a shift $g(\lambda, \Delta x)$ of order $\Delta x - f(\lambda) \simeq 12$ is clearly visible. (Observe that the rightmost full curve, representing $\tilde{H}_{\tilde{\phi}}$ at a large time, is related to the dashed curve, representing $\tilde{H}_{\tilde{\phi}_B}$ at the same time, by a mere translation along the ξ axis.)

Once the functions $\tilde{H}_{\tilde{\phi}}$ and $\tilde{H}_{\tilde{\phi}_B}$ are known for different values of Δx, ψ_λ, and Q follow from Eqs. (4) and (9); $G^{(0)}_{\Delta x}(\lambda)$ and $G_{\Delta x}(\lambda)$ can then be computed from the exact formulas (12) and (13) respectively. However, since we are interested in the $t \to +\infty$ and $|x - m_t| \gg 1$ limits, it is easier to measure numerically the front positions m_t for some large time, deduce the shift function $g(\lambda, \Delta x)$ and then derive $G^{(0)}_{\Delta x}(\lambda)$ and $G_{\Delta x}(\lambda)$ using Eqs. (22) and (24), respectively. In order to better approach the infinite-time limit, we can extrapolate the finite-t measurements of $g(\lambda, \Delta x)$ by assuming that the leading corrections[6] vanish as $\sim 1/t$ [15]. Extrapolating in this way from different times enables us to check the stability of the procedure.

III. ASYMPTOTICS OF THE SHIFT AND PROPERTIES OF THE GENERATING FUNCTIONS

A. General strategy

It will prove useful to change function $H \to 1 - H$. Therefore, for a generic function $\mathcal{F}(z)$ taking values between 0 and 1, we introduce the following notation:

$$\tilde{\mathcal{F}}(z) \equiv 1 - \mathcal{F}(z). \tag{33}$$

Then the function $\tilde{\phi}$, which is the initial condition for the time evolution of $\tilde{H}_{\tilde{\phi}} = 1 - H_\phi$ when ϕ is defined as in Eq. (3), is just the sum of the functions $\tilde{\phi}_A$ and $\tilde{\phi}_B$.

Let us decompose $\tilde{H}_{\tilde{\phi}}$ as the sum of two functions corresponding to the evolution either of $\tilde{\phi}_A$ or of $\tilde{\phi}_B$. We may perform this decomposition in two ways. First, we write

$$\tilde{H}_{\tilde{\phi}} = \tilde{H}_{\tilde{\phi}_B} + \Delta\tilde{H}_{\tilde{\phi}_A}. \tag{34}$$

If we require the first term in the r.h.s. to obey the FKPP equation with $\tilde{\phi}_B$ as an initial condition, then, since $\tilde{H}_{\tilde{\phi}}$ also obeys the FKPP equation, the functions $\tilde{H}_{\tilde{\phi}_B}$ and $\Delta\tilde{H}_{\tilde{\phi}_A}$ solve the following hierarchical system of equations:

$$\begin{aligned} \partial_t \tilde{H}_{\tilde{\phi}_B}(x,t) &= \partial_x^2 \tilde{H}_{\tilde{\phi}_B}(x,t) + \tilde{H}_{\tilde{\phi}_B}(x,t) - \tilde{H}^2_{\tilde{\phi}_B}(x,t) \\ \partial_t \Delta\tilde{H}_{\tilde{\phi}_A}(x,t) &= \partial_x^2 \Delta\tilde{H}_{\tilde{\phi}_A}(x,t) + [1 - 2\tilde{H}_{\tilde{\phi}_B}(x,t)]\Delta\tilde{H}_{\tilde{\phi}_A}(x,t) \\ &\quad - \Delta\tilde{H}^2_{\tilde{\phi}_A}(x,t). \end{aligned} \tag{35}$$

The solution of the first equation reads, in the limit of infinite time:

$$\tilde{H}_{\tilde{\phi}_B}(\xi + m_t, t) \underset{t\to+\infty}{\simeq} \tilde{F}(\xi) \equiv 1 - F(\xi). \tag{36}$$

$\Delta\tilde{H}_{\tilde{\phi}_A}$ obeys an equation with a new term with respect to the FKPP equation, proportional to $\tilde{H}_{\tilde{\phi}_B}$. This term vanishes when $\tilde{H}_{\tilde{\phi}_B} \ll 1$, while it damps the linear growth term when $\tilde{H}_{\tilde{\phi}_B} \sim 1$. This way of decomposing the equation for $\tilde{H}_{\tilde{\phi}}$ will prove

[6]This is because the terms that vanish with t in the large-t expansion of m_t form a series in powers of $t^{-1/2}$. The first term, of order $t^{-1/2}$ has a universal coefficient (independent of the initial conditions) [27]; Therefore, the first time-dependent correction in $g(\lambda, \Delta x)$, which is the difference of the position of two FKPP fronts starting from distinct initial conditions, is expected to be of order $1/t$.

useful when $\tilde{H}_{\tilde{\phi}_B}$ eventually dominates $\tilde{H}_{\tilde{\phi}}$, and $\Delta\tilde{H}_{\tilde{\phi}_A}$ can be considered a small perturbation.

But we can also write

$$\tilde{H}_{\tilde{\phi}} = \tilde{H}_{\tilde{\phi}_A} + \Delta\tilde{H}_{\tilde{\phi}_B}, \tag{37}$$

where

$$\begin{aligned}\partial_t \tilde{H}_{\tilde{\phi}_A}(x,t) &= \partial_x^2 \tilde{H}_{\tilde{\phi}_A}(x,t) + \tilde{H}_{\tilde{\phi}_A}(x,t) - \tilde{H}^2_{\tilde{\phi}_A}(x,t)\\ \partial_t \Delta\tilde{H}_{\tilde{\phi}_B}(x,t) &= \partial_x^2 \Delta\tilde{H}_{\tilde{\phi}_B}(x,t) + [1-2\tilde{H}_{\tilde{\phi}_A}(x,t)]\Delta\tilde{H}_{\tilde{\phi}_B}(x,t)\\ &\quad - \Delta\tilde{H}^2_{\tilde{\phi}_B}(x,t).\end{aligned} \tag{38}$$

Here, $\tilde{H}_{\tilde{\phi}_A}$ obeys the FKPP equation, but with the initial condition $\tilde{\phi}_A$, which is a rectangle of width Δx and height $1-\lambda$. Its large-time solution is also a traveling wave $\tilde{F}$, but shifted by a function of λ and Δx. Since $\tilde{H}_{\tilde{\phi}_A}$ starts out very small, $\Delta\tilde{H}_{\tilde{\phi}_B}$ dominates in the beginning of the evolution, and thus this decomposition may seem irrelevant. However, we will see that there are cases in which $\tilde{H}_{\tilde{\phi}_A}$ eventually dominates while $\Delta\tilde{H}_{\tilde{\phi}_B}$ can be considered a small perturbation, and those will turn out to be the most interesting cases.

All functions appearing in these equations take values between 0 and 1. Both systems of equations (35),(38) linearize and decouple in regions in which $\tilde{H}_{\tilde{\phi}} \ll 1$. In these regions, the functions $\tilde{H}_{\tilde{\Phi}}$, $\Delta\tilde{H}_{\tilde{\Phi}}$ (with $\tilde{\Phi} = \tilde{\phi}_A$ or $\tilde{\phi}_B$) approximately obey an equation of the form

$$\partial_t \tilde{h}(x,t) = \partial_x^2 \tilde{h}(x,t) + \tilde{h}(x,t), \tag{39}$$

where $\tilde{h}$ stands for a solution to Eq. (39) constrained by the appropriate initial condition.

This equation can easily be solved. However, even in the regions in which the linear approximation is valid, the nonlinearities may act as effective absorptive boundary conditions, deforming the solution of the initial-value problem. It is known that by putting an absorptive boundary on the linearized equation, one can get an approximate solution to the FKPP equation, which converges to the exact solution at large time (see, e.g., Ref. [27]). Our approach will essentially consist of replacing the nonlinearities appearing in Eqs. (35) and (38) by suitable boundary conditions.

The first step is to solve Eq. (39) with the initial condition $\tilde{\phi}_A$ and an absorptive (moving) boundary to effectively represent the nonlinearities. Once this is done, we will be able to compute $\tilde{H}_{\tilde{\phi}}$ in limiting cases in which the $\Delta\tilde{H}_{\tilde{\Phi}}$ functions in the decompositions bring very small contributions to $\tilde{H}_{\tilde{\phi}}$ in the limit of large times.

B. Linear evolution of a flat rectangle

1. Free boundaries

Let us solve the linearized equation (39). We start with the localized initial condition $\delta(x-x_0)$ at some positive time t_0, without imposing boundary conditions at this stage. The exact solution at time $t > t_0$ reads

$$\tilde{h}_{x_0}(x,t-t_0) = \frac{1}{\sqrt{4\pi(t-t_0)}} e^{t-t_0} \exp\left[-\frac{(x-x_0)^2}{4(t-t_0)}\right]. \tag{40}$$

Let us go to a frame comoving with the traveling-wave solution to the FKPP equation for Q, by changing coordinates $x \to \xi = x - m_t$, $x_0 \to \xi_0 = x_0 - m_{t_0}$, and let us keep only the leading terms for $\xi \ll t$. In the two cases which will be relevant to what follows,

$$\tilde{h}_{\xi_0+m_{t_0}}(\xi+m_t, t-t_0) = \begin{cases} \frac{t}{\sqrt{4\pi}} e^{-(\xi-x_0)} \exp\left[-\frac{(\xi-x_0)^2}{4t}\right] & \text{if } t_0 = 0,\\ \frac{1}{\sqrt{4\pi(t-t_0)}}\left(\frac{t}{t_0}\right)^{3/2} e^{-(\xi-\xi_0)} \exp\left[-\frac{(\xi-\xi_0)^2}{4(t-t_0)}\right] & \text{if } t_0 \gg 1,\end{cases} \tag{41}$$

where it is understood that $m_{t=0} = 0$; The large-time asymptotical formula (20) has also been used for m_t, as well as for m_{t_0} in the case $t_0 \gg 1$.

We now turn to the case in which the initial condition $\tilde{\phi}_A(x)$ is taken at $t_0 = 0$. This case will prove relevant since the solution to the linearized equation (39) with this initial condition is an approximation to the solution to the full nonlinear equations for $\tilde{H}_{\tilde{\phi}_A}$ valid when $\tilde{H}_{\tilde{\phi}_A} \ll 1$, and for $\Delta\tilde{H}_{\tilde{\phi}_A}$ when $\Delta\tilde{H}_{\tilde{\phi}_A} \ll 1$ and $\tilde{H}_{\tilde{\phi}_B} \ll 1$ simultaneously. We get $\tilde{h}_{\tilde{\phi}_A}$ by superposing the solutions with initial conditions localized at x_0, see Eq. (40), for all $x_0 \in [0, \Delta x]$, and by weighting them uniformly by $1-\lambda$:

$$\begin{aligned}\tilde{h}_{\tilde{\phi}_A}(x,t) &= \int_0^{\Delta x} dx_0\,(1-\lambda)\,\tilde{h}_{x_0}(x,t)\\ &= \frac{1}{2}(1-\lambda)e^t\left[\operatorname{erfc}\frac{x-\Delta x}{\sqrt{4t}} - \operatorname{erfc}\frac{x}{\sqrt{4t}}\right],\end{aligned} \tag{42}$$

where $z \mapsto \operatorname{erfc}(z)$ is the complementary error function, defined with the following normalization [29]:

$$\operatorname{erfc}(z) \equiv \frac{2}{\sqrt{\pi}} \int_z^{+\infty} dw\, e^{-w^2}. \tag{43}$$

We will eventually be interested in the shift $g(\lambda, \Delta x)$, which we get focusing on the region of x where $\tilde{h}_{\tilde{\phi}_A} \sim \mathcal{O}(1)$, namely for $x \sim 2t$. In this region, obviously, $x \gg \sqrt{t}$, and Δx is finite, such that both error functions may be expanded using

$$\operatorname{erfc}(z) \underset{z\to+\infty}{\sim} \frac{e^{-z^2}}{z\sqrt{\pi}}. \tag{44}$$

Since furthermore $\Delta x \gg 1$, only the Δx-dependent term in Eq. (42) is relevant. Hence

$$\tilde{h}_{\tilde{\phi}_A}(x,t) \simeq \frac{1-\lambda}{\sqrt{\pi}} \frac{\sqrt{t}}{x-\Delta x} e^t \exp\left[-\frac{(x-\Delta x)^2}{4t}\right]. \tag{45}$$

Going to the comoving frame defined above and keeping again only the leading terms for $\xi \ll t$, we find

$$\tilde{h}_{\tilde{\phi}_A}(\xi + m_t, t) \simeq \frac{1}{\sqrt{4\pi}}\,(1-\lambda)\,t\,e^{-(\xi-\Delta x)}\exp\left[-\frac{(\xi-\Delta x)^2}{4t}\right]. \tag{46}$$

We note that the following identity holds in these approximations:

$$\tilde{h}_{\tilde{\phi}_A}(\xi + m_t, t) = (1-\lambda)\,\tilde{h}_{x_0=\Delta x}(\xi + m_t, t). \tag{47}$$

In words, the rectangular initial condition leads asymptotically to the same traveling wave as an initial condition localized at the rightmost edge of the rectangle, namely at $x = \Delta x$.

Let us stress that $\tilde{h}_{\tilde{\phi}_A}$ becomes of order 1 when

$$\mathcal{E} \equiv -\ln\frac{1}{1-\lambda} + \ln t - (\xi - \Delta x) - \frac{(\xi-\Delta x)^2}{4t} \tag{48}$$

goes to zero. The equation $\mathcal{E} = 0$ has a solution if $t > t_-$, where t_- obeys

$$t_- + \ln t_- = \ln\frac{1}{1-\lambda}. \tag{49}$$

By iteration, keeping only the large terms when $\lambda \to 1$, we can get a closed expression for t_-:

$$t_- \simeq f(\lambda), \quad \text{with} \quad f(\lambda) \equiv \ln\frac{1}{1-\lambda} - \ln\ln\frac{1}{1-\lambda}. \tag{50}$$

For $t > t_-$, the rightmost point for which unitarity is reached is at the position

$$\xi = \Delta x - 2t\left[1 - \sqrt{1 - \frac{1}{t}\left(\ln\frac{1}{1-\lambda} - \ln t\right)}\right]. \tag{51}$$

2. Generic absorptive boundary

We now add an absorptive boundary condition to the linear equation, which will later be appropriately chosen to represent the relevant nonlinearities. Let us denote the position of the boundary in the comoving frame by X_t. Mathematically, the absorptive boundary condition reads

$$\tilde{h}(x = X_t + m_t, t) = 0. \tag{52}$$

The solution to this new problem is obtained through the method of images.[7]

We start with the localized initial condition $\delta(x - x_0)$ at time t_0. The method of images amounts to modifying Eq. (41) by replacing the Gaussian factor therein as follows:

$$\exp\left[-\frac{(\xi-\xi_0)^2}{4(t-t_0)}\right] \longrightarrow \exp\left[-\frac{(\xi-\xi_0)^2}{4(t-t_0)}\right] - \exp\left[-\frac{(\xi+\xi_0-2X_t)^2}{4(t-t_0)}\right]. \tag{53}$$

The latter can be rewritten as

$$\exp\left[-\frac{(\xi-X_t)^2}{4(t-t_0)}\right]\left\{\exp\left[\frac{(\xi_0-X_t)(\xi-X_t)}{2(t-t_0)}\right] - \exp\left[-\frac{(\xi_0-X_t)(\xi-X_t)}{2(t-t_0)}\right]\right\}\exp\left[-\frac{(\xi_0-X_t)^2}{4(t-t_0)}\right]. \tag{54}$$

For t such that none of the two Gaussian factors give a large suppression, i.e., for $|\xi_0 - X_t|$ and $|\xi - X_t|$ small compared to $t - t_0$, which, we anticipate, can both be realized by taking t large enough and by choosing appropriately ξ, one may linearize the difference of exponentials. This results in the following expression:

[7] The method of images would be straightforward if the boundary were enforced at a fixed x. Here, the position of the boundary has a t dependence, so a little more care is needed: See, for example, Ref. [30] for the first application of this method in the context of high-energy physics, to derive the solution of an equation for the hadronic scattering amplitudes at very high energies.

$$\tilde{h}_{\xi_0+m_{t_0},X_t}(\xi + m_t, t - t_0) \simeq \begin{cases} \frac{1}{\sqrt{4\pi}}\,(x_0 - X_t)(\xi - X_t)\,e^{-\xi+x_0}\exp\left[-\frac{(\xi-X_t)^2}{4t} - \frac{(x_0-X_t)^2}{4t}\right] & \text{if } t_0 = 0,\\ \frac{1}{\sqrt{4\pi}}\left[\frac{t}{t_0(t-t_0)}\right]^{3/2}(\xi_0 - X_t)(\xi - X_t)\,e^{-\xi+\xi_0} \times\exp\left[-\frac{(\xi-X_t)^2}{4(t-t_0)} - \frac{(\xi_0-X_t)^2}{4(t-t_0)}\right] & \text{if } t_0 \gg 1, \end{cases} \tag{55}$$

where we kept the Gaussian factors as cutoffs.

We turn to the case of the rectangular initial condition $\tilde{\phi}_A$ at $t_0 = 0$. Actually, we may get the answer without any further calculation, by taking advantage of the identification (47). For reasons that will become clear below, we introduce $\delta_t \equiv \Delta x - X_t$ in order to parametrize the position of the boundary. Simple replacements in Eq. (55) lead to

$$\tilde{h}_{\tilde{\phi}_A,\delta_t}(\xi + m_t, t) \simeq \frac{1}{\sqrt{4\pi}}(1-\lambda)\,\delta_t\,(\xi - \Delta x + \delta_t)\,e^{-\xi+\Delta x} \times \exp\left[-\frac{(\xi-\Delta x+\delta_t)^2}{4t} - \frac{\delta_t^2}{4t}\right]. \tag{56}$$

In the present context, we will basically replace the nonlinearities in Eqs. (35) and (38) by appropriate moving boundaries. The latter equations boil then down to the linear equation (39) that governs the evolution of $\tilde{h}$, supplemented by boundary conditions.

C. Limiting cases

1. Small shift

If $\lambda \to 1$ while Δx is kept fixed, one does not expect that $\tilde{H}_{\tilde{\phi}}$ be very different from $\tilde{H}_{\tilde{\phi}_B}$. In this case, it seems appropriate to address the problem starting from the system of equations (35). $\Delta\tilde{H}_{\tilde{\phi}_A}$ in that decomposition can be regarded as a perturbation. Consequently, the shift $g(\lambda, \Delta x)$ will be small for such values of the parameters. (We will soon determine the parametric region in which this is true.)

The function $\tilde{H}_{\tilde{\phi}_B}(x, t)$ obeys the ordinary FKPP equation, and thus its solution is a traveling-wave front at position $x = m_t$, i.e., at $\xi = 0$ by definition of the comoving frame; See Eq. (36). In the initial stages of the evolution and for $\xi > 0$, the equation for $\Delta\tilde{H}_{\tilde{\phi}_A}$ can be thought of as a linear equation with an absorptive boundary at the position $\xi = 0$ of the front B. Hence $\Delta\tilde{H}_{\tilde{\phi}_A} \propto \tilde{h}_{\tilde{\phi}_A,\delta_t=\Delta x}$, and thus, from Eq. (56),

$$\Delta\tilde{H}_{\tilde{\phi}_A}(\xi + m_t, t) \simeq C_0 \times (1-\lambda)\Delta x\, e^{\Delta x - \Delta x^2/4t} \times \xi\, e^{-\xi - \xi^2/4t}. \tag{57}$$

We do not have control over the overall factor C_0, which we, however, expect to be of order 1.

We note that $\Delta\tilde{H}_{\tilde{\phi}_A}$ remains of order 1 or less for any $\xi > 0$ throughout the time evolution whenever the parameters Δx and λ satisfy

$$(1-\lambda)\Delta x\, e^{\Delta x} < 1, \quad \text{namely when} \quad \Delta x < f(\lambda), \tag{58}$$

where $f(\lambda)$ is the time t_- introduced in Eq. (50).

Let us assume that the condition (58) is strongly satisfied, i.e., $f(\lambda) - \Delta x \gg 1$, so that $\Delta\tilde{H}_{\tilde{\phi}_A} \ll 1$ throughout and for all $\xi > 0$. Then the second nonlinear term in its evolution equation (35) can always be neglected. Therefore, in this limit, the infinite-time solution of the system of equations reads

$$\begin{aligned} \tilde{H}_{\tilde{\phi}_B}(\xi + m_t, t) &\underset{\text{Eq. (36)}, t\to+\infty}{\simeq} C \times \xi e^{-\xi}, \\ \Delta\tilde{H}_{\tilde{\phi}_A}(\xi + m_t, t) &\underset{\text{Eq. (57)}, t\to+\infty}{\simeq} C_0 \times (1-\lambda)\Delta x\, e^{\Delta x} \times \xi\, e^{-\xi}. \end{aligned} \tag{59}$$

The sum of these two functions amounts to $\tilde{H}_{\tilde{\phi}}$, which as we know becomes, in the infinite-time limit, the same traveling wave as $\tilde{H}_{\tilde{\phi}_B}$ but shifted by $g(\lambda, \Delta x)$:

$$\begin{aligned} &\tilde{H}_{\tilde{\phi}_B} + \Delta\tilde{H}_{\tilde{\phi}_A} = \tilde{H}_{\tilde{\phi}} \\ &\Rightarrow C \times \xi e^{-\xi}\left[1 + \frac{C_0}{C} \times (1-\lambda)\Delta x\, e^{\Delta x}\right] \\ &\qquad\simeq C \times [\xi - g(\lambda, \Delta x)]e^{-\xi + g(\lambda,\Delta x)}. \end{aligned} \tag{60}$$

This equality should hold for $\xi \gg 1$. Since $g(\lambda, \Delta x)$ is small compared to 1, taking ξ much larger than 1, the ξ dependence cancels left and right. This equation turns into a closed expression for $g(\lambda, \Delta x)$, up to the unknown constant $c \equiv C_0/C$:

$$g(\lambda, \Delta x) \simeq \ln[1 + c(1-\lambda)\Delta x\, e^{\Delta x}] \simeq c(1-\lambda)\Delta x\, e^{\Delta x}. \tag{61}$$

This expression is valid when the parameters satisfy the condition (58) strongly. The generating functions $G^{(0)}_{\Delta x}(\lambda)$ and $G_{\Delta x}(\lambda)$ are then very close to 1.

According to Eq. (22) with $g(\lambda, \Delta x)$ replaced by the expression in Eq. (61), sticking to the $\Delta x \gg 1$ limit, $G^{(0)}_{\Delta x}(\lambda)$ reads

$$G^{(0)}_{\Delta x}(\lambda) \simeq 1 - (1-\lambda)\, c\, \Delta x\, e^{\Delta x}. \tag{62}$$

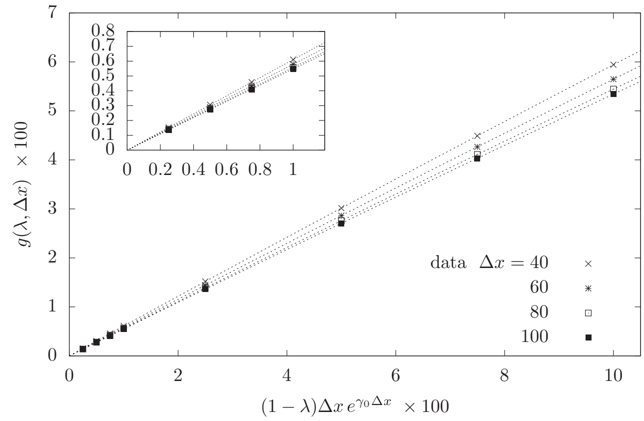

FIG. 3. Asymptotic shift $g(\lambda, \Delta x)$ as a function of the variable $(1-\lambda)\Delta x\, e^{\gamma_0 \Delta x}$, for different values of Δx. Note that this variable, as well as the shift, are both scaled by a factor 100, for the sake of a better visibility. The *inset* is a zoom, by a factor 10, on the small values of the variable. The *dotted lines* represent quadratic fits to the numerical data for fixed Δx, intended to guide the eye. The slopes at the origin for the two largest values of Δx differ by about 2%.

According to Eqs. (24) and (61), $G_{\Delta x}(\lambda)$ reads

$$G_{\Delta x}(\lambda) \simeq \begin{cases} 1 - (1-\lambda)c'\, e^{\Delta x} & \text{for } x > m_t \\ 1 - (1-\lambda)c\sqrt{2}\,\Delta x\, e^{\Delta x} & \text{for } x < m_t. \end{cases} \tag{63}$$

c' is another constant, *a priori* distinct from c, that we cannot determine due to our lack of control of the terms of order $(1-\lambda)e^{\Delta x}$ in Eq. (61).

The explicit expressions (62) and (63) will enable us to derive *mean* particle numbers in the interval $[x - \Delta x, x]$, see Sec. IV A below.

Comparison to a numerical solution of the FKPP equation. We can easily check these formulas for the generating functions numerically. It is actually enough to check the form (61) for the shift[8] $g(\lambda, \Delta x)$: The expressions for the generating functions indeed follow from the one for the shift through the exact relations (63) and (62) in the infinite-time limit.

Hence, for different choices of λ and Δx such that $f(\lambda) - \Delta x \gg 1$, we solve the FKPP equation starting from the initial condition ϕ, and we compare, at some large time t, the position of the obtained front with that of the front that results from the evolution of the initial condition ϕ_B. (Actually, in practice, we measure g at time $t = 9000$, and again at time $t = 12000$, and use the two obtained values to extrapolate to $t = +\infty$ under the assumption that the difference to the asymptotical value decreases as $1/t$. We checked that we get indeed very close to the asymptotics by comparing to extrapolations from smaller times.)

Our numerical results are displayed in Fig. 3: They show a good agreement with the formula (61), which improves as Δx grows larger and λ gets closer to 1.

[8]We actually include the factor γ_0 in the exponent in Eq. (61) in order to take into account the difference between the FKPP equation and its discretized version we are solving, see the caption of Fig. 3.

2. Large shift

If $\Delta x > f(\lambda)$ instead, then the linearized evolution drives $\Delta\tilde{H}_{\tilde{\phi}_A}$ to values larger than 1. We observe that this happens at a time on the order of

$$\tau \equiv \frac{\Delta x^2}{4[\Delta x - f(\lambda)]}. \tag{64}$$

τ turns out to be the main timescale in this problem: It is actually the time at which the evolution of A reaches unitarity at the position of the front B. When this happens, we cannot regard $\Delta\tilde{H}_{\tilde{\phi}_A}$ as a perturbation to $\tilde{H}_{\tilde{\phi}_B}$. Instead, the shift $g(\lambda, \Delta x)$ of the position of the asymptotic traveling wave will be substantial. In this situation, it may be more appropriate to address the problem from the point of view of the system of equations (38). Indeed, we expect that the evolution of the front A brings a significant contribution to $\tilde{H}_{\tilde{\phi}}$. Therefore, we shall study the limit in which, at large times, the front A dominates $\tilde{H}_{\tilde{\phi}}$, while the front B is a small perturbation.

The first equation of the system (38) is the FKPP equation for $\tilde{H}_{\tilde{\phi}_A}$. For times less than t_- defined in Eq. (50), the evolution is essentially linear. For larger times, we replace the nonlinearity by a moving absorptive boundary at $\Delta x - \delta_t$, as in Eq. (56): $\tilde{H}_{\tilde{\phi}_A} \propto \tilde{h}_{\tilde{\phi}_A,\delta_t}$. Now, we set δ_t in such a way that the maximum of $\tilde{H}_{\tilde{\phi}_A}$ be 1. A straightforward calculation leads to the following expression for δ_t:

$$\delta_t = 2t\left(1 - \sqrt{1 - \frac{1}{t}\ln\frac{1}{1-\lambda}}\right) - \frac{1}{\sqrt{1 - \frac{1}{t}\ln\frac{1}{1-\lambda}}} \times \ln\left[2t\left(1 - \sqrt{1 - \frac{1}{t}\ln\frac{1}{1-\lambda}}\right)\right]. \tag{65}$$

Expanding for large t and $\lambda \to 1$,

$$\delta_t = \left(\ln\frac{1}{1-\lambda} - \ln\ln\frac{1}{1-\lambda}\right) + \frac{1}{4t}\ln^2\frac{1}{1-\lambda} + \mathcal{O}\left(\frac{1}{t}\ln\frac{1}{1-\lambda}\ln\ln\frac{1}{1-\lambda}\right). \tag{66}$$

In particular, $\delta_{t\to\infty} = f(\lambda)$, where $f(\lambda)$ was defined in Eq. (50). The asymptotic front is then described by

$$\tilde{H}_{\tilde{\phi}_A}(x = \xi + m_t, t) \underset{t\to+\infty}{\simeq} C \times [\xi - \Delta x + f(\lambda)]\, e^{-\xi+\Delta x - f(\lambda)}$$
$$\text{with} \quad f(\lambda) = \ln\frac{1}{1-\lambda} - \ln\ln\frac{1}{1-\lambda}. \tag{67}$$

These expressions are valid in the limit $t \to +\infty$, $x - m_t \gg 1$, $|\ln(1-\lambda)| \gg 1$, $\xi - \Delta x + f(\lambda) \gg 1$. We have recovered a result obtained by Brunet and Derrida [14] for the "delay" of a traveling wave when the initial condition for the FKPP equation is the small step $(1-\lambda)\Theta(\Delta x - x)$, with respect to a traveling wave evolved from the full step function $\Theta(-x)$ connecting 0 and 1.

Let us now turn to the discussion of $\Delta\tilde{H}_{\tilde{\phi}_B}$. So long as $t < \tau$, where τ is given by Eq. (64), its evolution is driven by the plain FKPP equation: Indeed, the term involving $\tilde{H}_{\tilde{\phi}_A}$ in the second equation of the system (38) can be neglected. This means that

$$\Delta\tilde{H}_{\tilde{\phi}_B}(\xi + m_t, t) \underset{t<\tau}{\simeq} C \times \xi\, e^{-\xi - \xi^2/4t}. \tag{68}$$

But for $t > \tau$, the term involving $\tilde{H}_{\tilde{\phi}_A}$ in Eq. (38) is no longer negligible: Front A starts to "cut off" front B. The evolution of B ahead of this effective cutoff may then be approximated by the linearized equation supplemented by an absorptive boundary at the position of front A, namely at $\xi = \Delta x - f(\lambda)$. But the size of front B at the time τ at which the evolution of A starts to cut it off is of order $\sqrt{\tau}$, see Eq. (68), that is $\Delta\tilde{H}_{\tilde{\phi}_B}$ is negligible in the region $\xi > \sqrt{\tau}$. Consequently, if

$$\sqrt{\tau} \ll \Delta x - f(\lambda), \quad \text{namely} \quad \Delta x - f(\lambda) \gg \Delta x^{2/3}, \tag{69}$$

then the evolution of $\tilde{H}_{\tilde{\phi}}$ coincides with a good approximation with the evolution of front A: for $t > \tau$, $\tilde{H}_{\tilde{\phi}} \simeq \tilde{H}_{\tilde{\phi}_A}$. In this case, the asymptotic shift is simply the position of front A in the comoving frame, at infinite time, namely

$$g(\lambda, \Delta x) \underset{\Delta x - f(\lambda) \gg (\Delta x)^{2/3}}{\simeq} \Delta x - f(\lambda). \tag{70}$$

If instead $1 \ll \Delta x - f(\lambda) \ll \Delta x^{2/3}$, then, at time τ, front B is cut off by front A in a region in which $\tilde{H}_{\tilde{\phi}_B} \sim 1$. Thus for $\xi \gg \Delta x - f(\lambda)$, $\tilde{H}_{\tilde{\phi}}$ is just the sum of two FKPP fronts

$$\tilde{H}_{\tilde{\phi}}(\xi + m_t, t) \underset{t\to+\infty}{\simeq} C \times \xi\, e^{-\xi + \Delta x - f(\lambda)} + C \times \xi\, e^{-\xi}. \tag{71}$$

In the present limit in which $\Delta x - f(\lambda)$ is large, it is clear that the second term (front B) is small compared to the first one (front A): Front A eventually dominates, while front B can be considered a perturbation. The shift is calculated by solving the equation $\tilde{H}_{\tilde{\phi}}[\xi - g(\lambda, \Delta x) + m_t, t] = \tilde{H}_{\tilde{\phi}_B}(\xi + m_t, t)$ in the limit of infinite time and large ξ. We find

$$g(\lambda, \Delta x) = \Delta x - f(\lambda) + e^{-\Delta x + f(\lambda)}. \tag{72}$$

Introducing a function $\alpha(\lambda, \Delta)$, we may summarize the properties of the shift in the region $\Delta x > f(\lambda)$ by the following formula:

$$g(\lambda, \Delta x) = \Delta x - f(\lambda) + \alpha(\lambda, \Delta x)\, e^{-\Delta x + f(\lambda)}$$
$$\text{with} \quad \begin{cases} \alpha(\lambda, \Delta x) \to 1 & \text{when } \Delta x - f(\lambda) \ll \Delta x^{2/3} \\ \alpha(\lambda, \Delta x) \to 0 & \text{when } \Delta x - f(\lambda) \gg \Delta x^{2/3}. \end{cases} \tag{73}$$

Using Eqs. (22) and (24) to relate the shift to the generating functions, we find that $G^{(0)}_{\Delta x}(\lambda)$ and $G_{\Delta x}(\lambda)$ read

$$\begin{cases} G^{(0)}_{\Delta x}(\lambda) = \left[\alpha(\lambda, \Delta x) - \frac{\partial\alpha(\lambda,\Delta x)}{\partial\Delta x}\right]\exp\left[-\Delta x + f(\lambda)\right] \\ G_{\Delta x}(\lambda) = \left[\alpha(\lambda, \Delta x) - \frac{\partial\alpha(\lambda,\Delta x)}{\partial\Delta x}\right]\exp\left[\alpha(\lambda, \Delta x) e^{-\Delta x + f(\lambda)}\right]. \end{cases} \tag{74}$$

It is clear from these formulas that $G_{\Delta x}(\lambda)$ is of order 1 for any $(\lambda, \Delta x)$ satisfying $1 \ll \Delta x - f(\lambda) \lesssim \Delta x^{2/3}$ (and actually also when the first inequality is released), while it tends to zero for $\Delta x - f(\lambda) \gg \Delta x^{2/3}$. This is in sharp contrast with $G^{(0)}_{\Delta x}(\lambda)$ for which the parametric region where it is of order 1 is reduced to $\Delta x - f(\lambda) \lesssim 1$, because of the exponential factor.

D. Intermediate region: Scaling law for the generating function $G_{\Delta x}(\lambda)$

Let us now try and pin down more properties of the function α introduced in Eq. (73). We have identified τ defined in Eq. (64) as a relevant time scale: It is the time at which front A starts to cut off front B. Since the position of front A eventually reads $\Delta x - f(\lambda)$, see Eq. (67), the square of the latter quantity is another relevant time scale: It is the time needed for a FKPP front to extend over that distance. We define

$$t_{\text{diff}} \equiv \frac{1}{4}[\Delta x - f(\lambda)]^2. \quad (75)$$

It is then natural to expect the generating function $G_{\Delta x}(\lambda)$ to scale with the ratio $\sigma^2 \equiv t_{\text{diff}}/\tau$, namely

$$G_{\Delta x}(\lambda) \text{ is a function of } \sigma^2(\lambda, \Delta x) \equiv \frac{[\Delta x - f(\lambda)]^3}{\Delta x^2} \text{ only.} \quad (76)$$

A full calculation of $G_{\Delta x}(\lambda)$ in the relevant parametric region would be out of reach. However, as will be explained in Subsec. III D 1, such a scaling emerges from an analytical calculation in the context of a toy model for the system of equations (38), and can be searched for numerically, as will be shown in Subsec. III D 2.

1. Scaling in a toy model

In the parametric regime of interest in this section, the shift function is essentially determined by the A front, while the B front, at large time, brings a small perturbation to the latter. What makes, however, this case complicated is that in the beginning of the evolution, until a time $t_0 \sim \tau$, the evolution is dominated by front B; Namely, $\tilde{H}_{\tilde{\phi}}(x,t) \underset{t<t_0}{\simeq} \Delta\tilde{H}_{\tilde{\phi}_B}(x,t)$. This is because $\tilde{H}_{\tilde{\phi}_A}$ starts out very small, and remains so in the region $\xi > 0$ until t_0.

As discussed above, for t much larger than t_0 and in the region $\xi - [\Delta x - f(\lambda)] \gg 1$, the time evolution of $\Delta\tilde{H}_{\tilde{\phi}_B}$ can be approximated by a linear equation with a boundary at the position where the nonlinearity in the equation for $\tilde{H}_{\tilde{\phi}_A}$ cuts off the linear evolution, namely at $\xi = \Delta \equiv \Delta x - f(\lambda)$ in the moving frame, which is, to a good accuracy, the position of the asymptotic front. In order to be able to conduct a complete calculation, we shall take the model assumption that the transition occurs instantaneously at time t_0. Therefore, we first write

$$\Delta\tilde{H}_{\tilde{\phi}_B}(\xi + m_{t_0}, t_0) = \tilde{H}_{\tilde{\phi}_B}(\xi + m_{t_0}, t_0)$$
$$= 1 - F_{t_0}(\xi) \simeq C \times \xi\, e^{-\xi - \xi^2/4t_0}. \quad (77)$$

At time t such that $t - t_0 \gg \mathcal{O}(\sqrt{t_0})$, since $\Delta\tilde{H}_{\tilde{\phi}_B}(\xi, t_0)$ results from the evolution of the linearized FKPP equation with a boundary that moves at the velocity of a FKPP front, it must have the following shape in the region $\xi > \Delta$:

$$\Delta\tilde{H}_{\tilde{\phi}_B}(\xi + m_t, t) \simeq c_\Delta \times \xi_\Delta e^{-\xi_\Delta - \xi_\Delta^2/4t} \quad \text{with} \quad \xi_\Delta \equiv \xi - \Delta. \quad (78)$$

The only new unknown in this formula is the constant c_Δ, that we shall express as a function of C, Δ, and t_0.

Since our simplified model consists technically of solving a linear equation with boundaries, we may use a completeness relation at t_0: Each point at coordinate $\xi_0 > \Delta$ at time t_0 evolves independently into a front through the linearized equation supplemented with an absorptive boundary at position Δ in the comoving frame, and the resulting front is the superposition of all these partial fronts weighted by $\Delta\tilde{H}_{\tilde{\phi}_B}(\xi_0 + m_{t_0}, t_0)$. In equations,

$$\Delta\tilde{H}_{\tilde{\phi}_B}(\xi + m_t, t) = \int_\Delta^\infty d\xi_0\, \Delta\tilde{H}_{\tilde{\phi}_B}(\xi_0 + m_{t_0}, t_0) \times \tilde{h}_{\xi_0 + m_{t_0}, \Delta}(\xi + m_t, t - t_0). \quad (79)$$

Using Eq. (56) and Eq. (77), and going to the $t \to +\infty$ limit while keeping t_0 and Δ fixed,

$$\Delta\tilde{H}_{\tilde{\phi}_B}(\xi + m_t, t)_{t\to+\infty} = \frac{C}{\sqrt{4\pi}} \times e^{-\Delta}\, \xi_\Delta e^{-\xi_\Delta} \left[\frac{1}{t_0^{3/2}} \int_\Delta^\infty d\xi_0\, (\xi_0 - \Delta)\xi_0 e^{-\xi_0^2/4t_0}\right]. \quad (80)$$

The ξ_0 integral can be performed exactly: One may notice, e.g., that the factor $\xi_0\, e^{-\xi_0^2/4t_0}$ may also be written as $-2t_0 \times \frac{d}{d\xi_0} e^{-\xi_0^2/4t_0}$, and integrate by parts. We arrive at the elegant expression

$$\Delta\tilde{H}_{\tilde{\phi}_B}(\xi + m_t, t) = C \times e^{-\Delta} \operatorname{erfc} \frac{\Delta}{2\sqrt{t_0}}\, \xi_\Delta\, e^{-\xi_\Delta}. \quad (81)$$

The constant c_Δ is readily identified by comparison of Eq. (81) with Eq. (78) in the $t \to +\infty$ limit.

In order to match this simplified calculation with our initial problem, we replace Δ by $\Delta x - f(\lambda)$, and set

$$t_0 = \kappa^2 \tau, \quad (82)$$

where κ is a number of order 1: This arbitrary constant takes into account the fact that in the initial problem, the effective absorptive boundary does not exhibit a discontinuity at time τ. Instead, the transition occurs smoothly in a time window of size of order τ from $t = \tau$. Then,

$$\frac{c_\Delta}{C} e^\Delta = \operatorname{erfc} \frac{[\Delta x - f(\lambda)]^{3/2}}{\kappa \Delta x}. \quad (83)$$

As in the case of a small shift discussed in Sec. III C 1, the function $g(\lambda, \Delta x)$ is obtained by writing $\tilde{H}_{\tilde{\phi}}$ as the sum of $\tilde{H}_{\tilde{\phi}_A}$ and $\Delta\tilde{H}_{\tilde{\phi}_B}$, and by computing the position of the resulting traveling wave in the reference frame comoving with the front $\tilde{H}_{\tilde{\phi}_B}$. Keeping only the leading terms, we arrive at the following expression:

$$g(\lambda, \Delta x) \simeq \Delta x - f(\lambda) + e^{-[\Delta x - f(\lambda)]} \operatorname{erfc} \frac{[\Delta x - f(\lambda)]^{3/2}}{\kappa \Delta x}. \quad (84)$$

Again, we can deduce an expression for the generating function $G_{\Delta x}(\lambda)$ by using Eq. (24). The latter takes a very simple

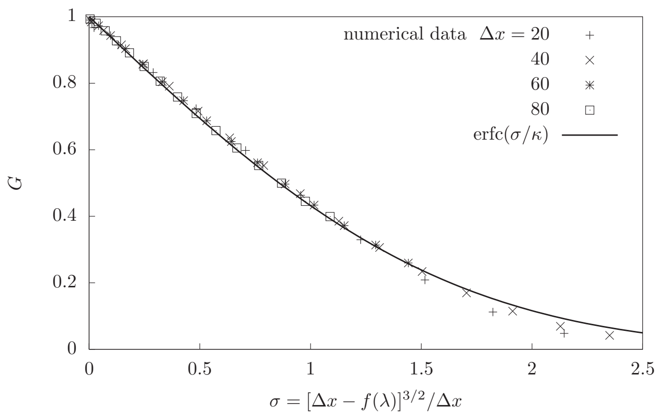

FIG. 4. Generating function $G_{\Delta x}(\lambda)$ extracted from the numerical solution of the FKPP equation as a function of the scaling variable $[\Delta x - f(\lambda)]^{3/2}/\Delta x$. The numerical data is compared to Eq. (85) with κ set to 1.8.

form when $1 \ll [\Delta x - f(\lambda)]^{3/2} \lesssim \Delta x$:

$$G_{\Delta x}(\lambda) \simeq \text{erfc}\, \frac{[\Delta x - f(\lambda)]^{3/2}}{\kappa\, \Delta x}. \tag{85}$$

We do not expect this one-free-parameter simple functional form to be exact, since it was derived in a simplified model for the actual system of equations from which $G_{\Delta x}(\lambda)$ was deduced. However, it may be a valuable expression to start with, and in particular, to compare with numerical solutions to the FKPP equation.

2. *Numerical check of the scaling*

We can compute numerically $G_{\Delta x}(\lambda)$ for different values of the parameters Δx and λ, and check the scaling (85). We employ the method based on the measurement of the shift function $g(\lambda, \Delta x)$ explained in Sec. II D.

Although the numerical evaluation of $g(\lambda, \Delta x)$ is, in principle, straightforward, it is not so easy in practice to get $G_{\Delta x}(\lambda)$ to a fair accuracy. Indeed, we need to evolve the FKPP equation to large times to make sure that the traveling wave has approached its asymptotic shape. We set the maximum evolution time to 8000 in all our calculations. A further difficulty is that for large $g(\lambda, \Delta x)$, the formula (24) becomes the product of a large term, the exponential, by a small prefactor, which is numerically awkward. This puts limits on the range of the parameters we are able to explore with our current implementation: The largest value of Δx we shall consider is 80, and we shall stick to the region $\sigma \lesssim \mathcal{O}(1)$.

The result of the numerical calculation is displayed in Fig. 4: The generating function is plotted against the scaling variable σ, where for the latter we used the definition in Eq. (76) with an additional overall factor $1/\gamma_0$ in $f(\lambda)$ meant to take into account the effect of the discretization, see the discussion in Sec. II D. We estimate that the numerical accuracy is on the order of the size of the data points.

The numerical data exhibit the scaling (76) quite spectacularly, especially given that the values of Δx for which we are able to produce data barely allow us to satisfy the constraints $1 \ll \Delta x - f(\lambda) \lesssim \Delta x^{2/3}$, which we would expect necessary for the scaling to be verified. Interestingly enough, the formula (85) for $G_{\Delta x}(\lambda)$ established in the context of the toy model reproduces fairly well the numerical data, with the parameter κ arbitrarily set to the value 1.8, which, as expected, is indeed of order unity.

IV. PARTICLE NUMBERS IN THE TIP

In this section, we apply the results we got for the shift $g(\lambda, \Delta x)$ and for the generating function $G_{\Delta x}(\lambda)$ to the characterization of the particle number distribution within a distance Δx from the lead particle.

A. Mean particle numbers from small shifts

The expected number of particles

$$\bar{n}(\Delta x) \equiv n^{(1)}(\Delta x) = \sum_n n\, p_n(\Delta x) \tag{86}$$

in an interval of size Δx from the lead particle can be obtained from the coefficient of the term of order $\lambda - 1$ in the $\lambda \to 1$ expansion of the generating function $G^{(0)}_{\Delta x}(\lambda)$ (if the position of the lead particle is not constrained, namely if one considers typical events) or $G_{\Delta x}(\lambda)$ (if the position of the lead particle is fixed). Expressions (63) and (62) are accurate enough to enable us to read off these expectation values:

(1) The mean number of particles in typical realizations of the BBM (namely in which the lead particle is not constrained to be at a given position) can be calculated from $G^{(0)}_{\Delta x}(\lambda)$. Identifying Eq. (62) to the two first terms in Eq. (14), we find, in the limit $\Delta x \gg 1$ in which our analytical expressions are expected to be exact:

$$\bar{n}(\Delta x) \simeq c\, \Delta x\, e^{\Delta x}. \tag{87}$$

We have recovered a result already obtained by Brunet and Derrida in Ref. [15].

(2) The mean number of particles when the position of the lead particle is fixed is deduced from $G_{\Delta x}(\lambda)$ in Eq. (63). Two cases must be distinguished:

(a) The position of the lead particle, x, is chosen far to the left of its expectation value, i.e., $m_t - x \gg 1$. Then, comparing Eqs. (63) and (14) we get

$$\bar{n}(\Delta x) \simeq c\sqrt{2}\, \Delta x\, e^{\Delta x}. \tag{88}$$

(b) Finally, in the case of main interest in this paper, when the lead particle is unusually ahead its expected position, $x - m_t \gg 1$, we get

$$\bar{n}(\Delta x) \simeq c'\, e^{\Delta x}. \tag{89}$$

We note that $\bar{n}(\Delta x)$ is smaller in the last case than in the case when one averages over the position of the lead particle, but only by a factor Δx. The leading factor, $e^{\Delta x}$, is the same. Since $\bar{n}(\Delta x)$ is computed by weighting $p_n(\Delta x)$ by n, the configurations that contribute most are the ones which have a large number of particles in the tip. Requiring n to be large is tantamount to asking for a robust front extending at least from $x - \Delta x$ to x, which in the phenomenological model [8], can only be generated by a front fluctuation if Δx is large. Therefore, we expect the number of particles in a typical tip fluctuation to be less than $\bar{n}(\Delta x) \propto e^{\Delta x}$.

B. Particle distributions in rare tip fluctuations: Tentative estimate

Let us now investigate the consequences of the scaling (76) of the generating function that we have found.

1. Particle numbers

Let us define λ_0 and λ_ζ as solutions of the equations

$$\Delta x - f(\lambda_0) = 0 \quad \text{and} \quad \Delta x - f(\lambda_\zeta) = \zeta\, \Delta x^{2/3}, \tag{90}$$

where ζ is a constant of order unity. The explicit solutions read

$$\frac{1}{1-\lambda_0} \simeq \Delta x\, e^{\Delta x} \quad \text{and} \quad \frac{1}{1-\lambda_\zeta} \simeq \Delta x\, e^{\Delta x - \zeta\, \Delta x^{2/3}}. \tag{91}$$

λ_0 is the point where the scaling variable σ is 0, while λ_ζ is the point at which $\sigma = \zeta^{3/2}$. According to the discussion in Secs. III C and III D (see also Fig. 4), the transition region in which $G_{\Delta x}(\lambda)$ drops significantly from 1 coincides with the parametric region $0 < \sigma \lesssim \zeta^{3/2}$. For fixed Δx, this corresponds to λ describing the interval $[\lambda_\zeta, \lambda_0]$. According to Eq. (26), the probability $p_n(\Delta x)$ is significant for $1/(1-\lambda_\zeta) < n < 1/(1-\lambda_0)$, namely for

$$\Delta x\, e^{\Delta x - \zeta\, \Delta x^{2/3}} < n < \Delta x\, e^{\Delta x}. \tag{92}$$

Note that the situation is very different for the number of particles in an interval Δx from the lead particle in *typical* realizations. It is easy to see from the formula in Eq. (74) that the generating function of the corresponding probabilities, $G^{(0)}_{\Delta x}(\lambda)$, has its transition region around $1/(1-\lambda)$ of the order of $\Delta x\, e^{\Delta x}$: More precisely, this region roughly extends over the λ interval $[\lambda'_0, \lambda_0]$, where λ'_0 solves $\Delta x - f(\lambda'_0) = 1$. This implies that the typical number of particles in an interval of size Δx from the lead particle in typical realizations coincides, in order of magnitude, with the mean number of particles, see Eq. (87).

Equation (92) shows that the particle number may be smaller by a factor on the order of $e^{\zeta\, \Delta x^{2/3}}$ in rare tip fluctuations than it would be in typical tips. This would be enough for the main assumption of the phenomenological model for fluctuations in branching random walks to prove correct: The latter indeed assumes that the particle density in tip fluctuations is less than $e^{\Delta x}$. However, the dominant behavior of typical particle numbers in the limit of large interval sizes is still given by the exponential $e^{\Delta x}$, which may seem a bit surprising.

2. Mean distances between nearby particles

Let us recall that Brunet and Derrida computed rigorously the large-n asymptotics of the mean distances between particles number n and $n+1$ from the lead particle when the position of the latter is left free. The result reads [14]

$$\langle d_{n,n+1}\rangle = \frac{1}{n} - \frac{1}{n \ln n}. \tag{93}$$

They observed that these distances are smaller than in the case of a Poisson process of exponential intensity e^x, for which $\langle d_{n,n+1}\rangle = 1/n$. However, they turn out to coincide with what would be found in the case of a Poisson process on a line of intensity $x\, e^x$ [15].

Let us come back to rare tip fluctuations. If, in these realizations, the particles were distributed according to a Poisson process on a line of intensity $x\, e^{x-\zeta\, x^{2/3}}$, then the mean distance between particle number n and particle number $n+1$ from the lead particle would read

$$\langle d_{n,n+1}\rangle = \frac{1}{n} + \frac{2\zeta}{3}\frac{1}{n(\ln n)^{1/3}}. \tag{94}$$

We see that this distance is *larger* than that found both in the case in which the lead particle is not an unusual fluctuation, and in the case of the Poisson process with exponential intensity. However, we know that there should exist strong correlations, especially in the rare realizations we select, which would *a priori* invalidate a Poisson process assumption.

V. CONCLUSIONS AND OUTLOOK

With the aim of understanding tip fluctuations in branching random walks, our main thrust was the study, in the $t \to +\infty$ limit, of the generating function $G_{\Delta x}(\lambda)$ of the particle-number probabilities $p_n(\Delta x)$ to observe n particles in an interval of size Δx from the lead particle, the position x of the latter being fixed to some large number, in such a way that the difference between x and its expectation value m_t be large. In this limit, the x dependence of $G_{\Delta x}(\lambda)$ vanishes. Our main result can be summarized as follows for the branching Brownian motion:

$$\text{Defining } \sigma^2 \equiv \frac{[\Delta x - f(\lambda)]^3}{\Delta x^2},$$

$$G_{\Delta x}(\lambda) = \text{func}\big(\sigma^2\big) \simeq \begin{cases} 1 & \text{for } \Delta x < f(\lambda) \\ \mathcal{O}(1) & \text{for } \sigma \sim 1 \\ 0 & \text{for } \sigma \gg 1, \end{cases} \tag{95}$$

where $f(\lambda)$ is the Brunet-Derrida "delay function" [14]

$$f(\lambda) = \ln\frac{1}{1-\lambda} - \ln\ln\frac{1}{1-\lambda}\,. \tag{96}$$

We gave analytical arguments in support of this scaling, and checked it numerically, see Fig. 4. Proving this scaling and finding an accurate expression for $G_{\Delta x}(\lambda)$ are outstanding challenges.

We have also found that the expectation value of the particle number grows like $e^{\Delta x}$ with the size of the interval. However, mean particle numbers are probably dominated by front fluctuations, at variance with typical particle numbers, which we expect to build up in late stages of the evolution.

The scaling for the generating function $G_{\Delta x}(\lambda)$ has enabled us to estimate heuristically the typical values of the number of particles in a rare tip fluctuation as

$$n(\Delta x)|_{\text{typical}} \sim \Delta x\, e^{\Delta x - \zeta\, \Delta x^{2/3}}. \tag{97}$$

If this guess is confirmed, the phenomenological model for fluctuations in branching random walks is indeed justified, since this number is much less than $e^{\Delta x}$. However, it lies surprisingly close to the exponential asymptotically in the limit of large Δx.

Knowing the scaling form of the generating function is not enough to allow one to fully calculate the probability $p_n(\Delta x)$ of observing n particles in an interval of given size from the tip when the position of the latter is fixed, or the mean distance between two nearby particles. Such quantities would help us to better assess the particle distribution in these fluctuations.

It was not possible either to calculate the particle number probabilities $p_n(\Delta x)$ from our numerical implementation: The latter provides the generating function $G_{\Delta x}(\lambda)$, $p_n(\Delta x)$ would be related to the nth λ derivative of $G_{\Delta x}(\lambda)$, which cannot be computed in practice for n much larger than one. A Monte Carlo implementation of the stochastic process would be more useful, but the latter is not straightforward since we are interested in extremely rare realizations of the branching Brownian motion. (Note that algorithms for generating specifically rare events have been designed in a different subfield of statistical physics, see, e.g., Ref. [31]). We leave this for further investigations.

We also plan to address the correlations of the positions of the lead particles at different times, which will give valuable information on the way tip fluctuations develop over time.

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