Rapid Communications

Quasideterministic dynamics, memory effects, and lack of self-averaging in the relaxation of quenched ferromagnets

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We discuss the interplay between the degree of dynamical stochasticity, memory persistence, and violation of the self-averaging property in the aging kinetics of quenched ferromagnets. We show that, in general, the longest possible memory effects, which correspond to the slowest possible temporal decay of the correlation function, are accompanied by the largest possible violation of self-averaging and a quasideterministic descent into the ergodic components. This phenomenon is observed in different systems, such as the Ising model with long-range interactions, including the mean-field, and the short-range random-field Ising model.

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Introduction. When computing thermodynamic properties one must, in principle, consider the full statistical-mechanical average $\langle\cdot\rangle$, namely over the realizations of the stochastic trajectories, the initial conditions, and, if present, over the quenched disorder distribution. However, if the sample has specific self-averaging properties, the latter two averages are not necessary because they are realized by the system itself in the thermodynamic limit. Restricting for the moment the discussion to clean samples, i.e., without quenched disorder, this occurs when the system is ergodic. In this case after some time a large part of phase space is visited, and the memory of the initial condition is fully lost: Therefore the fate of a thermodynamical process does not depend on the specific initial microstate belonging to the same macrostate.

The situation is more subtle when phase space breaks into ergodic components [1], namely mutually nonaccessible regions. In this case, if the initial state is well inside one of these components its memory cannot be deleted because the other cannot be accessed. This is trivial for a uniaxial ferromagnet below the critical temperature T_c, where the equilibrium magnetization M takes the two possible values $M_\pm = \pm M_{\rm eq}$. A sample prepared with a macroscopic $M(t=0)>0$ (<0) evolves towards the positive (negative) equilibrium value and self-averaging is not operating.

A different situation occurs when the system is initially on the boundary $\mathcal{B}$ between ergodic components. In ferromagnets, $\mathcal{B}$ is the set of configurations with $M\simeq 0$, and this happens when the initial state is sampled from a high-temperature ($T \geqslant T_c$) equilibrium state. The evolution in this case proceeds by the coarsening of domains of the competing equilibrium phases [2], whose typical size $L(t)$, at time t, grows unbounded. Aging is manifested [3] and the dynamics remains on $\mathcal{B}$ forever. This is strictly true if the thermodynamic limit is taken before letting time t become large. However, in all physical situations, one deals with a large but finite system. Therefore the initial state, due to thermal fluctuations, will have some offset $M(0)$ from $\mathcal{B}$ and one can ask how this may change the destiny of the system.

The different options can be appreciated in terms of the exponent λ controlling the decay of the autocorrelation function and also related [4] to the growth in time of the magnetization $M(t)\sim L(t)^{d-\lambda}$, where d is the spatial dimension. The Fisher-Huse inequality [5,6] fixes the bounds for λ,

$$\frac{d}{2} \leqslant \lambda \leqslant d. \qquad (1)$$

If the system stays close to $\mathcal{B}$ forever [7] the magnetization does not amplify ($\lambda = d$), self-averaging is at work, and memory of the initial condition is retained the least possible [8]. In the opposite situation the system deterministically falls in the ergodic component selected by the sign of $M(0)$. In this case the offset $M(0)$ is strongly amplified and $M(t)$ grows as fast as possible, i.e., $\lambda = d/2$. This process is associated with the longest possible memory of the initial condition and with the strongest violation of self-averaging. In between these two extrema there is a continuum of options, with $d/2 < \lambda < d$.

Existing analytical [9–12] and numerical [10,13–17] determinations of λ suggest that the maximum of memory, $\lambda = d/2$, is only approached in unphysical limits, diverging space dimension limit $d\to\infty$ or diverging order parameter component limit $\mathcal{N}\to\infty$. Instead, upon associating the origin of the lower bound $\lambda = d/2$ with some deterministic properties of the dynamics, in this Rapid Communication we show that it is possible to toggle among all the three situations above and that the case with $\lambda = d/2$ is found also for finite d and $\mathcal{N}$ in the presence of long-range interactions or in the presence of quenched disorder.

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The model and the two limiting regimes. In order to set the stage with a specific example, let us start our discussion by considering the one-dimensional clean ferromagnet described by the Hamiltonian

$$H = -\frac{1}{2}\sum_{i,j} J(|i-j|)s_i s_j, \tag{2}$$

where $s_i = \pm 1$ are N Ising variables, and $J(r) = \delta_{r,1}$ for nearest-neighbor (NN) couplings, and $J(r) = 1/r^{1+\sigma}$ in the case of long-range interactions. We will focus on the case $\sigma > 0$ where additivity and extensivity hold [18]. The model has a ferromagnetic phase below a finite critical temperature $T_c(\sigma) > 0$ for $\sigma < 1$ [19,20]; it has a Kosterlitz-Thouless transition [21] for $\sigma = 1$; finally, $T_c = 0$ for $\sigma > 1$.

Let us now discuss the relaxation of the model with a nonconserved order parameter after a quench from $T_i = \infty$ to a low T. We consider Glauber dynamics where a random spin is reversed with probability $w = [1 + \exp(\Delta E/T)]^{-1}$, where ΔE is the energy difference due to the spin flip. Not only do the static properties but also the nonequilibrium kinetics change crossing $\sigma = 1$. $L(t) \sim t^{1/z}$ grows with a dynamical exponent [22,23] $z = 1 + \sigma$ for $0 < \sigma \leqslant 1$ or $z = 2$ for $\sigma > 1$ and NN. This behavior is captured by a single domain model. The distance $X(t)$ between two neighboring domain walls satisfies an overdamped Langevin equation $\dot{X}(t) = -F(X) + \xi(t)$, where $F(X)$ is a force determined by Eq. (2) and $\xi(t)$ is a Gaussian white noise. The force is given by $F(X) = -U'(X)$, where $U(X) = \sum_{i=1}^{X}(\sum_{j=-\infty}^{0} + \sum_{j=X+1}^{\infty})J(|i-j|)$. For large X we can replace discrete summations with integrals and evaluating the integrals in the parentheses we obtain $U(X) \simeq (2/\sigma)\int_1^X ds/s^\sigma$, therefore $F(X) \sim -1/X^\sigma$. Given that $F(X)$ is the average speed of the domain wall, the closure time of a domain of initial size $X(0) = L$ is $t = \int_L^0 dX/F(X) \propto L^z$ with $z = 1 + \sigma$ for $\sigma \leqslant 1$ and $z = 2$ for $\sigma > 1$. The difference between these two regimes is due to the deterministic force $F(X)$, that affects the coarsening process in the former ($\sigma \leqslant 1$) while it is irrelevant in the latter ($\sigma > 1$). For this reason these regimes will be called *convective* and *diffusive* regimes, respectively.

These two regimes can be clearly distinguished by considering the fluctuating magnetization $M(t) = \sum_{i=1}^N s_i$, which is shown in Fig. 1 for systems prepared with a fixed condition $M(0) \sim \sqrt{N}$ equal for all σ values. In the convective regime $M(t)$ asymptotically diverges and it typically has the same sign as $M(0)$ [24]. In the diffusive regime it fluctuates around $M(0)$. This means that the convective regime keeps the memory of the initial condition, while the diffusive does not. This implies that decorrelation is slower in the first case and, actually, we will show in a moment that it occurs in the slowest possible way. Self-averaging with respect to initial conditions is broken for $0 < \sigma \leqslant 1$ (convective regime) while it holds for $\sigma > 1$ (diffusive regime).

With this example in mind, we now turn to a more general discussion. Let us consider the correlation function which, using a continuous picture for a scalar field [25] $\phi(\mathbf{x}, t)$, reads $S(r; t_1, t_2) \equiv \langle \phi(\mathbf{x} + \mathbf{r}, t_1)\phi(\mathbf{x}, t_2)\rangle$, where $t_2 > t_1$ and $\langle \cdots \rangle$ is the full nonequilibrium statistical average. We focus on the scaling regime where the autocorrelation function $C(t_1, t_2) =$

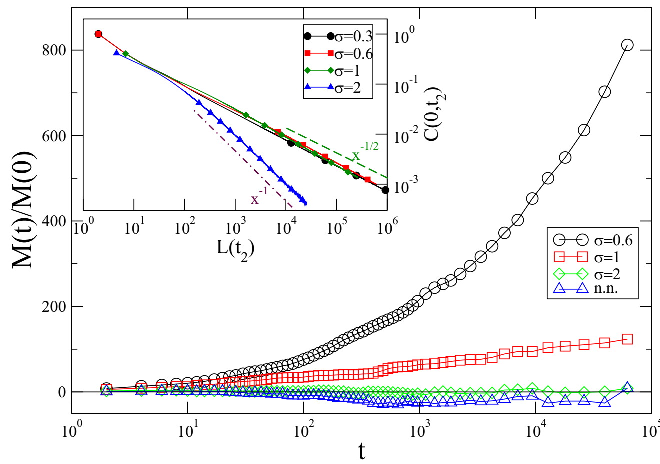

FIG. 1. The fluctuating magnetization $M(t)$ for a single realization starting from the same initial condition. The system size is $N = 10^6$ and the quench temperature is $T = 0.1$. In the inset, $C(0, t_2)$ is plotted against $L(t_2)$ for different σ after a quench to $T = 0.1$. The system size is $N = 2 \times 10^7$. The dashed straight lines are the decays $x^{-\lambda}$ with $\lambda = 1$ and $\lambda = 1/2$.

$S(r = 0; t_1, t_2)$ behaves as [26]

$$C(t_1, t_2) \simeq [L(t_1)/L(t_2)]^\lambda, \tag{3}$$

where, as it will be discussed around Eq. (9), λ is the same exponent introduced before which therefore obeys Eq. (1).

The inequalities for λ. A derivation of Eq. (1) is now provided following Ref. [6]. We indicate with $u_l = \phi_l(\mathbf{q}, t_1)$ the Fourier transform of the field $\phi_l(\mathbf{x}, t_1)$ evaluated at the time t_1 during the lth realization of the dynamics. Similarly, we define $v_l = \phi_l(\mathbf{q}, t_2)$ at the time t_2. We can therefore define the scalar product as $\vec{u} \cdot \vec{v} \equiv (2\tilde{N})^{-1}\sum_l (u_l v_l^* + \text{c.c.}) = \frac{1}{2}[S(q, t_1, t_2) + S^*(q, t_1, t_2)]$, where $\tilde{N}$ is the number of realizations and $S(q, t_1, t_2) \equiv \langle \phi(\mathbf{q}, t_1)\phi(-\mathbf{q}, t_2)\rangle$ is the Fourier transform of $VS(r; t_1, t_2)$, with V the system volume. We can now apply the Cauchy-Schwarz inequality $|\vec{u} \cdot \vec{v}| \leqslant |u||v|$ and obtain

$$\frac{1}{2}|S(q, t_1, t_2) + S^*(q, t_1, t_2)| \leqslant \sqrt{S(q, t_1)S(q, t_2)}, \tag{4}$$

where, for ease of notation, $S(q, t) \equiv S(q, t, t)$. If we integrate over $\mathbf{q}$ we find

$$C(t_1, t_2) \leqslant \frac{1}{V(2\pi)^d}\int d\mathbf{q}\,\sqrt{S(q, t_1)S(q, t_2)}. \tag{5}$$

Using Eq. (3) and the scaling form $S(q, t) = L^d(t)f(qL)$, with $f(x) \simeq 1$ for $x \ll 1$ and $f(x)$ negligibly small for $x \gg 1$, we find the lower bound of Eq. (1) [27].

We now originally prove that the same lower bound can be derived from the term $q = 0$ only of Eq. (4),

$$S(0, t_1, t_2) \leqslant \sqrt{S(0, t_1)S(0, t_2)}. \tag{6}$$

Using the scaling form for $S(q, t)$ [see below Eq. (5)] it is straightforward to rewrite the previous equation as

$$S(0, t_1, t_2) \leqslant f(0)(L_1 L_2)^{d/2}, \tag{7}$$

where we used the shorthand $L_1 \equiv L(t_1)$, and similarly for L_2. The left-hand side of Eq. (5) can be worked out expressing the two-time correlation function as follows, $C(t_1,t_2) = \frac{1}{V(2\pi)^d}\int d\mathbf{q}\, S(q,t_1,t_2) = \frac{L_2^d}{V(2\pi)^d}\int d\mathbf{q}\, F(qL_2, L_1/L_2)$, where we have used the scaling hypothesis $S(q,t_1,t_2) = L_2^d F(qL_2, L_1/L_2)$, valid when both times t_1 and t_2 are in the scaling regime. In the limit of large L_2 (i.e., of large t_2) only wave vectors $q < 1/L_2$ contribute to the integral. If $S(q \to 0, t_1, t_2)$ goes to a constant, which is the case for quenches below T_c or to $T = 0$, we can finally write

$$C(t_1,t_2) \simeq \frac{1}{V(2\pi)^d}\frac{S(0,t_1,t_2)}{L_2^d}. \tag{8}$$

Using this relation and Eq. (7) we find $C(t_1,t_2) \leqslant \mathrm{const}(L_1/L_2)^{d/2}$ and the scaling form (3) gives $\lambda \geqslant d/2$. Therefore Eq. (6) is equivalent to the lower bound (1).

The upper bound in Eq. (1) is defined in Ref. [5] as a "suggestive bound" because it cannot be proved as rigorously as the lower bound. In order to derive it, starting from the straightforward relation $\langle M(t_1)M(t_2)\rangle = S(0,t_1,t_2)$, and using Eqs. (8) and (3), one arrives at

$$\langle M(t_1)M(t_2)\rangle = \mathrm{const}\, L_1^{\lambda} L_2^{d-\lambda}. \tag{9}$$

The authors of Ref. [5] argue that $\lambda \leqslant d$ because "forgetting of an initial bias appears unlikely." In other words, the strongest memory loss corresponds to the limit $\lambda = d$.

Averaging and memory. We now consider the role of the different statistical averages. The full one $\langle\cdots\rangle$ is taken over the stochastic trajectories $\langle\cdots\rangle_{\mathrm{tr}}$, the initial condition $\langle\cdots\rangle_i$, and, if present, over the quenched disorder $\langle\cdots\rangle_q$. Let us consider, to begin with, a clean system. We can split the fluctuating magnetization as $M(t) = \langle M(t)\rangle_{\mathrm{tr}} + \psi(t)$, where $\psi(t)$ is the stochasticity left over after taking the partial averaging $\langle M(t)\rangle_{\mathrm{tr}}$, so that $\langle\psi(t)\rangle_{\mathrm{tr}} \equiv 0$. Then we have $\langle M(t_1)M(t_2)\rangle = \langle\langle M(t_1)\rangle_{\mathrm{tr}}\langle M(t_2)\rangle_{\mathrm{tr}}\rangle_i + \langle\psi(t_1)\psi(t_2)\rangle$. If we now fix t_1 and let t_2 diverge, $\langle\psi(t_1)\psi(t_2)\rangle = \langle\psi(t_1)\rangle\langle\psi(t_2)\rangle = 0$, and from Eq. (9) we obtain

$$\langle\langle M(t_1)\rangle_{\mathrm{tr}}\langle M(t_2)\rangle_{\mathrm{tr}}\rangle_i \simeq L_2^{d-\lambda}. \tag{10}$$

Next, we argue that, if the quench is made in a ferromagnetic phase, due to the presence of two ergodic components, for large t_1 it is $\mathrm{sgn}[M(t_1)] = \mathrm{sgn}[M(t_2)]$. This is very well observed for $\sigma < 1$ (see Fig. 1). Hence it is also $\mathrm{sgn}\langle M(t_1)\rangle_{\mathrm{tr}} = \mathrm{sgn}\langle M(t_2)\rangle_{\mathrm{tr}}$, therefore Eq. (10) (valid for t_1 fixed) amounts to

$$\langle M(t)\rangle_{\mathrm{tr}} \simeq L(t)^{d-\lambda}, \tag{11}$$

where we have denoted t_2 as t to ease the notation. Notice that the equation above is more general and applies to systems without a proper ferromagnetic phase, such as the one-dimensional (1D) Ising model with $\sigma > 1$ or with NN, because in this case there is no development of magnetization starting from a given state (see Fig. 1), and indeed it is $\lambda = d$.

Equation (11) shows that the slowest possible decorrelation, $\lambda = d/2$, is accompanied by the fastest possible growth of the magnetization developed from an initial condition [4]. Let us observe that such maximum growth is the one expected upon assuming a random arrangements of a number $\sim VL^{-d}$ of domains of size L each contributing a magnetization $\sim L^d$. Equation (11) for $\lambda = d/2$ then derives from the central limit theorem.

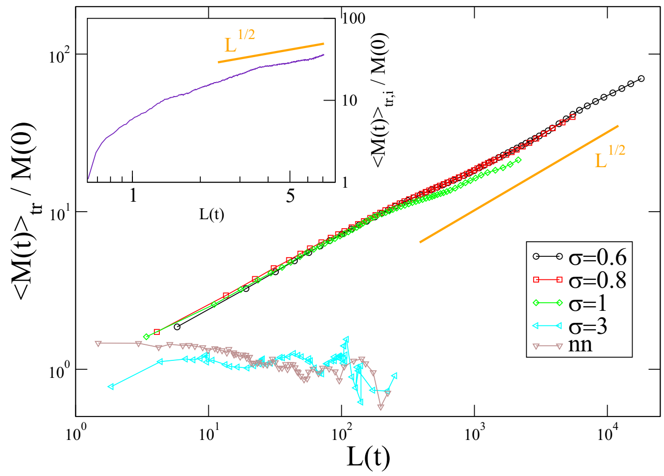

FIG. 2. $\langle M(t)\rangle_{\mathrm{tr}}$ [normalized by its typical initial value $M(0)$] is plotted against $L(t)$ for different σ. The system size is $N = 10^6$ and the quenching temperature is $T = 0.1$. The orange straight line is the behavior $L(t)^{1/2}$. In the inset a similar plot is shown for $\langle M(t)\rangle_{\mathrm{tr},i}$ in the 1D RFIM quenched to $T \to 0$ with $h/T = 1/2$. The system size is $N = 10^5$.

The result (11) implies also that there is breaking of self-averaging with respect to initial conditions if $\lambda < d$, as reflected by the fact that, for large N, the observable *magnetization* does not attain its average value $\lim_{N\to\infty}\langle M(t)\rangle = 0$ unless the average over initial conditions is performed. The most severe self-averaging breakdown occurs when λ is at the lower bound in (1), while it is fully restored when it is at the upper bound.

Let us put these arguments to the test in different models, starting from the 1D model of Eq. (2). Let us recap what is known about λ. For NN there is the exact result [28] $\lambda = 1$, the upper bound of Eq. (1) is saturated, and self-averaging holds. For the long-range case it was shown in Ref. [29] that there are two universality classes associated with the values $\lambda = 1$ (for $\sigma > 1$) and $\lambda = 1/2$ (for $\sigma \leqslant 1$). Since it is known [6] that for the nonconserved order parameter this exponent is independent of t_1, the best determination can be obtained by letting $t_1 = 0$. This is displayed in the inset of Fig. 1, where $C(0,t_2)$ is shown for various choices of σ, showing that $\lambda = 1$ for $\sigma > 1$ and $\lambda = 1/2$ for $\sigma \leqslant 1$.

In Fig. 2 we plot $\langle M(t)\rangle_{\mathrm{tr}}$ as a function of $L(t)$, for different σ and *the same initial condition*. This shows very clearly that in the convective regime ($0 < \sigma \leqslant 1$) where $\lambda = d/2$ it is $\langle M(t)\rangle_{\mathrm{tr}} \sim \sqrt{L(t)}$ while in the diffusive case ($\sigma > 1$ or NN) it is $\langle M(t)\rangle_{\mathrm{tr}} \sim M(0)$, as expected after Eq. (11). Hence $\sigma = 1$ separates the two opposite situations in which the dynamics occurs on the boundary $\mathcal{B}$ of the ergodic components (for $\sigma > 1$) from the one where it deterministically sinks into such components (for $\sigma \leqslant 1$). We should stress that $T_c = 0$ is not a sufficient condition to have $\lambda = d$, as attested by the 2D *XY* model where $\lambda \simeq 1.17 < d = 2$ even if $T_c = 0$ [30].

In our model (2) determinism can be ascribed to the convective character of domain wall motion [29,31]. Let us suppose to have two close domains of sizes ℓ_1, ℓ_2, with ℓ_2 slightly larger than ℓ_1. In the diffusive case the average closure time of ℓ_1, $\overline{t_1}$, is slightly smaller than the one of ℓ_2, $\overline{t_2}$, but the probability that $t_1 < t_2$ is only slightly larger than

$1/2$. In the convective regime, instead, the dominance of the deterministic force makes a domain wall always move towards the closest one [29], so that t_1 is *always* smaller than t_2. This induces a memory effect, since domains which are eliminated have a larger probability to be antialigned with $M(t)$ and their removal further increases $M(t)$. Summarizing, in the convective regime there is a reduced degree of stochasticity and an increased memory with respect to the diffusive one, and this is the physical origin of the saturation of λ to the lower bound.

Same ideas, other models. Let us now apply these ideas to different systems, starting from the short-range ferromagnetic model in $d > 1$. In this case we have strict inequalities for any d, $d/2 < \lambda < d$ [11]. Hence self-averaging is spoiled, $\lambda < d$, in opposition to $d = 1$. This is because in $d > 1$ interfaces do not freely diffuse, and there is a deterministic drift induced by the curvature. However, the fate of the system is not fully determined by such a deterministic force because the shape of the percolating cluster plays a major role in the subsequent dynamics [32]. Hence there is only a weak drift from $\mathcal{B}$ towards the ergodic components and λ stays larger than $d/2$.

When long-range interactions are present, results in $d > 1$ are rare [33] and studies of λ are almost absent [34]. However, it is interesting that for the NN case in the limit $d \to \infty$, which corresponds to the, so to say, *longest possible range of interactions*, the mean field, one has $\lambda \to d/2$ [11] and $M(t) \sim L(t)^{d/2}$ [31,35], as expected on the basis of our previous argument. In this limit there are no interfaces and therefore the strong memory effects leading to $\lambda = d/2$ cannot be associated with the determinism of their motion, as in finite dimension. Instead, it can be observed that the mean field amounts to an averaging procedure which makes the evolution, in a sense, more deterministic. Again, this reduction of the stochastic degree is perhaps the physical origin of the saturation of λ to the lower bound of Eq. (1).

There is another well-known limit in which phase ordering has a similar character. This is the case of a vectorial order parameter $\vec{\phi}(\mathbf{x}, t)$ with a large number $\mathcal{N}$ of components and short-range interactions. In the $\mathcal{N} \to \infty$ limit (a model sometimes denoted also as a *spherical model*) one finds [12] $\lambda = d/2$ for any d [36]. By choosing an initially magnetized state it can be shown [37] that the magnetization evolves deterministically as $M(t) \sim L(t)^{d/2}$, as expected after Eq. (11). It must be recalled that the large-$\mathcal{N}$ limit effectively amounts to replacing ϕ^2 with its mean value [12]. Then, similarly to the mean field, the model realizes a sort of averaging which tames the stochasticity and sets λ to the minimum possible value.

Up to now we have only considered clean systems. It is now interesting to discuss the case with quenched disorder focusing, as a paradigm, on the random-field Ising model (RFIM). The RFIM Hamiltonian is given by Eq. (2), plus a contribution $-\sum_i h_i s_i$ due to a quenched random external field that in the following we will consider with zero average and bimodal distribution $h_i = \pm h$. We will focus on the NN case. In order to discuss the role of the different averages, as done before, we must now take into account that in this case also the quenched one $\langle \cdots \rangle_q$ comes into the game. Splitting the magnetization as $M(t) = \langle M(t) \rangle_{\text{tr},i} + \psi(t)$, similarly to what done previously for the clean case but where now $\langle \cdots \rangle_{\text{tr},i}$ is a partial average taken over both dynamical trajectories and initial conditions, one can follow the same line of reasoning as before, arriving at the same results, replacing everywhere $\langle M(t) \rangle_{\text{tr}}$ with $\langle M(t) \rangle_{\text{tr},i}$.

Let us start discussing the case with $d = 1$, for which some analytical arguments are available. The model is characterized [38] by a value of λ at its minimum, $\lambda = 1/2$. Hence, one should expect $\langle M(t) \rangle_{\text{tr},i} \sim L(t)^{1/2}$. In the inset of Fig. 2 we plot $\langle M(t) \rangle_{\text{tr},i}$ versus the average size of domains $L(t)$ [which grows as $(\ln t)^2$]. The result nicely confirms our expectation. In this case the growth of $\langle M(t) \rangle_{\text{tr},i}$ can be traced back to the fact that the sum of the random fields in a given quenched realization is of order $N^{-1/2}$ and hence there is an explicit breaking of the up-down spin symmetry. Hence, here it is the random field which causes the deterministic fall into the ergodic components. Interestingly, this effect seems not to be limited to one dimension. For $d > 1$ the RFIM can only be studied numerically. For $d = 2$ one observes [39] that $\lambda = d/2 = 1$ is still at the lowest possible value, as for $d = 1$. This suggests that the mechanism found in $d = 1$ might be a general feature with random fields.

Conclusions. We have interpreted the exponent λ and its bounds, $d/2 \leqslant \lambda \leqslant d$, in terms of stochasticity, memory effects, ergodicity breaking, and self-averaging. When $\lambda = d$ memory is lost as fast as possible, magnetization does not develop, and there is no breaking of self-averaging. This occurs, for instance in the 1D Ising model with NN, or in the 2D $O(2)$ model E [40]. When $\lambda = d/2$ memory is maintained as much as possible, magnetization grows as $M(t) \sim [L(t)]^{d/2}$ and there is a strong breaking of self-averaging. This occurs in the 1D long-range Ising model with $\sigma \leqslant 1$, in the mean-field and spherical model limits, in the RFIM. Between the two limiting cases, a continuum exists.

It would be interesting to check if some model contrasts these ideas, starting from long-range systems in $d > 1$ [34]. The case of aging without ergodicity breaking, as in the case of a ferromagnet quenched to the critical temperature, is also another test bench where the relation between stochasticity, memory effects, and self-averaging ought to be considered. In this case the Fisher-Huse lower bound generalizes [6] to $\lambda \geqslant (d + \beta)/2$, where β is an exponent characterizing the small q behavior of the structure factor. It would be interesting to check if in this case it is still possible to relate the bounds on λ to specific features of the dynamics.

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[24] In our simulations this occurs in ∼80% of the thermal histories. This fraction does not seem to depend significantly on the size N of the system. What is mostly important is that for *all* thermal histories there exists a finite time t^* beyond which $M(t)$ does not change sign and whose absolute value increases in time.
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