Quantum corrections to plasma kinetic equations: A deformation approach

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Plasmas, as well as several other many-body systems of technological interest, have been studied mostly as a purely classical subject. However, in dense plasmas, and in some semiconductor devices, metallic nanostructures and thin metal films, when the de Broglie wavelength of the charge carriers is comparable to the interparticle distance, quantum effects come into play. Because the classical kinetic equations are phase-space equations with positions and momenta as variables, which variables are noncommuting in quantum mechanics, kinetic equations are not directly applicable to quantum plasmas. Therefore, most treatments consider a full quantum many-body problem in Hilbert space and then, by reduction, obtain the quantum version of the kinetic equations. However, quantum mechanics may also be directly formulated in phase space by modifying the Poisson algebra into a new deformed algebra, hence the classical kinetic equations may also be deformed into their corresponding quantum versions. This is the approach followed here and applied to derive the quantum corrections to the Vlasov–Poisson, Vlasov–Maxwell, and Boltzmann equations (in the latter case also within the relaxation-time approximation).

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I. INTRODUCTION

Plasma physics phenomena have been studied mostly by using purely classical descriptions, yet, if the de Broglie wavelength λ_B of the charge carriers becomes comparable to the interparticle distance (i.e., if $V/N \lesssim \lambda_B = (h/\bar{p})^3$, with N/V the particle density, h Planck's constant, and $\bar{p}$ some mean moment), quantum effects become important [1–6]. We are thus speaking of dense plasmas, as are present in white dwarfs, the atmosphere of neutron stars, and intense laser-solid plasma interactions. In general, many-body charged particle systems cannot be treated by purely classical-physics equations when there is considerable overlap of the wave functions. This is the case, not only of dense plasmas, but also of liquid metals and semiconductor devices, for which kinetic equations might also be used, quantum plasma effects being also relevant for the physics of metallic nanostructures and thin metal films.

The classical kinetic equations, relevant to plasma physics, are phase-space equations and, because in quantum mechanics positions and momenta are noncommuting variables, kinetic equations have not been considered as immediately applicable to quantum plasmas. Therefore, most treatments start from a full quantum many-particle problem in Hilbert space, making use, for instance, of the Wigner function or the Hartree formulation, and then, via a reduction process, obtain the quantum version of the kinetic equations [1–5].

However, quantum mechanics may also be directly formulated in phase-space with a modification of the Poisson algebra to a new deformed algebra. This suggests that the quantum version of the kinetic equations might also be directly obtained by deformation of the classical kinetic equations. It turns out that this is indeed possible and simpler than the traditional approaches. Of particular interest for the applications are the leading quantum corrections to the kinetic equations, which may change, for instance, the stability conditions of their solutions [7].

The paper is organized as follows: In Sec. II we briefly recall how quantum mechanics can be obtained as a deformation of the classical Poisson algebra; in Sec. III, which contains the main physics results of our work, we deform the classical kinetic equations of plasma physics (i.e., the Vlasov–Poisson, Vlasov–Maxwell, and Boltzmann equations) to obtain the appropriate quantum corrections to them; finally, in Sec. IV, we draw a few conclusions.

II. QUANTUM MECHANICS AND DEFORMATION THEORY

The phase space of classical mechanics is a symplectic manifold $W = (T^*M, \omega)$, where T^*M is named the cotangent bundle over the configuration space M and ω is a symplectic form [8]. In local (Darboux) coordinates (q_j, p_j) the symplec-

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tic form reads

$$\omega = \sum_j dq_j \wedge dp_j, \tag{1}$$

and the Poisson bracket gives a Lie algebra structure to the C^∞ functions on W, namely,

$$\{f, g\} = \sum_j \left(\frac{\partial f}{\partial q_j}\frac{\partial g}{\partial p_j} - \frac{\partial f}{\partial p_j}\frac{\partial g}{\partial q_j} \right) \tag{2}$$

in local coordinates. The transition to quantum mechanics can now be regarded as a deformation of this Poisson algebra [9]. For example, if we let $T^*M = \mathbb{R}^{2n}$, then

$$\omega = \sum_{j=1}^{2n}\sum_{k=j}^{2n} \omega_{j,k} dx_j \wedge dx_k = \sum_{j=1}^{n} dx_j \wedge dx_{j+n}$$
$$= \sum_{j=1}^{n} dx_j \wedge dp_j, \tag{3}$$

with $\omega_{j,k} = \delta_{|k-j|,n}\,\mathrm{sgn}(k-j)$, $\delta_{j,k}$ being the Kronecker symbol and $\mathrm{sgn}(x)$ the signum function (i.e., extracting the signal of x), and henceforth with $p_j = x_{j+n}$. Introducing the bidifferential operator $P(f, g)$ whose powers are defined according to

$$P^r(f, g) = \sum_{j_1,\cdots,j_r,k_1,\cdots,k_r=1}^{2n} \omega_{j_1,k_1}\cdots\omega_{j_r,k_r} \times \frac{\partial^r f}{\partial x_{j_1}\cdots\partial x_{j_r}}\frac{\partial^r g}{\partial x_{k_1}\cdots\partial x_{k_r}}, \tag{4}$$

$P(f, g)$ is the usual Poisson bracket for functions of $\mathbf{x}$ and $\mathbf{p}$,

$$P(f, g) = \{f, g\} = \left(\frac{\partial_f}{\partial x_1}\frac{\partial_g}{\partial p_1} + \cdots + \frac{\partial_f}{\partial x_n}\frac{\partial_g}{\partial p_n} - \frac{\partial_f}{\partial p_1}\frac{\partial_g}{\partial x_1} - \cdots - \frac{\partial_f}{\partial p_n}\frac{\partial_g}{\partial x_n} \right) fg \tag{5}$$

and $P^3(f, g)$ is the nontrivial 2-cocycle

$$P^3(f, g) = \left(\frac{\partial_f}{\partial x_1}\frac{\partial_g}{\partial p_1} + \cdots + \frac{\partial_f}{\partial x_n}\frac{\partial_g}{\partial p_n} - \frac{\partial_f}{\partial p_1}\frac{\partial_g}{\partial x_1} - \cdots - \frac{\partial_f}{\partial p_n}\frac{\partial_g}{\partial x_n} \right)^3 fg$$
$$= \sum_{\substack{j_1,\cdots,j_n,k_1,\cdots,k_n=0\\ j_1+\cdots+j_n+k_1+\cdots+k_n=3}}^{3} \frac{(-1)^{k_1+\cdots+k_n}3!}{j_1!\cdots j_n!k_1!\cdots k_n!} \times \frac{\partial^3 f}{\partial x_1^{j_1}\cdots\partial x_n^{j_n}\partial p_1^{k_1}\cdots\partial p_n^{k_n}} \times \frac{\partial^3 g}{\partial p_1^{j_1}\cdots\partial p_n^{j_n}\partial x_1^{k_1}\cdots\partial x_n^{k_n}}. \tag{6}$$

Barring obstructions, we expect the existence of nontrivial deformations of the Poisson algebra, and existence of nontrivial deformations have indeed been proved in a very general context [10–13]. They always exist if W is finite-dimensional and, for a flat Poisson manifold, they are all equivalent to the Moyal bracket [14,15]

$$[f, g]_{\mathrm{M}} = \frac{2}{\hbar}\sin\left[\frac{\hbar}{2}P(f, g)\right]$$
$$= \sum_{r=0}^{\infty} \frac{(-1)^r}{(2r+1)!}\left(\frac{\hbar}{2}\right)^{2r} P^{2r+1}(f, g)$$
$$= \{f, g\} - \frac{\hbar^2}{24}P^3(f, g) + \cdots, \tag{7}$$

with $\hbar = h/2\pi$. Moreover, $[f, g]_{\mathrm{M}} = (1/i\hbar)(f *_\hbar g - g *_\hbar f)$, where $f *_\hbar g$ is an associative star product,

$$f *_\hbar g = \exp\left[\frac{i\hbar}{2}P(f, g)\right]. \tag{8}$$

Correspondence with quantum mechanics formulated in Hilbert space is obtained by the Weyl quantization prescription [16]. Let $f(\mathbf{x}, \mathbf{p})$ be a function in a $2n$ phase space. Then, if to the function f we associate the Hilbert-space operator

$$\widehat{\boldsymbol{\Omega}}(f) = \frac{1}{(2\pi\hbar)^{2n}}\int d\mathbf{x}\int d\mathbf{p}\int d\boldsymbol{\alpha}\int d\boldsymbol{\beta} \times f(\mathbf{x}, \mathbf{p})e^{-\frac{i}{\hbar}[\boldsymbol{\alpha}\cdot(\mathbf{x}-\widehat{\mathbf{X}})+\boldsymbol{\beta}\cdot(\mathbf{p}-\widehat{\mathbf{P}})]}, \tag{9}$$

where the operators $\widehat{\mathbf{X}}$ and $\widehat{\mathbf{P}}$ are such that $\widehat{\mathbf{X}}\psi(\mathbf{x}) = \mathbf{x}\psi(\mathbf{x})$ and $\widehat{\mathbf{P}}\psi(\mathbf{x}) = -i\hbar(\partial/\partial\mathbf{x})\psi(\mathbf{x})$, we find the quantum commutation relation

$$[\widehat{\boldsymbol{\Omega}}(f), \widehat{\boldsymbol{\Omega}}(g)] = i\hbar\widehat{\boldsymbol{\Omega}}([f, g]_{\mathrm{M}}). \tag{10}$$

On the left-hand side (LHS) of Eq. (10), we have the usual commutator for Hilbert-space operators and, on the right-hand side (RHS), the Moyal bracket defined in Eq. (7). Therefore, quantum mechanics may be described either by associating self-adjoint operators in Hilbert space to the observables or, equivalently, staying in the classical setting of phase-space functions, but deforming their product to a $*_\hbar$ product and their Poisson brackets to Moyal brackets. Recalling that Hamiltonian systems are endowed with a Hamiltonian functional, a bracket, and a Jacobi identity, it is worth saying a few words regarding the latter. The full formal deformation series in powers of $\hbar$ that corresponds to the Moyal bracket Eq. (7) does obey the said identity, but the $P^r(f, g)$ in Eq. (4) (sometimes referred to as higher Poisson brackets) do not, even if they satisfy identities of the Jacobi type [17]. More precisely, we have [18]

$$\sum_{s=0}^{r} \frac{1}{(2s+1)!(2r-2s+1)}\big\{P^{2r-2s+1}[P^{2s+1}(f, g), h] + P^{2r-2s+1}[P^{2s+1}(h, f), g] + P^{2r-2s+1}[P^{2s+1}(g, h), f]\big\}$$
$$= 0, \tag{11}$$

the standard Jacobi identity following for $s = 0$.

III. KINETIC EQUATIONS AND QUANTUM CORRECTIONS

A kinetic equation deals with a probability density $f(\mathbf{x}, \mathbf{p}, t)$ of particles in phase space. The typical form is

$$\frac{\partial f}{\partial t} + \frac{\mathbf{p}}{m}\cdot\frac{\partial f}{\partial \mathbf{x}} + \mathbf{F}_{\mathrm{ext}}\cdot\frac{\partial f}{\partial \mathbf{p}} = S(f), \tag{12}$$

the LHS being a drift term defining the characteristics along which particles with mass m move between collisions under the effect of some external force $\mathbf{F}_{\text{ext}}$, and the RHS representing a collision term. It is therefore an equation involving a probability distribution in the $(\mathbf{x}, \mathbf{p})$ phase space. In quantum mechanics, $f(\mathbf{x}, \mathbf{p}, t)$ cannot be a classical probability distribution because $\mathbf{x}$ and $\mathbf{p}$ are noncommuting variables. However, f may be interpreted as a functional of elements in an algebra with a deformed product and, as discussed before, this leads to the correct quantum results. It is thus tempting to obtain the quantum corrections to Eq. (12) by simply replacing all products by deformed products. Nevertheless, recalling that at the basis of this interpretation of quantum mechanics is the deformation of a Poisson algebra, it is more appropriate to deform the kinetic equation when their (canonical or noncanonical) Hamiltonian structure is exhibited. This is the approach that we follow below.

A. The Vlasov–Poisson equation

The Vlasov–Poisson equation describing a collisionless plasma with purely electrostatic interactions is

$$\frac{\partial f}{\partial t} + \frac{\mathbf{p}}{m} \cdot \frac{\partial f}{\partial \mathbf{x}} - e\frac{\partial \phi}{\partial \mathbf{x}} \cdot \frac{\partial f}{\partial \mathbf{p}} = 0, \tag{13}$$

with e the elementary charge and $\phi(\mathbf{x}, t)$ the electrostatic potential, which obeys Poisson's equation

$$\frac{\partial}{\partial \mathbf{x}} \cdot \frac{\partial}{\partial \mathbf{x}}\phi(\mathbf{x}, t) = -\frac{e}{\varepsilon_0} \int d\mathbf{p}\, f(\mathbf{x}, \mathbf{p}, t), \tag{14}$$

with ε_0 the vacuum permitivity. This constitutes a noncanonical Hamiltonian system [8,19], with the Hamiltonian functional

$$H_{\text{VP}}(f) = \int d\mathbf{x} \int d\mathbf{p}\left[\frac{|\mathbf{p}|^2}{2m} + e\phi(\mathbf{x}, t)\right] f(\mathbf{x}, \mathbf{p}, t), \tag{15}$$

the time evolution of arbitrary phase-space functionals $F[f]$ being given by the Poisson structure

$$\frac{dF}{dt} = \int d\mathbf{x}' \int d\mathbf{p}' f\left\{\frac{\delta F}{\delta f}, \frac{\delta H_{\text{VP}}}{\delta f}\right\}. \tag{16}$$

Remark that, in this continuous, infinite-dimensional formulation, the set of variables is the function $f(\mathbf{x}, \mathbf{p}, t)$ itself, with $\mathbf{x}$ and $\mathbf{p}$ playing the role of indices. Hereabove $\{\cdot, \cdot\}$ stands for the Poisson bracket in Eq. (5) and, for some functional $F[f]$, the functional derivative $\delta F/\delta f$ is defined according to [8,19]

$$\left.\frac{dF[f + \epsilon \delta f]}{d\epsilon}\right|_{\epsilon=0} = \int d\mathbf{x} \int d\mathbf{p}\, \delta f(\mathbf{x}, \mathbf{p}) \frac{\delta F}{\delta f(\mathbf{x}, \mathbf{p})}. \tag{17}$$

Hence, taking into account that

$$\frac{\delta f(\mathbf{x}, \mathbf{p}, t)}{\delta f(\mathbf{x}', \mathbf{p}', t)} = \delta(\mathbf{x}' - \mathbf{x})\delta(\mathbf{p}' - \mathbf{p}), \tag{18}$$

with $\delta(\mathbf{x})$ the Dirac δ distribution, and

$$\frac{\delta H_{\text{VP}}}{\delta f(\mathbf{x}', \mathbf{p}', t)} = \frac{|\mathbf{p}'|^2}{2m} + e\phi(\mathbf{x}', t), \tag{19}$$

using Eq. (16) we obtain the classical Vlasov–Poisson equation Eq. (13) [20]

$$\begin{aligned}\frac{\partial f}{\partial t} &= \int d\mathbf{x}' \int d\mathbf{p}' f\left\{\frac{\delta f}{\delta f}, \frac{\delta H_{\text{VP}}}{\delta f}\right\} \\ &= -\frac{\mathbf{p}}{m} \cdot \frac{\partial f}{\partial \mathbf{x}} + e\frac{\partial \phi}{\partial \mathbf{x}} \cdot \frac{\partial f}{\partial \mathbf{p}}.\end{aligned} \tag{20}$$

For the quantum version, all we have to do is to replace in Eq. (16) the Poisson bracket Eq. (5) by the Moyal bracket Eq. (7),

$$\begin{aligned}\frac{\partial f}{\partial t} &= \int d\mathbf{x}' \int d\mathbf{p}' f\left[\frac{\delta f}{\delta f}, \frac{\delta H_{\text{VP}}}{\delta f}\right]_{\text{M}} \\ &= \int d\mathbf{x}' \int d\mathbf{p}' f \frac{2}{\hbar} \sin\left[\frac{\hbar}{2} P\left(\frac{\delta f}{\delta f}, \frac{\delta H_{\text{VP}}}{\delta f}\right)\right],\end{aligned} \tag{21}$$

$P(f, g)$ being the bidifferential operator defined in Eq. (4) via its powers $P^r(f, g)$. Of special interest, of course, is the leading quantum correction. With $n = 3$, the $\omega_{j,k}$ matrix in the symplectic form Eq. (3) has $\omega_{j,k+3} = -\omega_{j+3,k} = 1$, with all the other elements being zero. Because $\delta H_{\text{VP}}/\delta f$ in Eq. (19) is quadratic in $\mathbf{p}$, all terms in $\omega_{j,j+3}\omega_{k,k+3}\omega_{l,l+3}$ vanish or, equivalently, those terms in Eq. (6) with $j_1 + j_2 + j_3 = 3$ vanish. In addition, since each of the two terms in Eq. (19) depends either on $\mathbf{x}$ or on $\mathbf{p}$, all cross derivatives come to nought, which means that only those terms in Eq. (6) with $k_1 + k_2 + k_3 = 3$ survive. Therefore, we obtain, in the leading $\hbar^2$ order,

$$\begin{aligned}\frac{\partial f}{\partial t} &= -\frac{\mathbf{p}}{m} \cdot \frac{\partial f}{\partial \mathbf{x}} + e\frac{\partial \phi}{\partial \mathbf{x}} \cdot \frac{\partial f}{\partial \mathbf{p}} \\ &\quad - \frac{e\hbar^2}{24} \sum_{j,k,l=1}^{3} \frac{\partial^3 \phi}{\partial x_j \partial x_k \partial x_l} \frac{\partial^3 f}{\partial p_j \partial p_k \partial p_l} + O(\hbar^4).\end{aligned} \tag{22}$$

We must say that we can directly obtain this electrostatic result by setting $\mathbf{B} = 0$ in a previously derived semiclassical electromagnetic kinetic equation [2], and that its one-dimensional version coincides with published results for the Vlasov–Poisson system [1,2,21].

B. The Vlasov–Maxwell equation

The Vlasov–Maxwell equation [22]

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \frac{\partial f}{\partial \mathbf{x}} + \frac{e}{m}(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \frac{\partial f}{\partial \mathbf{v}} = 0, \tag{23}$$

which describes a classical collisionless plasma in an electromagnetic field $[\mathbf{E}(\mathbf{x}, t), \mathbf{B}(\mathbf{x}, t)]$, is also a noncanonical Hamiltonian system. There are several variational formulations of the Vlasov–Maxwell system, the most complete one being probably that in which part of the dynamics is coded in the Poisson structure rather than in the Hamiltonian [8,23]. As a consequence, and to apply the deformation theory for the transition to quantum mechanics, we would have to handle not just the replacement of the Poisson bracket involving the position and momentum of the particles, but also the deformation of the electromagnetic-field dynamics. Hence, and because here we only want to obtain the quantum corrections to the f dynamics, it is more convenient to use the so-called phase-space Hamiltonian, which has been derived from an appropriate action principle using a Legendre transform [24,25]

$$H_{\mathrm{VM}}(f) = \int d\mathbf{x} \int d\mathbf{p}\left[\frac{1}{2m}|\mathbf{p} - e\mathbf{A}(\mathbf{x}, t)|^2 + e\phi(\mathbf{x}, t)\right] \times f(\mathbf{x}, \mathbf{p}, t) + \frac{\varepsilon_0}{2}\int d\mathbf{x}[|\mathbf{E}(\mathbf{x}, t)|^2 + |c\mathbf{B}(\mathbf{x}, t)|^2]. \tag{24}$$

In Eq. (24), c is the speed of light and $\mathbf{E}(\mathbf{x}, t) = -\partial\phi(\mathbf{x}, t)/\partial\mathbf{x} - \partial\mathbf{A}(\mathbf{x}, t)/\partial t$ and $\mathbf{B}(\mathbf{x}, t) = (\partial/\partial\mathbf{x}) \times \mathbf{A}(\mathbf{x}, t)$ are the electric and magnetic fields written in terms of the potential, independent variables $[\phi(\mathbf{x}, t), \mathbf{A}(\mathbf{x}, t)]$. Because of the gauge invariance of the electromagnetic field, ϕ and A are not uniquely specified, but this we correct by the choice of the Coulomb gauge $(\partial/\partial\mathbf{x}) \cdot \mathbf{A} = 0$, which pins down ϕ to the charge density via Poisson's equation Eq. (14). The latter is to be complemented with the equation for $\mathbf{A}$, which reads

$$\left(\frac{\partial}{\partial\mathbf{x}} \cdot \frac{\partial}{\partial\mathbf{x}} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\mathbf{A} - \frac{1}{c^2}\frac{\partial}{\partial\mathbf{x}}\frac{\partial\phi}{\partial t} = -\frac{e}{\varepsilon_0 c^2}\int d\mathbf{p}\mathbf{v}\, f(\mathbf{x}, \mathbf{p}, t), \tag{25}$$

with $\mathbf{v}$ given here in terms of the canonical momentum $\mathbf{p}$ as

$$\mathbf{v} = \frac{\mathbf{p} - e\mathbf{A}}{m}. \tag{26}$$

The Poisson structure is the same as in Eq. (16) for the f dynamics and, for this Hamiltonian,

$$\frac{\delta H_{\mathrm{VM}}}{\delta f(\mathbf{x}', \mathbf{p}', t)} = \frac{|\mathbf{p}|^2}{2m} - \frac{e}{2m}[\mathbf{p} \cdot \mathbf{A}(\mathbf{x}, t) + \mathbf{A}(\mathbf{x}, t) \cdot \mathbf{p}] + \frac{e^2}{2m}|\mathbf{A}(\mathbf{x}, t)|^2 + e\phi(\mathbf{x}, t). \tag{27}$$

Then, recalling Eqs. (5), (16), and (18), we obtain for the classical Vlasov–Maxwell equation

$$\begin{aligned}\frac{\partial f}{\partial t} &= -\frac{\mathbf{p} - e\mathbf{A}}{m} \cdot \frac{\partial f}{\partial\mathbf{x}} + e\left[\frac{\partial\phi}{\partial\mathbf{x}} - \left(\frac{\mathbf{p}}{m} \cdot \frac{\partial}{\partial\mathbf{x}}\right)\mathbf{A} + \frac{e}{2m}\frac{\partial|\mathbf{A}|^2}{\partial\mathbf{x}}\right] \cdot \frac{\partial f}{\partial\mathbf{p}} \\ &= -\frac{\mathbf{p} - e\mathbf{A}}{m} \cdot \frac{\partial f}{\partial\mathbf{x}} - e\left(\mathbf{E} + \mathbf{v} \times \mathbf{B} + \frac{d\mathbf{A}}{dt}\right) \cdot \frac{\partial f}{\partial\mathbf{p}},\end{aligned} \tag{28}$$

where we have used $(1/2)(\partial|\mathbf{A}|^2/\partial\mathbf{x}) = (\mathbf{A} \cdot \partial/\partial\mathbf{x})\mathbf{A} + \mathbf{A} \times [(\partial/\partial\mathbf{x}) \times \mathbf{A}]$, and with

$$\frac{d\mathbf{A}}{dt} = \frac{\partial\mathbf{A}}{\partial t} + \left(\mathbf{v} \cdot \frac{\partial}{\partial\mathbf{x}}\right)\mathbf{A}. \tag{29}$$

Equation (28) is the same as Eq. (23), but written in the variables $(\mathbf{x}, \mathbf{p})$ instead of $(\mathbf{x}, \mathbf{v})$, and so more convenient for the Moyal bracket deformation, which acts on the former variables.

The complete quantum Vlasov–Maxwell equation becomes then

$$\frac{\partial f}{\partial t} = \int d\mathbf{x}' \int d\mathbf{p}' f \frac{2}{\hbar}\sin\left[\frac{\hbar}{2}P\left(\frac{\delta f}{\delta f}, \frac{\delta H_{\mathrm{VM}}}{\delta f}\right)\right], \tag{30}$$

being worthy to notice that, in the Hamiltonian Eq. (24), products must also be replaced by $*_\hbar$–products. Specifically, from Eqs. (4) and (8) we find

$$\begin{aligned}\mathbf{p} *_\hbar \mathbf{A} + \mathbf{A} *_\hbar \mathbf{p} &= 2\mathbf{p} \cdot \mathbf{A} + \frac{i\hbar}{2}[P^1(\mathbf{p}, \mathbf{A}) + P^1(\mathbf{A}, \mathbf{p})] \\ &= 2\mathbf{p} \cdot \mathbf{A},\end{aligned} \tag{31}$$

where the linearity in $\mathbf{p}$, the single dependency on $\mathbf{x}$ of $\mathbf{A}$, and the antisymmetry of $P^1(f, g)$ in Eq. (5) have been accounted for. Proceeding to compute the leading quantum corrections, we obtain

$$\begin{aligned}\frac{\partial f}{\partial t} = &-\frac{\mathbf{p} - e\mathbf{A}}{m} \cdot \frac{\partial f}{\partial\mathbf{x}} - e\left(\mathbf{E} + \mathbf{v} \times \mathbf{B} + \frac{d\mathbf{A}}{dt}\right) \cdot \frac{\partial f}{\partial\mathbf{p}} \\ &- \frac{e\hbar^2}{24}\sum_{j,k,l=1}^{3}\left\{\frac{\partial^3}{\partial x_j\partial x_k\partial x_l}\left(\phi + \frac{e}{2m}|\mathbf{A}|^2\right)\right. \\ &\times \frac{\partial^3 f}{\partial p_j\partial p_k\partial p_l} - \frac{1}{m}\left(\mathbf{p} \cdot \frac{\partial^3\mathbf{A}}{\partial x_j\partial x_k\partial x_l}\frac{\partial^3 f}{\partial p_j\partial p_k\partial p_l}\right. \\ &\left.\left. - 3\frac{\partial^2 A_j}{\partial x_k\partial x_l}\frac{\partial^3 f}{\partial x_j\partial p_k\partial p_l}\right)\right\} + O(\hbar^4).\end{aligned} \tag{32}$$

As far as we know, this is the first time such corrections are explicitly given in terms of the scalar and vector potentials, although their gauge invariant form written in terms of the electromagnetic fields is already known [2]. Also, and as a useful verification, it is easy to check that the Vlasov–Poisson result Eq. (22) is recovered if we set $\mathbf{A} = 0$ in Eq. (32).

C. The Boltzmann equation

In what follows we are concerned with the kinetics of a low-density gas of N identical particles described, classically, by the Boltzmann equation [26]

$$\frac{\partial f_1}{\partial t} + \frac{\mathbf{p}}{m} \cdot \frac{\partial f_1}{\partial\mathbf{x}} = Q(f_1, f_1) \tag{33}$$

for the one-particle distribution function $f_1(\mathbf{x}, \mathbf{p}, t)$. Any interaction between particles (e.g., electrostatic) enters via the collision term $Q(f_1, f_1)$ [27–29]

$$\begin{aligned}Q(f_1, f_1)(\mathbf{x}, \mathbf{p}, t) &= \frac{N}{m}\int d\mathbf{p}_1 \int d\Omega'|\mathbf{p} - \mathbf{p}_1|\sigma(|\mathbf{p} - \mathbf{p}_1|, \Omega') \\ &\times[f_1(\mathbf{x}, \mathbf{p}', t)f_1(\mathbf{x}, \mathbf{p}_1', t) - f_1(\mathbf{x}, \mathbf{p}, t)f_1(\mathbf{x}, \mathbf{p}_1, t)].\end{aligned} \tag{34}$$

In Eq. (34), $\mathbf{p}'$ and $\mathbf{p}_1'$ are the outgoing momenta after a two-particle collision, and $\mathbf{p}$ and $\mathbf{p}_1$ the incoming momenta, which, for an elastic process, are linked according to

$$\begin{aligned}\mathbf{p}' &= \frac{\mathbf{p} + \mathbf{p}_1}{2} + \frac{|\mathbf{p} - \mathbf{p}_1|}{2}\mathbf{n}' \\ \mathbf{p}_1' &= \frac{\mathbf{p} + \mathbf{p}_1}{2} - \frac{|\mathbf{p} - \mathbf{p}_1|}{2}\mathbf{n}',\end{aligned} \tag{35}$$

with $\mathbf{n}'$ the unit vector pointing along the direction of the unit-sphere solid-angle element $d\Omega'$, which is that into which the relative velocity $(\mathbf{p} - \mathbf{p}_1)/m$ is deflected after the collision. In addition, $|\mathbf{p} - \mathbf{p}_1|/m$ is the relative speed, which does not change if the binary encounter between alike particles is elastic, and $\sigma(|\mathbf{p}-\mathbf{p}_1|, \Omega')$ is the scattering differential cross section, which is a characteristic of the interaction potential. The set Eq. (35) can be replaced with

$$\begin{aligned}\mathbf{p}' &= \mathbf{p} - [(\mathbf{p}-\mathbf{p}_1)\cdot\mathbf{n}]\mathbf{n}\\ \mathbf{p}'_1 &= \mathbf{p}_1 + [(\mathbf{p}-\mathbf{p}_1)\cdot\mathbf{n}]\mathbf{n},\end{aligned} \tag{36}$$

where $\mathbf{n}$ is some unit vector, in which choice is coded the interaction potential, and which must verify the condition $(\mathbf{p}'-\mathbf{p}'_1)\cdot\mathbf{n} = -(\mathbf{p}-\mathbf{p}_1)\cdot\mathbf{n}$ [30]. For a gas of hard elastic spheres with radius r, Eq. (34) is more conveniently written as [31–35]

$$\begin{aligned}&Q(f_1, f_1)(\mathbf{x}, \mathbf{p}, t)\\ &= \frac{4Nr^2}{m}\int d\mathbf{p}_1 \int d\Omega(\mathbf{p}-\mathbf{p}_1)\cdot\mathbf{n}\\ &\quad\times[f_1(\mathbf{x}, \mathbf{p}', t)f_1(\mathbf{x}, \mathbf{p}'_1, t) - f_1(\mathbf{x}, \mathbf{p}, t)f_1(\mathbf{x}, \mathbf{p}_1, t)],\end{aligned} \tag{37}$$

where $\mathbf{n}$ points in the direction of $d\Omega$ (from the test particle at $\mathbf{x}$ toward the target particle at $\mathbf{x}_1$) and lies on the line connecting the centers of the two spheres at time of contact, with the integration in $d\Omega$ restricted by the condition $(\mathbf{p}-\mathbf{p}_1)\cdot\mathbf{n} \geqslant 0$ [36].

To obtain the quantum version of the Boltzmann equation, basically two approaches have been followed. The first is just to solve the scattering problem in quantum mechanics and then to replace, in the classical Boltzmann equation, the classical by the quantum cross section. For weakly coupled gases, the Born approximation has been used. The second, more sophisticated approach starts from the Schrödinger equation for a many-body problem, writes the evolution equation for the corresponding Wigner function, and then goes through series expansions and limiting steps very much analogous to the classical ones to obtain an equation for the one-particle Wigner function [37]. The quantum mechanical computation of the cross section is always, of course, a necessary step and it depends on the particular interaction potential, but we have nothing to say about it here. Our attention is to be focused on, say, the structural term $[f_1(\mathbf{x}, \mathbf{p}', t)f_1(\mathbf{x}, \mathbf{p}'_1, t) - f_1(\mathbf{x}, \mathbf{p}, t)f_1(\mathbf{x}, \mathbf{p}_1, t)]$, as opposed to the scattering kernel $|\mathbf{p}-\mathbf{p}_1|\sigma(|\mathbf{p}-\mathbf{p}_1|, \Omega')$, and on the quantum corrections to it.

The Boltzmann equation does not follow directly from a Hamiltonian, its irreversible nature arising mostly from the choice of uncorrelated incoming configurations to represent the collisions [38]. More precisely, it is found that the missing (reversible) information would be contained in the higher-order cumulants of the distribution function [33,35,39]. Because the quantum deformation is only defined for Hamiltonian systems, our approach is to proceed as far as possible in a reversible Hamiltonian framework and only in the final stage, after the quantum deformation has been taken into account, do we make use of the specific approximations leading to the Boltzmann equation. Consider a system of N particles interacting by a two-body potential $\phi(\mathbf{x}_\alpha - \mathbf{x}_\beta)$. The Hamiltonian that drives the evolution of the N-particle distribution function $f_N(\mathbf{x}_1, \mathbf{p}_1, \dots, \mathbf{x}_j, \mathbf{p}_j, \dots, \mathbf{x}_N, \mathbf{p}_N, t)$ is

$$\begin{aligned}H(f_N) &= \prod_{\alpha=1}^{N}\int d\mathbf{x}_\alpha \int d\mathbf{p}_\alpha\\ &\quad\times\left[\sum_{\alpha=1}^{N}\frac{\mathbf{p}_\alpha^2}{2m} + \sum_{\alpha=1}^{N-1}\sum_{\beta=\alpha+1}^{N}\phi(\mathbf{x}_\alpha - \mathbf{x}_\beta)\right]\\ &\quad\times f_N(\mathbf{x}_1, \mathbf{p}_1, \dots, \mathbf{x}_\alpha, \mathbf{p}_\alpha, \dots, \mathbf{x}_N, \mathbf{p}_N, t).\end{aligned} \tag{38}$$

Taking into account the quantum deformation, the time evolution of f_N is, as before, given by

$$\frac{\partial f_N}{\partial t} = \prod_{\alpha=1}^{N}\int d\mathbf{x}'_\alpha \int d\mathbf{p}'_\alpha\, f_N \frac{2}{\hbar}\sin\left[\frac{\hbar}{2}P\left(\frac{\delta f_N}{\delta f_N}, \frac{\delta H}{\delta f_N}\right)\right], \tag{39}$$

which, in leading order, yields

$$\begin{aligned}\frac{\partial f_N}{\partial t} &= -\sum_{\alpha=1}^{N}\frac{\mathbf{p}_\alpha}{m}\cdot\frac{f_N}{\partial \mathbf{x}_\alpha}\\ &\quad+\sum_{\alpha=1}^{N-1}\sum_{\beta=\alpha+1}^{N}\frac{\partial\phi(\mathbf{x}_\alpha - \mathbf{x}_\beta)}{\partial(\mathbf{x}_\alpha - \mathbf{x}_\beta)}\cdot\left(\frac{\partial f_N}{\partial \mathbf{p}_\alpha} - \frac{\partial f_N}{\partial \mathbf{p}_\beta}\right)\\ &\quad-\frac{\hbar^2}{24}\sum_{\alpha,\beta,\gamma=1}^{N}\sum_{j,k,l=1}^{3}\frac{\partial^3}{\partial x_{\alpha j}\partial x_{\beta k}\partial x_{\gamma l}}\\ &\quad\times\left[\sum_{\alpha'=1}^{N-1}\sum_{\beta'=\alpha'+1}^{N}\phi(\mathbf{x}_{\alpha'} - \mathbf{x}_{\beta'})\right]\\ &\quad\times\frac{\partial^3 f_N}{\partial p_{\alpha j}\partial p_{\beta k}\partial p_{\gamma l}} + O(\hbar^4),\end{aligned} \tag{40}$$

where we have used the fact that $\partial\phi(\mathbf{x}_\alpha - \mathbf{x}_\beta)/\partial\mathbf{x}_\gamma = (\delta_{\alpha,\gamma} - \delta_{\beta,\gamma})\partial\phi(\mathbf{x}_\alpha - \mathbf{x}_\beta)/\partial(\mathbf{x}_\alpha - \mathbf{x}_\beta)$. A particular instance of $\phi(\mathbf{x}_\alpha - \mathbf{x}_\beta)$, and one that encompasses the so-called Coulomb model relevant for plasma physics [22], is the central-force potential $\phi(|\mathbf{x}_\alpha - \mathbf{x}_\beta|)$, in which case $\partial\phi(\mathbf{x}_\alpha - \mathbf{x}_\beta)/\partial(\mathbf{x}_\alpha - \mathbf{x}_\beta) = \partial\phi(|\mathbf{x}_\alpha - \mathbf{x}_\beta|)/\partial(\mathbf{x}_\alpha - \mathbf{x}_\beta) = \phi'(|\mathbf{x}_\alpha - \mathbf{x}_\beta|)(\mathbf{x}_\alpha - \mathbf{x}_\beta)/|\mathbf{x}_\alpha - \mathbf{x}_\beta|$. In Eq. (40), the LHS and the first two terms on the RHS constitute the classical Liouville equation, from which can be derived a whole hierarchy of kinetic equations governing joint particle distributions, from the N-particle down to the one-particle distribution function, the so-called BBGKY hierarchy [22,31–35]. Following thus a similar procedure, we integrate over the $N-1$ identical particles to obtain the equation for the one-particle marginal $f_1(\mathbf{x}, \mathbf{p}, t)$,

$$\begin{aligned}\frac{\partial f_1}{\partial t} &= -\frac{\mathbf{p}}{m}\cdot\frac{\partial f_1}{\partial \mathbf{x}} + (N-1)\int d\mathbf{x}_1 \int d\mathbf{p}_1\\ &\quad\times\frac{\partial\phi(\mathbf{x} - \mathbf{x}_1)}{\partial(\mathbf{x} - \mathbf{x}_1)}\cdot\frac{\partial}{\partial\mathbf{p}}f_2(\mathbf{x}, \mathbf{p}, \mathbf{x}_1, \mathbf{p}_1, t)\\ &\quad-\frac{\hbar^2}{24}(N-1)\sum_{j,k,l=1}^{3}\int d\mathbf{x}_1 \int d\mathbf{p}_1 \frac{\partial^3\phi(|\mathbf{x} - \mathbf{x}_1|)}{\partial x_j\partial x_k\partial x_l}\\ &\quad\times\frac{\partial^3 f_2(\mathbf{x}, \mathbf{p}, \mathbf{x}_1, \mathbf{p}_1, t)}{\partial p_j\partial p_k\partial p_l} + O(\hbar^4).\end{aligned} \tag{41}$$

Choosing now a spherical coordinate system centered at $\mathbf{x}$, such that $d\mathbf{x}_1 = |\mathbf{x}_1 - \mathbf{x}|^2 d|\mathbf{x}_1 - \mathbf{x}| d\Omega$, we make the assumption that the two-body potential is a hardcore potential whose gradient reads

$$\frac{\partial \phi(\mathbf{x}-\mathbf{x}_1)}{\partial(\mathbf{x}-\mathbf{x}_1)} = \mu(|\mathbf{p}-\mathbf{p}_1|,\Omega)\delta(|\mathbf{x}-\mathbf{x}_1|-2r)\frac{\mathbf{x}-\mathbf{x}_1}{|\mathbf{x}-\mathbf{x}_1|} = -\mu(|\mathbf{p}-\mathbf{p}_1|,\Omega)\delta(|\mathbf{x}-\mathbf{x}_1|-2r)\mathbf{n}. \quad (42)$$

In Eq. (42), $\mu(|\mathbf{p}-\mathbf{p}_1|,\Omega)$ allows both for the intensity and any eventual dependence on relative speed and solid-angle, and we have made $\mathbf{n} = (\mathbf{x}_1 - \mathbf{x})/|\mathbf{x}_1 - \mathbf{x}|$. Equation (41) then reads

$$\frac{\partial f_1}{\partial t} = -\frac{\mathbf{p}}{m}\cdot\frac{\partial f_1}{\partial \mathbf{x}} - 4(N-1)r^2\int d\mathbf{p}_1\int d\Omega \times\mu(|\mathbf{p}-\mathbf{p}_1|,\Omega)\mathbf{n}\cdot\frac{\partial}{\partial \mathbf{p}}f_2(\mathbf{x},\mathbf{p},\mathbf{x}+2r\mathbf{n},\mathbf{p}_1,t) + \frac{\hbar^2}{24}4(N-1)r^2\sum_{j,k=1}^{3}\int d\mathbf{p}_1\int d\Omega\int d|\mathbf{x}_1-\mathbf{x}| \times\mu(|\mathbf{p}-\mathbf{p}_1|,\Omega)\delta(|\mathbf{x}-\mathbf{x}_1|-2r) \times\mathbf{n}\cdot\frac{\partial}{\partial \mathbf{p}}\frac{\partial^4 f_2(\mathbf{x},\mathbf{p},\mathbf{x}_1,\mathbf{p}_1,t)}{\partial x_{1j}\partial x_{1k}\partial p_j\partial p_k} + O(\hbar^4), \quad (43)$$

which, in the so-called Boltzmann–Grad limit $N \to \infty$ and $r \to 0$, but $(N-1)r^2 \to \lambda^{-1}$ with λ constant [31–35], becomes

$$\frac{\partial f_1}{\partial t} = -\frac{\mathbf{p}}{m}\cdot\frac{\partial f_1}{\partial \mathbf{x}} - 4(N-1)r^2\int d\mathbf{p}_1\int d\Omega \times\mu(|\mathbf{p}-\mathbf{p}_1|,\Omega)\mathbf{n}\cdot\frac{\partial}{\partial \mathbf{p}}f_2(\mathbf{x},\mathbf{p},\mathbf{x},\mathbf{p}_1,t) + \frac{\hbar^2}{24}4(N-1)r^2\sum_{j,k=1}^{3}\int d\mathbf{p}_1\int d\Omega\int d|\mathbf{x}_1-\mathbf{x}| \times\mu(|\mathbf{p}-\mathbf{p}_1|,\Omega)\delta(|\mathbf{x}-\mathbf{x}_1|) \times\mathbf{n}\cdot\frac{\partial}{\partial \mathbf{p}}\frac{\partial^4 f_2(\mathbf{x},\mathbf{p},\mathbf{x}_1,\mathbf{p}_1,t)}{\partial x_{1j}\partial x_{1k}\partial p_j\partial p_k} + O(\hbar^4). \quad (44)$$

Up to this point we have stayed within the Hamiltonian framework and its quantum deformation, the limit $(N-1)r^2 \to \lambda^{-1}$ being simply a scaling low-density limit. Basically, it states that the ratio between the effective volume of all gas molecules and the constant volume V of the gas container behaves like $Nr^3/V \to r/\lambda V \to 0$, which is also the same scaling for the ratio between the interaction-potential range and the mean free path [40]. Incidentally, being still within the Hamiltonian framework shows that the Boltzmann–Grad limit can only have a marginal effect on the irreversibility of the Boltzmann equation, a fact already noticed in the past [38]. Rather, the irreversible collision term of the Boltzmann equation is obtained by the restriction to two-body collisions, factorization of the two-particle marginal into the product of two one-particle distributions, restriction of the integration to the incoming particles, and neglect of higher-order correlations [31,33–35,39]. The last choices are actually related, because factorization implies statistical independence and, after the collision, the particles are certainly correlated.

So, moving into the irreversible realm of Boltzmann's equation, we proceed with the factorization $f_2(\mathbf{x},\mathbf{p},\mathbf{x}_1,\mathbf{p}_1,t) \simeq f_1(\mathbf{x},\mathbf{p},t)f_1(\mathbf{x}_1,\mathbf{p}_1,t)$ and rewrite Eq. (44) accordingly,

$$\frac{\partial f_1}{\partial t} = -\frac{\mathbf{p}}{m}\cdot\frac{\partial f_1}{\partial \mathbf{x}} - 4(N-1)r^2\int d\mathbf{p}_1\int d\Omega \times\mu(|\mathbf{p}-\mathbf{p}_1|,\Omega)\mathbf{n}\cdot\frac{\partial}{\partial \mathbf{p}}\left[f_1(\mathbf{x},\mathbf{p},t)f_1(\mathbf{x},\mathbf{p}_1,t) - \frac{\hbar^2}{24}\sum_{j,k=1}^{3}\frac{\partial^2 f_1(\mathbf{x},\mathbf{p},t)}{\partial p_j\partial p_k}\frac{\partial^2 f_1(\mathbf{x},\mathbf{p}_1,t)}{\partial x_j\partial x_k}\right] + O(\hbar^4). \quad (45)$$

We subsequently notice that the collision physics is fully contained in the integral

$$\int d\mathbf{p}_1\int d\Omega\mu(|\mathbf{p}-\mathbf{p}_1|,\Omega)\mathbf{n}\cdot\frac{\partial}{\partial \mathbf{p}}f_1(\mathbf{x},\mathbf{p},t)f_1(\mathbf{x},\mathbf{p}_1,t), \quad (46)$$

which enters both the classical and the deformed part of Eq. (45). We could then immediately write down the final result, stated in Eq. (49), by invoking the extensive literature on hard spheres and short-range potentials, and by directly transposing the results therein [31–35,38,39]. In these works, the short-range potential is treated by restricting the calculation to the outer free space beyond the interaction range, which gives a contribution analogous to the hard-sphere problem, and then showing that the inner space contribution vanishes in the Boltzmann–Grad limit. This having been said, a simple heuristic reasoning might shed some light on the meaning of their rigorous derivation. Hence, in the evolution equation Eq. (45), the term $\mathbf{n}\cdot(\partial/\partial\mathbf{p})f_1(\mathbf{x},\mathbf{p},t)f_1(\mathbf{x},\mathbf{p}_1,t)$ stands for a continuous rate of variation of the phase-space density product $f_1 f_1$ when the momentum $\mathbf{p}$ changes under the action of the force $-\partial\phi/\partial\mathbf{x}$. However, in the limit of the hardcore potential Eq. (42), the variation of the momenta, before and after the collision, is discontinuous (as in a hard-sphere encounter), and the momentum gradient $\partial(f_1 f_1)/\partial\mathbf{p}$ in the term above may be appropriately replaced with a finite difference $\Delta(f_1 f_1)/\Delta\mathbf{p}$. Incidentally, bear in mind that, because of momentum conservation, as $\mathbf{p}$ changes to $\mathbf{p}'$, so must $\mathbf{p}_1$ change to $\mathbf{p}'_1$, according to Eq. (36). Putting it differently, the $\partial/\partial\mathbf{p}$ operator in Eqs. (45) and (46) acts not only directly on $\mathbf{p}$, but also indirectly on $\mathbf{p}_1$, since $\mathbf{p}+\mathbf{p}_1$ must be kept constant. Therefore, we can write

$$f_1(\mathbf{x},\mathbf{p}+\Delta\mathbf{p},t)f_1(\mathbf{x},\mathbf{p}_1-\Delta\mathbf{p},t) \simeq f_1(\mathbf{x},\mathbf{p},t)f_1(\mathbf{x},\mathbf{p}_1,t) + \Delta\mathbf{p}\cdot\frac{\partial}{\partial\mathbf{p}}f_1(\mathbf{x},\mathbf{p},t)f_1(\mathbf{x},\mathbf{p}_1,t), \quad (47)$$

so that, with $\Delta\mathbf{p} = -[(\mathbf{p}-\mathbf{p}_1)\cdot\mathbf{n}]\cdot\mathbf{n}$ as follows from Eq. (36),

$$\mathbf{n}\cdot\frac{\partial}{\partial\mathbf{p}}f_1(\mathbf{x},\mathbf{p},t)f_1(\mathbf{x},\mathbf{p}_1,t) \simeq \frac{f_1(\mathbf{x},\mathbf{p},t)f_1(\mathbf{x},\mathbf{p}_1,t) - f_1(\mathbf{x},\mathbf{p}',t)f_1(\mathbf{x},\mathbf{p}'_1,t)}{(\mathbf{p}-\mathbf{p}_1)\cdot\mathbf{n}}. \quad (48)$$

Since we are only concerned with colliding particles (moving against each other), and recalling that $\mathbf{n}$ points from $\mathbf{x}$ toward $\mathbf{x}_1$, the restriction $(\mathbf{p}-\mathbf{p}_1)\cdot\mathbf{n} \geqslant 0$ applies.

Replacing thus Eq. (48) in both terms on the RHS of Eq. (45) leads to Eq. (33), but with the quantum-deformed collision term

$$\begin{aligned} Q(f_1, f_1)(\mathbf{x}, \mathbf{p}, t) &= 4(N-1)r^2 \int d\mathbf{p}_1 \int d\Omega \frac{\mu(|\mathbf{p}-\mathbf{p}_1|, \Omega)}{(\mathbf{p}-\mathbf{p}_1)\cdot\mathbf{n}} \\ &\times \Bigg\{ [f_1(\mathbf{x}, \mathbf{p}', t) f_1(\mathbf{x}, \mathbf{p}_1', t) - f_1(\mathbf{x}, \mathbf{p}, t) f_1(\mathbf{x}, \mathbf{p}_1, t)] \\ &- \frac{\hbar^2}{24} \sum_{j,k=1}^{3} \left[\frac{\partial^2 f_1(\mathbf{x}, \mathbf{p}', t)}{\partial p_j \partial p_k} \frac{\partial^2 f_1(\mathbf{x}, \mathbf{p}_1', t)}{\partial x_j \partial x_k} \right. \\ &\left. - \frac{\partial^2 f_1(\mathbf{x}, \mathbf{p}, t)}{\partial p_j \partial p_k} \frac{\partial^2 f_1(\mathbf{x}, \mathbf{p}_1, t)}{\partial x_j \partial x_k} \right] \Bigg\} + O(\hbar^4). \end{aligned} \tag{49}$$

In the thermodynamic limit $N \gg 1$, the first, classical term above recovers the hard-sphere form Eq. (37) if we choose $\mu(|\mathbf{p}-\mathbf{p}_1|, \Omega) = [(\mathbf{p}-\mathbf{p}_1)\cdot\mathbf{n}]^2/m$. The quantities that multiply $[f_1(\mathbf{x}, \mathbf{p}', t) f_1(\mathbf{x}, \mathbf{p}_1', t) - f_1(\mathbf{x}, \mathbf{p}, t) f_1(\mathbf{x}, \mathbf{p}_1, t)]$ in Eq. (49) depend both on the limit $(N-1)r^2 \to \lambda^{-1}$ and on the nature of the potential, via $\mu(|\mathbf{p}-\mathbf{p}_1|, \Omega)$. Hence, carrying out a *bona fide* generalization, we extend Eq. (49) to an arbitrary scattering kernel $|\mathbf{p}-\mathbf{p}_1|\sigma(|\mathbf{p}-\mathbf{p}_1|, \Omega')$, and replace Eq. (34) with its quantum-deformed version

$$\begin{aligned} Q(f_1, f_1)(\mathbf{x}, \mathbf{p}, t) &= \frac{N}{m} \int d\mathbf{p}_1 \int d\Omega' |\mathbf{p}-\mathbf{p}_1| \sigma(|\mathbf{p}-\mathbf{p}_1|, \Omega') \\ &\times \Bigg\{ [f_1(\mathbf{x}, \mathbf{p}', t) f_1(\mathbf{x}, \mathbf{p}_1', t) - f_1(\mathbf{x}, \mathbf{p}, t) f_1(\mathbf{x}, \mathbf{p}_1, t)] \\ &- \frac{\hbar^2}{24} \sum_{j,k=1}^{3} \left[\frac{\partial^2 f_1(\mathbf{x}, \mathbf{p}', t)}{\partial p_j \partial p_k} \frac{\partial^2 f_1(\mathbf{x}, \mathbf{p}_1', t)}{\partial x_j \partial x_k} \right. \\ &\left. - \frac{\partial^2 f_1(\mathbf{x}, \mathbf{p}, t)}{\partial p_j \partial p_k} \frac{\partial^2 f_1(\mathbf{x}, \mathbf{p}_1, t)}{\partial x_j \partial x_k} \right] \Bigg\} + O(\hbar^4). \end{aligned} \tag{50}$$

In the usual construction of the collision term in the classical Boltzmann equation, the approximations involved may formally correspond to neglecting terms that, for some practical configurations, may indeed be larger than the newly derived quantum corrections in Eqs. (49) or (50). Nevertheless, the essential point of our result is that, if all the standard Boltzmann assumptions are valid, then, not only the cross section, but also the form of the collision term itself must have quantum corrections for nonuniform densities.

In this work, we have aimed at deriving the quantum corrections to the very structure of the relevant kinetic equations in plasma physics, instead of simply infusing some quantumness in quantities entering an otherwise classical equation. This is why we have assumed classical, Maxwell–Boltzmann statistics for the identical particles. Hence the remark that, had we chosen to use bosonic or fermionic statistics, we would have simply to replace $[f_1(\mathbf{x}, \mathbf{p}', t) f_1(\mathbf{x}, \mathbf{p}_1', t) - f_1(\mathbf{x}, \mathbf{p}, t) f_1(\mathbf{x}, \mathbf{p}_1, t)]$ with [27,41]

$$\begin{aligned} \{ f_1(\mathbf{x}, \mathbf{p}', t)[1+\theta f_1(\mathbf{x}, \mathbf{p}, t)] f_1(\mathbf{x}, \mathbf{p}_1', t)[1+\theta f_1(\mathbf{x}, \mathbf{p}_1, t)] \\ - f_1(\mathbf{x}, \mathbf{p}, t)[1+\theta f_1(\mathbf{x}, \mathbf{p}', t)] f_1(\mathbf{x}, \mathbf{p}_1, t) \\ \times [1+\theta f_1(\mathbf{x}, \mathbf{p}_1', t)]\}, \end{aligned} \tag{51}$$

where $\theta = -1, 0, +1$ indicates, respectively, Fermi–Dirac, Maxwell–Boltzmann, or Bose–Einstein statistics, and a similar replacement should intervene in the term born of the quantum deformation. Notice, however, that the derivation of the bosonic and fermionic modifications to the Boltzmann equation is not a clearcut business, since the assumption of binary encounters (at the heart of Boltzmann's equation) becomes questionable for degenerate quantum gases [27]. In any case, in the low-density limit, where the Boltzmann equation is most relevant, the particles are too rare to make statistical correlations a dominant effect. It is also immediate to verify, from Eqs. (49) or (50), that the Boltzmann collision operator with the quantum corrections yields the same equilibrium distributions as its classical counterparts Eqs. (34) or (37). The reason is that, for an uniform gas (or plasma), which must be the case in equilibrium, the quantum deformation vanishes and the appropriate equilibrium distribution is given by equating Eq. (51) to nought.

Note still that any rigorous, formal derivation of the collision operator for the Boltzmann equation always assumes short-range, hard-sphere-like potentials, be it in the quantum-deformation approach followed above, or in the known classical treatments [31–35,38,39]. Moreover, also in the more heuristic derivations it is assumed that the range of the interaction potential must be short, typically much shorter than a mean free path (which is essential for the assumption of molecular chaos) [27,28]. Now, it is well recognized that in a plasma, where charged particles interact by means of the Coulomb potential, we must deal with the long-range $1/r$ dependency of the latter. It is precisely this long-range tail of the potential that gives rise to collective effects in a plasma. In fact, whenever two particles interact at a large distance from each other, their encounter is not really binary (which is another crucial assumption in deriving Boltzmann's equation), as they are under the electrostatic (or electromagnetic) influence of many other particles [22,27]. These collective effects are accounted for by the ensemble-averaged self-consistent fields (or potentials) that enter the different forms of the Vlasov equation (sometimes referred to as the collisionless Boltzmann equation [22,28]). But binary encounters, called Coulomb collisions, do take place in a plasma, during which two charged particles get close enough to each other for the divergent $1/r$ two-particle Coulomb potential to dominate the long-range electric field generated by the smoothed out distribution of the entire charge population. And, because of Debye shielding, there is indeed an effective exponential cutoff of the $1/r$ Coulomb potential, which brings the interaction range to a distance of the order of the Debye length λ_D [22,27]. These discrete-particle effects are taken care of by a collision term such as the one appearing on the RHS of the Boltzmann equation, where the cross-section entering it can be calculated according to Coulomb scattering. The simultaneous account of collective and discrete-particle (i.e., collisional) effects must then be included in a full plasma kinetic equation, which can then be seen as a combination of the Vlasov and Boltzmann equations, and deformed accordingly. For instance, the classical kinetic equation for the Coulomb model, in which the magnetic fields produced by the motion of charged particles are neglected, may read

$$\frac{\partial f_1}{\partial t} + \frac{\mathbf{p}}{m} \cdot \frac{\partial f_1}{\partial \mathbf{x}} + e\mathbf{E} \cdot \frac{\partial f_1}{\partial \mathbf{p}} = Q(f_1, f_1), \tag{52}$$

in which the RHS can be worked out to arrive at the Lenard–Balescu equation, the subsequent Landau form for the collision operator and, finally, the well-known Fokker–Planck operator for Coulomb collisions [22]. As for the quantum deformation of Eq. (52), our educated guess would say it would be accomplished by the combination of Eqs. (22) and (50). Having thus justified the inclusion of the Boltzmann equation in our paper on the quantum corrections to the fundamental kinetic equations of plasma physics, we should recall that the importance and conspicuousness of this equation goes much beyond the physics of fully ionized, high-temperature plasmas (where we only need to account for elastic, Coulomb collisions), encompassing also the fields of low-temperature plasmas and plasma chemistry (with their myriad of inelastic collisional processes). This, and the fact that the Boltzmann equation is key to the kinetic theory of gases and provides the starting point for a near-exact formulation of transport theory [27,28], would have justified, *per se*, our interest in deriving the quantum corrections to it.

Even in its classical form, solving the integro-differential Boltzmann equation Eqs. (33) and (34) easily becomes a daunting task, and things certainly do not get easier when using the quantum-deformed collision term Eq. (50). This is the reason why the collision operator is often written in the so-called relaxation-time approximation, which in the classical case means

$$Q(f_1, f_1) \approx -\frac{f_1 - f_1^{(0)}}{\tau}, \tag{53}$$

where $f_1^{(0)}$ is some local equilibrium distribution (locally restored because of collisions) and τ^{-1} some (possibly velocity dependent) collision frequency [27,28]. The RHS of Eq. (53) is known in plasma physics as the BGK form of the collision operator [42], so that, looking for the quantum corrections to it, we hope to give an alternative to Eq. (50) that would be easier to use in practical applications. So, looking at the scattering kernel in Eq. (50), we start by identifying the post-collision distributions $f_1(\mathbf{x}, \mathbf{p}', t)$ and $f_1(\mathbf{x}, \mathbf{p}_1', t)$ with the locally restored equilibrium distributions $f_1^{(0)}(\mathbf{x}, \mathbf{p}, t)$ and $f_1^{(0)}(\mathbf{x}, \mathbf{p}_1, t)$, whence

$$\begin{aligned}
&\frac{N}{m}\int d\mathbf{p}_1 \int d\Omega' |\mathbf{p} - \mathbf{p}_1| \sigma(|\mathbf{p} - \mathbf{p}_1|, \Omega') \\
&\quad \times [f_1(\mathbf{x}, \mathbf{p}', t) f_1(\mathbf{x}, \mathbf{p}_1', t) - f_1(\mathbf{x}, \mathbf{p}, t) f_1(\mathbf{x}, \mathbf{p}_1, t)] \\
&\approx \frac{N}{m}\int d\mathbf{p}_1 \int d\Omega' |\mathbf{p} - \mathbf{p}_1| \sigma(|\mathbf{p} - \mathbf{p}_1|, \Omega') \\
&\quad \times [f_1^{(0)}(\mathbf{x}, \mathbf{p}, t) f_1^{(0)}(\mathbf{x}, \mathbf{p}_1, t) - f_1(\mathbf{x}, \mathbf{p}, t) f_1(\mathbf{x}, \mathbf{p}_1, t)].
\end{aligned} \tag{54}$$

Subsequently, we recall the rigorous definition of τ as the inverse of an average collision rate (or frequency) [28],

$$\begin{aligned}
\tau^{-1}(\mathbf{x}, \mathbf{p}, t) = {}& \frac{N}{m}\int d\mathbf{p}_1 \int d\Omega' |\mathbf{p} - \mathbf{p}_1| \sigma(|\mathbf{p} - \mathbf{p}_1|, \Omega') \\
&\times f_1(\mathbf{x}, \mathbf{p}_1, t),
\end{aligned} \tag{55}$$

and assume that deviations from equilibrium are small enough that there are no significant differences if Eq. (55) is calculated using either f_1 or $f_1^{(0)}$. Therefore, Eq. (54) becomes [43]

$$\begin{aligned}
&\frac{N}{m}\int d\mathbf{p}_1 \int d\Omega' |\mathbf{p} - \mathbf{p}_1| \sigma(|\mathbf{p} - \mathbf{p}_1|, \Omega') \\
&\quad \times [f_1(\mathbf{x}, \mathbf{p}', t) f_1(\mathbf{x}, \mathbf{p}_1', t) - f_1(\mathbf{x}, \mathbf{p}, t) f_1(\mathbf{x}, \mathbf{p}_1, t)] \\
&\quad \approx \tau^{-1}(\mathbf{x}, \mathbf{p}, t)\big[f_1^{(0)}(\mathbf{x}, \mathbf{p}, t) - f_1(\mathbf{x}, \mathbf{p}, t)\big].
\end{aligned} \tag{56}$$

Putting together Eqs. (50) and (56) we then obtain the quantum-deformed version of the BGK collision operator Eq. (53), which reads

$$\begin{aligned}
Q(f_1, f_1) \approx {}& -\frac{f_1 - f_1^{(0)}}{\tau} + \frac{\hbar^2}{24}\sum_{j,k=1}^{3} \frac{\partial^2(1/\tau)}{\partial x_j \partial x_k} \frac{\partial^2\big(f_1 - f_1^{(0)}\big)}{\partial p_j \partial p_k} \\
&+ O\big(\hbar^4\big).
\end{aligned} \tag{57}$$

Of course, when solving the Boltzmann equation using the relaxation-time approximation, we assume the existence of τ but we have only rough means of calculating this quantity. A possibility is to iterate the solution of Eqs. (33) and (57) with the computation of τ in terms of f_1 via Eq. (55), a simpler approach being to fix τ by using $f_1^{(0)}$ in the latter.

IV. CONCLUSIONS

We have shown that, using the formulation of quantum mechanics as a (Moyal) deformation of the Poisson structure, it is possible to obtain a simple, direct derivation of the quantum corrections to phase-space kinetic equations. In the quantum kinetic equations that are thus obtained, the arguments $\mathbf{x}$ and $\mathbf{p}$ of the distributions $f(\mathbf{x}, \mathbf{p}, t)$ are elements of a deformed algebra and so, strictly speaking, are not classical variables. Consequently, when the leading $O(\hbar^2)$ correction is used in dense plasmas (where it may be eventually needed), it must be kept in mind that interpreting $f(\mathbf{x}, \mathbf{p}, t)$ as a classical quantity is an approximation. Nevertheless it is a controlled, educated approximation, in the sense that, not only are the quantum kinetic equations derived in a rigorous manner, but also the correct classical limit is retrieved by making Planck's constant go to zero.

The quantum modifications to the Vlasov–Poisson and Vlasov–Maxwell equations are certainly physically very relevant for materials involving dense assemblies of charged particles. As for the quantum corrections to the Boltzman equation, they are perhaps not so relevant from the physics point of view, since most of the approximations used (both in the classical and quantum derivations) apply to rarefied gases, with small statistical overlap of the wave functions. Nonetheless, we have included here this equation mostly to emphasize that, whenever quantum effects are relevant, in addition to the features encoded in the scattering kernel, also the structure of the collision term should be modified. Still concerning the Boltzmann equation, but within the framework of the relaxation-time approximation, we have also derived the quantum deformation of the BGK collision operator [42]. Finally, we wish to point out that, to the best of our knowledge, the quantum corrections obtained in this paper for the Boltzmann equation are not to be found elsewhere, whereas quantum modifications to the Vlasov–Poisson and Vlasov–Maxwell systems have been known [1,2], even if written differently in the latter case.

Furthermore, the deformation approach followed here provides a direct, straightforward procedure to derive the quantum corrections to any arbitrary order in $\hbar$. Notice that we have derived in Eqs. (21), (30), and (39) the full, formal series that correspond to the quantum, Moyal deformation of the Vlasov–Poisson, Vlasov–Maxwell, and Liouville equations (the latter constituting the starting point to derive the quantum-deformed Boltzmann equation, whose nature, even classically, is non-Hamiltonian). Whereas these series, which yield the correct quantum versions of the classical phase-space kinetic equations, do correspond to a complete Moyal bracket and thus obey the Jacobi identity [15,17], they may be of little practical use. That is why we have also given in Eqs. (22), (32), and (40) the lowest-order quantum corrections (in $\hbar^2$) to those same equations, knowing well that, by breaking the Lie structure of the bracket, these truncated versions most likely violate the original Hamiltonian structure. At this stage, it is still difficult to assess the possible effects this violation may have on the truncated formulas, yet our best conjecture is that they would be acceptable *vis-à-vis* the trade-off between the tractability of the deformed equations (so they can be effectively applied to problems of interest) and the preservation of their Hamiltonian structure. So much so that similarly truncated equations have been obtained, and proven to be useful, following a totally different approach, with no regard for the said structure [1,2].

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[20] Since $F[f]$ stands for a functional, dF/dt in Eq. (16) is a total time derivative, which is written as a partial derivative $\partial f/\partial t$ in Eq. (20) when applied to the functional $f[f] = \int f(\mathbf{x}', \mathbf{p}', t)\delta(\mathbf{x}' - \mathbf{x})\delta(\mathbf{p}' - \mathbf{p})d\mathbf{x}'d\mathbf{p}' = f(\mathbf{x}, \mathbf{p}, t)$, to account for the extra dependencies on $\mathbf{x}$ and $\mathbf{p}$.

[21] Incidentally, there is a possible typo in Eq. (4.7) of Ref. [1], since the sign of the $\hbar^2$ correction in it is incorrect, whereas Eq. (2.52) of Ref. [2] is written for a charge $-e$, hence an apparent discrepancy in signs with respect to Eq. (22) herein.

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[30] For elastic collisions amongst alike particles (i.e., with the same mass m), it is common knowledge that the total momentum $\mathbf{p} + \mathbf{p}_1$ is conserved (i.e., $\mathbf{p}' + \mathbf{p}_1' = \mathbf{p} + \mathbf{p}_1$) and that the relative momentum $\mathbf{p} - \mathbf{p}_1$ is simply rotated into $\mathbf{n}' = (\mathbf{p}' - \mathbf{p}_1')/|\mathbf{p}' - \mathbf{p}_1'|$ (i.e., $|\mathbf{p}' - \mathbf{p}_1'| = |\mathbf{p} - \mathbf{p}_1|$) [28], so Eq. (35) yields $2(\mathbf{p}' - \mathbf{p}) = |\mathbf{p} - \mathbf{p}_1|\mathbf{n}' - (\mathbf{p} - \mathbf{p}_1)$, the RHS of the latter representing the change in relative momentum, hence $|\mathbf{p} - \mathbf{p}_1|\mathbf{n}' - (\mathbf{p} - \mathbf{p}_1) = -2[(\mathbf{p} - \mathbf{p}_1) \cdot \mathbf{n}]\mathbf{n}$ and Eq. (36) ensues, with $\mathbf{n}$ defining the direction along which lies the component of the relative momentum that is reflected, all other components remaining unchanged.

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