PHYSICAL REVIEW E **102**, 013201 (2020)

Editors' Suggestion

Inconsistency of magnetic-moment conservation with entropy increase in collisionless shocks

Michael Gedalin
Department of Physics, Ben Gurion University of the Negev, Beer-Sheva, Israel

(Received 26 March 2020; accepted 11 May 2020; published 1 July 2020)

Collisionless shocks are multiscale objects. Energetic ion distributions are gyrotropic at sufficiently large distances upstream and downstream of the shock transition while at the transition itself the ion dynamics is significantly gyrophase dependent. Magnetic-moment conservation of an ion is widely used as a viable approximation during the shock crossing. It is shown that this approximation is inconsistent with the required entropy increase due to the loss of the gyrophase information.

DOI: 10.1103/PhysRevE.102.013201

I. INTRODUCTION

Collisionless shocks are efficient accelerators of charged particles. Acceleration to high energies occurs via the diffusive acceleration mechanism [1–8]. This is a two-stage mechanism which includes turbulent scattering of charged particle in the upstream and downstream regions as well as scatter-free shock crossing [5,7,9]. The energetic particle distributions in the diffusive regions are described using gyrophase averaging, that is, the distribution function is a function of the perpendicular $p_\perp$ and parallel $p_\parallel$ momenta or, alternatively, the momentum magnitude p and the pitch-angle cosine μ [7,8]. Here perpendicular and parallel refer to the direction of the mean magnetic field in the region under consideration, while the pitch angle is the angle between this mean field and the particle momentum. The above description is appropriate in the plasma frame or in the de Hoffman–Teller shock frame, in which the electric field vanishes upstream and downstream of the shock, otherwise the corresponding drifts should be added [9–13]. The scatter-free shock crossing plays the decisive role in establishing the particle spectra via matching conditions for the upstream and downstream gyro tropic distributions [4,5,7,14]. Ion dynamics during the shock crossing is essentially gyrophase-dependent [7,15–20]. An incident gyrotropic distribution becomes strongly nongyrotropic upon shock crossing. The latter may result in transmission or reflection, and in both cases gyrotropy is broken. In order to restore the description in terms of the gyrotropic distribution function one has to average over the gyrophases, thus losing this piece of information. Therefore, the coarse-grained entropy (defined precisely below) should increase even if the incident distribution is gyrotropic initially. Ion dynamics in the shock front is governed mainly by the macroscopic fields and to a lesser extent by fluctuations [20]. Even in this case the motion is nonadiabatic and the equations of motion cannot be integrated analytically. In the absence of a better solution, matching at the shock front is traditionally used applying the assumption of the magnetic-moment conservation and/or continuity of the distribution function [7,15,21–25]. Neither of these assumptions is correct except probably for strictly parallel or strictly perpendicular shocks. Recently, it was suggested that an appropriate description of the matching conditions may be done using a probabilistic approach to ion crossing of the shock front [19,20,26,27]. In this paper we show that magnetic-moment conservation is inconsistent with the coarse-grained entropy increase due to the loss of the gyrophase information.

II. CONSERVATION LAWS AND ENTROPY BEHAVIOR

The microscopic (fine-grained) entropy density s and flux $\boldsymbol{S}$ are defined using the distribution function as follows:

$$s = -\int f \ln f \, d^3\boldsymbol{p}, \tag{1}$$

$$\boldsymbol{S} = -\int \boldsymbol{v} f \ln f \, d^3\boldsymbol{p}, \tag{2}$$

where $f(\boldsymbol{p})$ is the exact distribution function. In the collisionless case the Vlasov equation

$$\frac{\partial f}{\partial t} + \boldsymbol{v}\cdot\nabla_{\boldsymbol{r}} f + q\left(\boldsymbol{E} + \frac{\boldsymbol{v}}{c}\times\boldsymbol{B}\right)\cdot\nabla_{\boldsymbol{p}} f = 0 \tag{3}$$

gives the fine-grained entropy conservation in the form

$$\frac{\partial s}{\partial t} + \nabla_{\boldsymbol{r}}\cdot\boldsymbol{S} = 0. \tag{4}$$

In an ideal shock the fields depend only on one coordinate along the shock normal and do not depend on time. In this case the fine-grained entropy conservation reads

$$S_n = \text{const} \tag{5}$$

or $S_{n,i} = S_{n,f}$ where i denotes the incident particles (initial distribution), f denotes the particles after interaction with the shock fields (final distribution), and n denotes the direction along the shock normal. These particles may be transmitted and reflected as well. Moreover, incident particles may come to the shock front from downstream constituting the leaked distribution.

In real shocks the fields in the transition layer can vary along the shock front and with time. Yet the mean fields in the upstream and downstream gyrotropic regions are time-independent. For the global shock normal along $\hat{n}$ the coarse-grained entropy increase means $S_{n,i} < S_{n,f}$ where the coarse-grained entropy and flux are defined as follows:

$$s = -\int F \ln F \, d\Omega, \tag{6}$$

$$S_n = -\int v\mu \cos\theta F \ln F \, d\Omega, \tag{7}$$

where $F = F(p, \mu)$ is the gyrophase-averaged gyrotropic distribution function and $d\Omega = 2\pi p^2 \, dp \, d\mu$ is the elementary volume in the gyrotropic momentum space.

At the scale of gyrotropy the shock is uniform on average, in both upstream and downstream regions. It is convenient to work in the de Hoffman–Teller frame, in which the average upstream and downstream electric fields vanish and the flow is along the average magnetic field. The averaging removes the small-scale fluctuations. Let the state of an ion be given by its momentum p and pitch-angle cosine μ. The velocity $V_n = v\mu \cos\theta$ is the velocity of the guiding center in the direction of the *global* shock normal, while θ is the angle between this normal and the average magnetic field [28,29]. Here $v = p/mc\gamma$, $\gamma = (1 + p^2/m^2c^2)^{1/2}$. If $F(p, \mu)$ is the distribution function of ions (guiding centers), that is, the gyrophase averaged distribution function of ions, then $F \, d\Omega$ is the number of guiding centers in $d\Omega$ and

$$dN_n = v\mu \cos\theta F \, d\Omega \tag{8}$$

is the particle number flux in the direction of the shock normal. This expression is equally valid both upstream and downstream. The number of guiding centers is conserved, which means that the total initial and final fluxes are equal, $N_{n,i} = N_{n,f}$. The corresponding differential entropy flux is

$$dS_n = -v\mu \cos\theta F \ln F \, d\Omega = -\ln F \, dN_n, \tag{9}$$

and the requirement of the coarse-grained entropy increase reads

$$S_{n,f} - S_{n,i} = -\int \ln\left(\frac{F_f}{F_i}\right) dN_n > 0. \tag{10}$$

Let the initial state of an ion be given as (p_i, μ_i). Upon scattering at the shock front the ion will be found in the state (p_f, μ_f) with the probability $w_{if}(p_i, \mu_i; p_f, \mu_f) \, d\Omega_f$, $\int_f w_{if} \, d\Omega_f = 1$. The flux conservation means

$$V_{n,f} F(p_f, \mu_f) = \int_i w_{if} V_{n,i} F(p_i, \mu_i) \, d\Omega_i, \tag{11}$$

$$F(p_f, \mu_f) = \int_i w_{if} \left(\frac{v_i \mu_i \cos\theta_i}{v_f \mu_f \cos\theta_f}\right) F(p_i, \mu_i) \, d\Omega_i. \tag{12}$$

This is a generalization of the approach described by Gedalin *et al.* [20,26]. The probability $w_{if}(p_i, \mu_i; p_f, \mu_f)$ can describe any shock transition including rippling, reformation, large amplitude waves, and small-scale turbulence as well. The possible initial and final states are defined relative to the uniform background and may be as follows: (a) initial upstream incident ions, $\mu_i > 0$, and final downstream transmitted ions, $\mu_f > 0$, or upstream backstreaming ions, $\mu_f < 0$, or (b) downstream ions moving toward the shock front, $\mu_i < 0$, and final leaked ions, $\mu_f < 0$. Thus, w_{if} completely determines $F(p_f, \mu_f)$ for any given $F(p_i, \mu_i)$. The scattering probability replaces the detailed information on ion dynamics inside the shock front. Since it is a statistical property, it can describe ion scattering at any shock, however complicated its structure is. Thus, $F(p_i, \mu_i) \mapsto F(p_f, \mu_f)$ is a coarse-grained Liouville mapping with gyrophase information removed.

In the de Hoffman–Teller frame the final momentum is a single-valued function of the initial momentum, $p_f = p_f(p_i)$. In the present paper we shall also assume that final pitch angle is a single-valued function of the initial pitch angle, $\mu_f = \mu_f(\mu_i)$. Then the conservation of the number of guiding centers may be written as equality of initial and final differential fluxes $dN_{n,i} = dN_{n,f}$:

$$v_i \mu_i \cos\theta_i F_i p_i^2 \, dp_i \, d\mu_i = v_f \mu_f \cos\theta_f F_f p_f^2 \, dp_f \, d\mu_f, \tag{13}$$

which immediately provides the distribution function in the final state as follows:

$$F_f = \left(\frac{v_i \mu_i \cos\theta_i p_i^2}{v_f \mu_f \cos\theta_f p_f^2}\right)\left(\frac{dp_i}{dp_f}\right)\left(\frac{d\mu_i}{d\mu_f}\right) F_i. \tag{14}$$

Here $\gamma_f = \gamma_i$ for reflected ions and $\gamma_f = \gamma_i - e\phi/mc^2$ for transmitted ions, where ϕ is the de Hoffman–Teller cross-shock potential. In addition, one has

$$v_f dp_f = v_i dp_i \to \frac{dp_i}{dp_f} = \frac{v_f}{v_i} \tag{15}$$

and therefore

$$F_f = \left(\frac{\cos\theta_i p_i^2}{\cos\theta_f p_f^2}\right)\left(\frac{d\mu_i^2}{d\mu_f^2}\right) F_i. \tag{16}$$

The magnetic moment (up to the constant multiplier) is

$$m = \frac{p_\perp^2}{B} = \frac{p^2(1-\mu^2)\cos\theta}{B_n}, \tag{17}$$

where B is the magnetic field magnitude, which changes upon the shock crossing, and B_n is the normal component of the magnetic field, $B_n = B\cos\theta$, which is constant throughout the shock. Therefore,

$$m_i = \frac{p_i^2 \cos\theta_i (1-\mu_i^2)}{B_n}, \tag{18}$$

$$m_f = \frac{p_f^2 \cos\theta_f (1-\mu_f^2)}{B_n}. \tag{19}$$

For a given p_i the final momentum p_f does not depend on μ_i, μ_f and

$$\frac{d\mu_i^2}{d\mu_f^2} = \frac{p_f^2 \cos\theta_f}{p_i^2 \cos\theta_i}, \tag{20}$$

$$\frac{dm_f}{dm_i} = \left(\frac{\cos\theta_i p_i^2}{\cos\theta_f p_f^2}\right)\left(\frac{d\mu_i^2}{d\mu_f^2}\right) \tag{21}$$

so that

$$F_f = F_i \left(\frac{dm_f}{dm_i}\right). \tag{22}$$

Thus, one has

$$dS_{n,f} - dS_{n,i} = -\left[\ln\left(\frac{dm_f}{dm_i}\right)\right] dN_n. \tag{23}$$

If an adiabatic invariant (magnetic moment) is conserved $m_f = m_i$, then $F_f = F_i$ and $dS_{n,f} = dS_{n,i}$ which means that the coarse-grained entropy is conserved. For gyrotropic distributions gyrophase information is lost, and therefore the coarse-grained entropy should increase upon shock crossing. Thus, adiabatic invariant cannot be conserved.

Since gyrophase information is lost for all particles, entropy increase should occur differentially for one-to-one mapping, which requires $dm_f/dm_i < 1$, that is, the magnetic moment should decrease. However, this is not possible for $m_i = 0$, that is, $\mu_i = 1$. Thus, one-to-one mapping $m_i \leftrightarrow m_f$ is inconsistent with the entropy increase due to the gyrophase information loss.

III. DISCUSSION AND CONCLUSIONS

The above analysis is valid for each separate combination of initial and final states. Namely, coarse-grained entropy should increase separately: (a) for incident upstream particles which are transmitted to proceed further downstream, (b) for upstream incident particles which are reflected at the shock and stream back upstream, and (c) for downstream incident particles which leak to the upstream. In all these cases the initial distribution is gyrotropic but the particle dynamics in the shock front is gyrophase-dependent. Therefore, the final gyrotropic distribution is obtained by averaging the exact gyrophase-dependent distribution over gyrophases. The loss of gyrophase information must result in the entropy increase. Thus, one-to-one mapping $\mu_i \mapsto \mu_f$ is not valid for any of these population conversions, although in certain cases it may appear a reasonable approximation. It has been shown [20] that the magnetic moment is approximately conserved for transmitted ions in nearly perpendicular shock and for backstreaming ions.

In a more general way, entropy increase would place restrictions on the probabilities. Let us denote, for brevity,

$$A_{if} = w_{if}\left(\frac{v_i\mu_i\cos\theta_i}{v_f\mu_f\cos\theta_f}\right), \quad (24)$$

then

$$F_f = \int A_{if}F_i\,d\Omega_i, \quad (25)$$

and the entropy increase means

$$\Delta S_n = -\int \ln\frac{\int A_{if}F_i d\Omega_i}{F_i}dN_n. \quad (26)$$

Finding implications for w_{if} requires a separate study. It has been shown that nonconservation of the magnetic moment affects the spectrum of accelerated ions even in the simplest case [26]. One can also expect that the pitch-angle distribution would be substantially affected.

ACKNOWLEDGMENT

The author is grateful to Mikhail Medvedev for raising the problem.

[1] G. F. Krymskii, A regular mechanism for the acceleration of charged particles on the front of a shock wave, Sov. Phys. Dok. **22**, 327 (1977).

[2] W. I. Axford, E. Leer, and G. Skadron, The acceleration of cosmic rays by shock waves, in *15th International Cosmic Ray Conference, Plovdiv, Bulgaria, 13–26 August, 1977*, Conference Papers, Vol. 11 (A79-44583 19-93) (Bulgarian Academy of Sciences, Sofia, 1978), pp. 132–137.

[3] A. R. Bell, The acceleration of cosmic rays in shock fronts. I, Mon. Not. R. Acad. Sci. **182**, 147 (1978).

[4] R. D. Blandford and J. P. Ostriker, Particle acceleration by astrophysical shocks, Astrophys. J. **221**, L29 (1978).

[5] I. N. Toptyghin, Acceleration of particles by shocks in a cosmic plasma, Space Sci. Rev. **26**, 157 (1980).

[6] J. R. Jokipii, Particle drift, diffusion, and acceleration at shocks, Astrophys. J. **255**, 716 (1982).

[7] L. O. Drury, An introduction to the theory of diffusive shock acceleration of energetic particles in tenuous plasmas, Rep. Progr. Phys. **46**, 973 (1983).

[8] R. Blandford and D. Eichler, Particle acceleration at astrophysical shocks: A theory of cosmic ray origin, Phys. Rep. **154**, 1 (1987).

[9] V. N. Vasilev, I. N. Toptygin, and A. G. Chirkov, Interaction between energetic particles and a shock front in a turbulent medium, Geomag. Aeron. **18**, 415 (1978).

[10] E. C. Roelof, Propagation of solar cosmic rays in the interplanetary magnetic field, in *Lectures in High Energy Astrophysics,* edited by H. Ögelmann and J. R. Wayland, NASA SP-199 (NASA Scientific and Technical Information Division, Washington, DC, 1969), p. 111.

[11] W. Droege, Transport of solar energetic particles, Astrophys. J. Suppl. Ser. **90**, 567 (1994).

[12] J. A. Le Roux, G. M. Webb, V. Florinski, and G. P. Zank, A focused transport approach to pickup ion shock acceleration: Implications for the termination shock, Astrophys. J. **662**, 350 (2007).

[13] W. Dröge, Y. Y. Kartavykh, B. Klecker, and G. A. Kovaltsov, Anisotropic three-dimensional focused transport of solar energetic particles in the inner heliosphere, Astrophys. J. **709**, 912 (2010).

[14] G. M. Webb, Boundary conditions for energetic particle transport at shocks, Astron. Astrophys. **124**, 163 (1983).

[15] T. Terasawa, Energy spectrum and pitch angle distribution of particles reflected by MHD shock waves of fast mode, Planet. Space Sci. **27**, 193 (1979).

[16] M. A. Lee, V. D. Shapiro, and R. Z. Sagdeev, Pickup ion energization by shock surfing, J. Geophys. Res. **101**, 4777 (1996).

[17] G. Zank, H. Pauls, I. Cairns, and G. Webb, Interstellar pickup ions and quasi-perpendicular shocks: Implications for the termination shock and interplanetary shocks, J. Geophys. Res. **101**, 457 (1996).

[18] D. Zilbersher and M. Gedalin, Pickup ion dynamics at the structured quasi-perpendicular shock, Plan. Space Sci. **45**, 693 (1997).

[19] M. Gedalin, W. Dröge, and Y. Y. Kartavykh, Scattering of high-energy particles at a collisionless shock front: Dependence on the shock angle, Astrophys. J. **807**, 126 (2015).

[20] M. Gedalin, W. Dröge, and Y. Y. Kartavykh, Dynamics of high energy ions at a structured collisionless shock front, Astrophys. J. **825**, 149 (2016).

[21] J. G. Kirk and P. Schneider, On the acceleration of charged particles at relativistic shock fronts, Astrophys. J. **315**, 425 (1987).

[22] J. G. Kirk, Pitch-angle anisotropy of low-energy ions at interplanetary shocks, Astrophys. J. **324**, 557 (1988).

[23] J. G. Kirk and A. F. Heavens, Particle acceleration at oblique shock fronts, Mon. Not. R. Acad. Sci. **239**, 995 (1989).

[24] U. D. J. Gieseler, J. G. Kirk, Y. A. Gallant, and A. Achterberg, Particle acceleration at oblique shocks and discontinuities of the density profile, Astron. Astrophys. **345**, 298 (1999).

[25] U. Keshet, Analytical Study of Diffusive Relativistic Shock Acceleration, Phys. Rev. Lett. **97**, 221104 (2006).

[26] M. Gedalin, W. Dröge, and Y. Y. Kartavykh, Dependence of the spectrum of Shock-Accelerated Ions on the Dynamics at the Shock Crossing., Phys. Rev. Lett. **117**, 275101 (2016).

[27] M. Gedalin, N. V. Pogorelov, and V. Roytershteyn, Rankine-Hugoniot relations including pickup ions, Astrophys. J. **889**, 116 (2020).

[28] L. Ofman and M. Gedalin, Rippled quasi-perpendicular collisionless shocks: Local and global normals, J. Geophys. Res. **118**, 5999 (2013).

[29] A. Johlander, S. J. Schwartz, A. Vaivads, Y. V. Khotyaintsev, I. Gingell, I. B. Peng, S. Markidis, P. A. Lindqvist, R. E. Ergun, G. T. Marklund *et al.*, Rippled Quasiperpendicular Shock Observed by the Magnetospheric Multiscale Spacecraft, Phys. Rev. Lett. **117**, 165101 (2016).