

Thermodiffusion-induced traveling and shock waves in a colloidal solutionPushpita Ghosh **Tata Institute of Fundamental Research Hyderabad, Telangana 500046, India*

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The formation of chemical waves in a nonlinear spatially extended system is one of the most fascinating far-from-equilibrium phenomena. An externally imposed thermal gradient in a liquid mixture may induce a concentration gradient generating a thermodynamic cross-flow, which is known as thermal diffusion or the Ludwig-Soret effect. The motion of the components of the mixture is governed by a nonlinear, partial differential equation for the density fraction in space and time. Here, we show that under an externally imposed constant thermal gradient, a traveling wave can emerge in a solution of self-propelled neutral colloid. An exact analytic solution of the spatially extended system is presented in one dimension for a constant thermal gradient to show the time development of a traveling wave. We analyze the effect of a small finite relaxation time of flux, which takes care of the finite inertia of the dispersing colloidal species. While the wave speed remains unaffected, the wave shape is significantly modified by the presence of the finite relaxation time of flux. Our result demonstrates that the traveling wave may reduce to a shock wave provided the product of the square of the wave speed and the relaxation time exactly balances the mass diffusion coefficient. This condition can be achieved by suitably adjusting the velocity of the emergent traveling wave by tuning the value of the constant thermal gradient and maintaining the appropriate boundary condition.

DOI: [10.1103/PhysRevE.102.012606](https://doi.org/10.1103/PhysRevE.102.012606)**I. INTRODUCTION**

The spontaneous occurrence of spatiotemporal order is a ubiquitous phenomenon in spatially extended systems in nature [1–3]. Nonlinearity present in the source kinetics or in the transport processes is the key to such occurrences. Thermal diffusion, thermodiffusion, or the Ludwig-Soret effect refers to the phenomenon of mass transport under the influence of a thermal gradient [4,5]. Thermodiffusion has been effectively used to separate the constituent components in a mixture by exposing them to an appropriate thermal gradient [6,7]. The phenomenon has played a significant role in understanding a number of widely varying natural processes, e.g., thermodiffusive convection in oceans [8], the onset of convective oscillations [9,10] in a binary mixture, colloidal suspensions [11–16], polymer solutions [17–20], micellar solutions [21], and component segregation in metallic alloys undergoing solidification [22,23]. The influence of a thermal gradient on reaction-diffusion systems [24,25] has also been explored to show that the imposed gradient may destabilize a chemical front giving rise to absolute and convective instability, initiating the propagation of waves and the formation of stationary Turing-like instabilities. For a sinusoidal temperature field [26,27] in a binary mixture and a large Ludwig-Soret effect, a moving front in the form of a shock wave has been experimentally realized and theoretically explained by ignoring the dissipative effects due to mass diffusion.

Mathematically, the Ludwig-Soret effect is governed by a nonlinear partial differential equation in space and time. Diffusion is the key physical process for mass transport in

the Ludwig-Soret effect. An elementary consideration based on the fundamental Gaussian solution of a diffusion equation with a point source of particles at some initial time asserts that there is always a finite nonzero density of particles at a very large distance from the source even when the time is very short. This implies that the particles have infinite speed. To avoid this difficulty, one may introduce, following Cattaneo [28–31], a nonzero relaxation time (τ) of the diffusive flux that takes care of the finite inertia of the dispersing species. At the microscopic level, this corresponds to the model of a correlated random walk introduced by Furth and others [32,33]. Although a number of mathematical and computational studies of thermodiffusion [34–36] have been carried out recently, the effect of the finite inertia of the particles has been ignored. The focal theme of the present paper is to find an exact analytic solution of the Ludwig-Soret equation for a constant temperature gradient in a binary mixture of self-propelling colloidal particles with finite inertia.

We first show that the thermal diffusion equation for a constant temperature gradient admits a moving front or traveling wave solution in the absence of any finite relaxation time of flux of the self-propelled colloidal system. The direction of the traveling wave depends on the sign of the Soret coefficient or the thermal diffusion constant of the colloidal system. It turns out from the theoretical analysis that the velocity of the moving front depends on the applied thermal gradient, the boundary conditions, and the self-propulsion velocity of the colloidal particles. However, we see that the aforementioned traveling wave can emerge even in the absence of any self-propulsion velocity of the colloidal system, which asserts that this wave instability originates purely as a result of thermodiffusion. Natural or synthetic self-propelling colloids show additional features and versatile spatiotemporal

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behavior under the influence of an applied thermal gradient. In the case of self-propelled colloidal particles showing negative thermodiffusion, i.e., movement toward the hot end of the one-dimensional channel under the imposed constant thermal gradient, we observe that a directional change of the moving front can occur with the increase of the applied thermal gradient when the self-propulsion velocity is positive. Moreover, a situation of a standing wave with zero velocity can arise for a particular value of the applied thermal gradient. Apparently, we find that the moving front takes the form of a shock wave in the limit of vanishing mass diffusion (or a large thermal diffusion coefficient). The extent to which thermal diffusion shocks are visible in the laboratory is clearly a function of how large the thermal diffusion factor can be made. In the present paper, we have considered that the applied temperature gradient does not cause any reactive changes of the colloidal systems, and also that the Soret coefficient does not appreciably depend on the temperature or concentration.

To this end, it is important to mention that our study is focused on the fact that we consider the roles of mass diffusion and thermodiffusion on an equal footing without assuming that thermal diffusion will be dominating over mass diffusion, as most of the earlier studies have considered. As follows, we also take into account the finite inertia of the diffusing particles by introducing a small relaxation time of the diffusive flux with the help of Cattaneo's modification of Fick's law [29,30]. Our results demonstrate that, while the velocity of the traveling wave does not depend on the relaxation time, the smoothening of the moving front depends significantly on the inertia of the diffusive particles. Moreover, a condition may arise in which the traveling wave reduces to a shock wave when the mass diffusion coefficient is exactly balanced by the product of the relaxation time of flux and the square of the wave velocity. This condition can be achieved by varying or adjusting the wave velocity, which is a function of the thermal gradient, the boundary condition, and the self-propulsion of the colloidal species. Overall, our study theoretically explores the emergence of spatiotemporal instability in a self-propelled colloidal system under the influence of an externally applied constant thermal gradient in a one-dimensional channel.

The rest of the paper is organized as follows: In Sec. II, we provide an exact solution of the transport equation for a constant thermal gradient, which demonstrates the development of a traveling wave of concentration of the self-propelled colloidal species. We analyze the direction and the magnitude of the wave velocity as a function of imposed temperature gradient for different cases. In Sec. III, we analyze and demonstrate the effect of a small but finite relaxation time of the flux of the dispersing colloidal species, which is attributed to an interesting dynamical scenario. The paper is concluded in Sec. IV.

II. SOLUTION OF THE TRANSPORT EQUATION OF A SELF-PROPELLED COLLOIDAL SYSTEM FOR A CONSTANT THERMAL GRADIENT

Under a constant thermal gradient, the transport equation, which accounts for mass diffusion, thermal diffusion, and drift motion altogether, can be written down in different forms [4]. We proceed with the traditional equation relating the mass flux

to the thermal and concentration gradients present in a self-propelled colloidal system neglecting hydrodynamic motion. For an incompressible colloidal solution, the total flux j is written by

$$j = j_{\text{diff}} + j_{\text{Soret}} + j_{\text{drift}} \\ = -[\rho D \nabla w + \rho w(1-w)D_T \nabla T + v_0 \rho w]. \quad (1)$$

Here j_{diff} refers to the diffusive mass flux, j_{Soret} is the mass flux contribution due to the Soret effect (or thermodiffusion) proportional to the constant temperature gradient ∇T , and j_{drift} represents the additional flux of the colloidal particles having a constant positive self-propelling velocity v_0 . ρ is the material density of the solution, w is the mass fraction of the solute, D represents the mass diffusion coefficient, and the phenomenological quantity D_T is called the coefficient of thermal diffusion or the thermal diffusion constant. D_T is a complex function of molecular parameters of individual species, their mass and size, and the associated interaction forces. Even for a particular mixture, it is a complex function of temperature and density, and there is no universal law to describe these features for liquids. The ratio of thermal to ordinary mass diffusion is called the Soret coefficient $S_T = D_T/D$. The Soret coefficient may be positive or negative depending on the sign of D_T or on the sense of migration of the reference component (to the cold or to the hot end). In absolute value, an order of magnitude of Soret coefficients for usual organic mixtures or aqueous solutions is $|S_T| \sim 10^{-3}-10^{-2} \text{ K}^{-1}$ but it can be several orders of magnitude larger for macromolecular and colloidal solutions (0.25–0.17 K^{-1}) [13,17]. We consider a dilute suspension of the colloidal particles having a small diameter to avoid intercolloidal and solvent interactions, so that S_T can be taken as a constant and independent of concentration in our present study. At the same time, we restrict the use of a large thermal gradient or temperature difference so that the Soret coefficient S_T could be assumed to remain constant for the range of temperature gradients studied for the present problem.

We use a similar mathematical approach to that used in [37] to solve the heat transport phenomenon equation for our present study to obtain a spatiotemporal wave instability induced by thermodiffusion as follows. Considering the x -component of the flux, Eq. (1) can be written as

$$j(x, t) = -D \frac{\partial c(x, t)}{\partial x} - c(x, t)[1 - \alpha c(x, t)]D_T \nabla T \\ + v_0 c(x, t), \quad (2)$$

where $\rho w = c(x, t)$ is the variable corresponding to the concentration of colloid, $\alpha = 1/\rho$. By substituting Eq. (2) into the equation of continuity,

$$\frac{\partial c(x, t)}{\partial t} + \frac{\partial j(x, t)}{\partial x} = 0, \quad (3)$$

we obtain

$$\frac{\partial c}{\partial t} = -\frac{\partial}{\partial x} \left[-D \left(\frac{\partial c}{\partial x} \right) - c(1 - \alpha c)D_T \nabla T + v_0 c \right] \\ = D \frac{\partial^2 c}{\partial x^2} + (\beta - v_0) \frac{\partial c}{\partial x} - 2\alpha \beta c \frac{\partial c}{\partial x}, \quad (4)$$

where ∇T is taken to be constant throughout and $D_T \nabla T$ is abbreviated as β .

Now using the traveling wave transformation with the variable $z = x - vt$, where v is the speed to be determined and writing $c(x, t) \equiv U(z)$, we obtain Eq. (5) in the following form:

$$-v \frac{\partial U}{\partial z} = D \frac{\partial^2 U}{\partial z^2} + (\beta - v_0) \frac{\partial U}{\partial z} - 2\alpha\beta U \frac{\partial U}{\partial z}. \quad (5)$$

Equation (5) can be rearranged as follows:

$$-\left(\frac{v + \beta - v_0}{2\alpha\beta}\right) \frac{\partial U}{\partial z} + U \frac{\partial U}{\partial z} - \left(\frac{D}{2\alpha\beta}\right) \frac{\partial^2 U}{\partial z^2} = 0. \quad (6)$$

Expressing $P = \frac{v + \beta - v_0}{2\alpha\beta}$ and $K = \frac{D}{2\alpha\beta}$, Eq. (6) can be rewritten as

$$-P \frac{\partial U}{\partial z} + U \frac{\partial U}{\partial z} - K \frac{\partial^2 U}{\partial z^2} = 0. \quad (7)$$

Physically at a very large and very small z , i.e., way behind or way before the wavefront, the concentration U should be constant so that the boundary conditions are maintained. We now impose the boundary conditions on $U(z)$ that it asymptotically tends to a constant value u_1 as $z \rightarrow -\infty$ and u_2 as $z \rightarrow \infty$ and $u_1 > u_2$. Nevertheless, this choice of boundary values does not necessarily mean that an infinitely elongated medium is needed to observe a moving chemical front. One can use a Soret cell of a finite length whose both the ends meet the boundary values as demanded by a wavefront.

A direct integration of Eq. (7) yields

$$\frac{\partial U}{\partial z} = \frac{1}{2K} (U^2 - 2PU - 2Q), \quad (8)$$

where Q is the integration constant.

If u_1 and u_2 are the roots of the quadratic equation $(U^2 - 2PU - 2Q) = 0$, then the constants P and Q can be obtained as

$$P = \frac{1}{2}(u_1 + u_2) \quad \text{and} \quad Q = -\frac{1}{2}u_1 u_2. \quad (9)$$

Therefore, the wave speed can be calculated by the following expression:

$$P = \frac{v + \beta - v_0}{2\alpha\beta} = \frac{1}{2}(u_1 + u_2)$$

so that we have

$$v = \beta[\alpha(u_1 + u_2) - 1] + v_0. \quad (10)$$

Equation (8) can then be expressed in the form

$$\frac{\partial U}{\partial z} = \frac{1}{2K} (U - u_1)(U - u_2), \quad (11)$$

which on direct integration gives

$$U(z) = \frac{u_1 + u_2 \exp\left[\frac{z}{2K}(u_1 - u_2)\right]}{1 + \exp\left[\frac{z}{2K}(u_1 - u_2)\right]}, \quad (12)$$

$$\text{or } U(z) = u_2 + \frac{u_1 - u_2}{1 + \exp\left[\frac{z}{2K}(u_1 - u_2)\right]}. \quad (13)$$

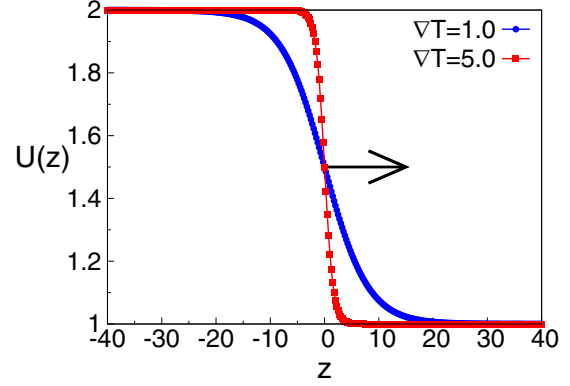


FIG. 1. Plot of the traveling wave $U(z)$ as a function of collective coordinate z for different values of imposed constant temperature gradients for the parameters $v_0 = 1.0$, $\alpha = 1.0$, $D = 1$, $D_T = 1.0$, $u_1 = 2.0$, $u_2 = 1.0$. The arrow represents the direction of the traveling wave.

Equation (13) can be written in another useful form as given by the expression

$$U(z) = \frac{1}{2}(u_1 + u_2) - \frac{1}{2}(u_1 - u_2) \tanh\left[\frac{z(u_1 - u_2)}{4K}\right]. \quad (14)$$

As $u_1 > u_2$, the wave profile $U(z)$ decreases monotonically with z from the constant value u_1 as $z \rightarrow -\infty$ to the constant value u_2 as $z \rightarrow \infty$ (provided K is a positive constant if the Soret coefficient is positive) as shown schematically in Fig. 1. This suggests the development of a traveling wave moving in a positive direction with a speed as given by $v = D_T \nabla T [\alpha(u_1 + u_2) - 1] + v_0$. The expression of the velocity reveals that it depends linearly on the coefficient of thermal diffusion and the applied constant temperature gradient. It also depends on the values of the boundary condition. Even in the case of the colloidal particles having no self-propulsion, i.e., $v_0 = 0$, we find the development of a traveling wave moving in a positive direction with a velocity $v = \beta[\alpha(u_1 + u_2) - 1]$. This velocity of the spatial inhomogeneity in the form of a moving front essentially appears due to the presence of thermodiffusion. In absence of the externally applied constant thermal gradient ($\nabla T = 0$), the wave profile does not exist and the velocity of the colloidal particles means the self-propulsion, i.e., $v = v_0$.

To demonstrate the quantitative relation between the wave velocity and the thermal gradient, we plot the velocity of the traveling wave v as a function of the externally imposed temperature gradient ∇T as depicted in Fig. 2. For the type of self-propelling colloids that move toward the cooler end as a response toward the applied thermal gradient, i.e., those having positive Soret coefficients, we obtain a right-moving traveling wave implying positive velocity. We observe that wave speed increases with an increase of the externally applied constant thermal gradient.

On the other hand, self-propelling colloidal species that show movement toward the hot end as a response to the applied thermal gradient, i.e., having a negative thermal diffusion or Soret coefficient, exhibit a conspicuous dynamical scenario. If K is negative, i.e., colloidal particles move toward the hot end in the presence of an externally applied

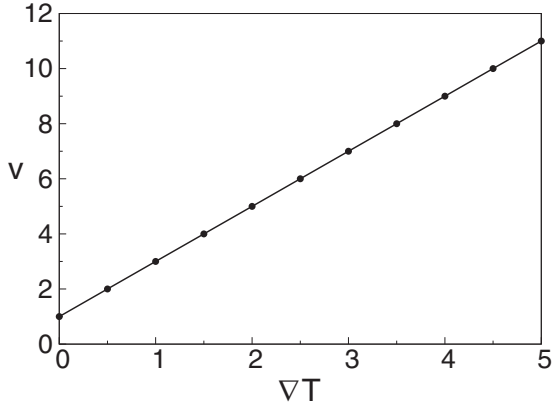


FIG. 2. Plot of speed of the traveling wave as a function of temperature gradient for the parameters $v_0 = 1.0$, $\alpha = 1.0$, $D = 1$, $D_T = 1.0$, $u_1 = 2.0$, $u_2 = 1.0$.

constant thermal gradient ∇T , the wave profile $U(z)$ increases monotonically with z from the constant value u_1 as $z \rightarrow -\infty$ to the constant value u_2 as $z \rightarrow \infty$ provided $u_1 < u_2$. A transition from a right-moving wave to a left-moving wave might occur with an increase of ∇T when D_T is negative. The underlying reason for this reversal is rooted in the presence of the self-propulsive motion of the colloidal particles. Initially, for the low values of imposed temperature gradients, the wave speed is positive and it decreases with an increase of ∇T . For $\nabla T = v_0/D_T[\alpha(u_1 + u_2) - 1]$, the wave profile stops moving in any direction and stalls, giving rise to a stationary instability. Again, with an increase of applied thermal gradient, the wave starts to move in the opposite direction as the velocity becomes negative. In Fig. 3, we depict the directional change of the traveling wave as a function of the constant thermal gradient in the case of negative thermodiffusion. Corresponding wave profiles for three different values of $\nabla T = 0.25, 0.5$, and 1.0 are shown in Fig. 4.

At $z = 0$, $U = (1/2)(u_1 + u_2)$. The shape of the wave form Eq. (14) is significantly affected by the constant $K = \frac{D}{2\alpha\beta}$. The expression of the wave form can be obtained by incorporating

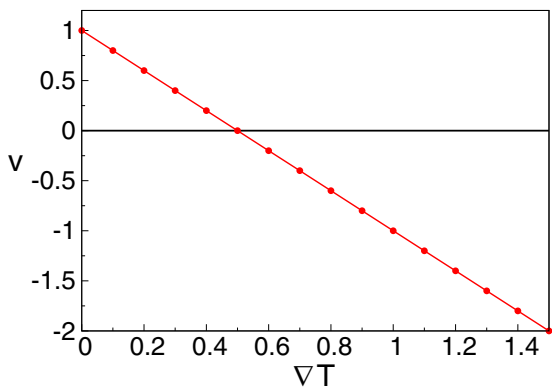


FIG. 3. Plot of the speed of the traveling wave as a function of temperature gradient for the parameters $v_0 = 1.0$, $\alpha = 1.0$, $D = 1$, $D_T = -1.0$, $u_1 = 1.0$, $u_2 = 2.0$.

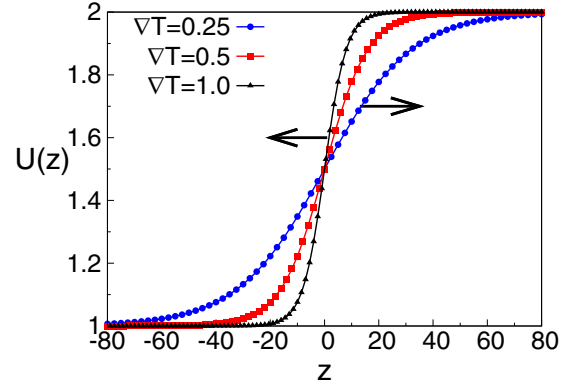


FIG. 4. A plot of the traveling waves $U(z)$ as a function of collective coordinate z for different values of imposed temperature gradients for the parameters $v_0 = 1.0$, $\alpha = 1.0$, $D = 1$, $D_T = -1.0$, $u_1 = 1.0$, $u_2 = 2.0$. For $\nabla T = 0.5$, the wave speed is zero. Arrows represent the direction of the movement of the traveling waves.

the explicit form of K to yield

$$U(z) = \frac{1}{2}(u_1 + u_2) - \frac{1}{2}(u_1 - u_2) \tanh \left[\frac{z(u_1 - u_2)D_T \nabla T}{2\rho D} \right]. \quad (15)$$

This solution connects the two asymptotic limits, and the presence of diffusion prevents the gradual distortion of the wave profile. It is also apparent from the expression (15) that in the absence of diffusion ($D = 0$) and with $D_T > 0$, the wave profile suffers from gradual distortion and steepening, and eventually it breaks up with the development of a shock. Shock waves are formed as a result of a balance between the steepening effect of the thermal diffusion (nonlinear) term and the smoothing effect of the linear diffusion term in the equation of motion. It is also noteworthy from Eq. (15) that in the absence of a thermal gradient $\nabla T = 0$, the solution is a constant and represents a flat wave profile. The outcome of the above analysis is that the nonlinear advection and diffusion terms exhibit opposite effects. The former introduces sharp discontinuity in the wave solution profile while the latter tends to spread out (smoothing) the discontinuity into a smooth profile. In view of this property, D represents the role of kinematic viscosity [38] in fluid flow, which measures the viscous dissipation.

Multiplying both the numerator and denominator of Eq. (12) by $\exp[\frac{z}{2K}(u_1 - u_2)]$, we can write the solution in the form

$$U(z) = \frac{u_2 + u_1 \exp \left[-\frac{z}{2K}(u_1 - u_2) \right]}{1 + \exp \left[-\frac{z}{2K}(u_1 - u_2) \right]}. \quad (16)$$

Expressing

$$\delta = \frac{K}{u_1 - u_2} = \frac{D}{2\alpha\beta}, \quad (17)$$

we observe that the exponential factor in the solution (16) indicates the existence of a thin transition layer of thickness δ . This thickness can be referred to as the shock thickness, which tends to zero as $D \rightarrow 0$ for a fixed u_1 and u_2 . Also δ increases as $u_1 \rightarrow u_2$ for a fixed D . If δ is small compared with other

typical length scales of the problem, the rapid shock transition can satisfactorily be approximated by a discontinuity. The solution (15) is a key result of this paper. It reveals that a traveling wave type inhomogeneous profile of concentration of tanh form propagates without distortion of its shape with a constant speed. A closer look at the result shows that while pure mass diffusion cannot sustain a traveling wave, the presence of a thermal gradient can give rise to a traveling wave instability due to the Soret effect. The generic origin of this type of instability lies in the nonlinearity of the Soret flux (thermodiffusion).

III. INCLUSION OF FINITE RELAXATION TIME OF FLUX

Diffusion is a key physical process for mass transport in the Ludwig-Soret effect. An elementary consideration based on the fundamental Gaussian solution of the diffusion equation with a point source of particles at some initial time asserts that there is always a finite nonzero density of particles at a very large distance from the source even when the time is very short. This implies that the particles have infinite speed. To avoid this difficulty, one may introduce, following Cattaneo [28], a nonzero relaxation time (τ) of the diffusive flux that takes care of the finite inertia of the dispersing species. At the microscopic level, this corresponds to the model of a correlated random walk introduced by Furth and others [32]. Our goal in this section is to include the effect of finite inertia of particles in the Ludwig-Soret diffusion equation. Following Cattaneo's modification of Fick's law, the total flux including thermodiffusion can be written in the form

$$j(x, t + \tau) = - \left[D \frac{\partial c(x, t)}{\partial x} + c(x, t) \{1 - \alpha c(x, t)\} D_T \nabla T \right] + v_0 c(x, t), \quad (18)$$

which takes care of the adjustment of a concentration gradient at time t with a flux $j(x, t + \tau)$ at a later time ($t + \tau$), and τ being the delay time of the particles in adopting one definite direction. Expanding $j(x, t + \tau)$ in Eq. (18) up to first order τ , we obtain

$$\begin{aligned} \tau \frac{\partial j(x, t)}{\partial t} + j(x, t) \\ = -D \frac{\partial c(x, t)}{\partial x} - c(x, t) \{1 - \alpha c(x, t)\} D_T \nabla T \\ + v_0 c(x, t). \end{aligned} \quad (19)$$

Differentiating Eq. (19) with respect to x and Eq. (3) with respect to t and eliminating j from the resulting equations, one arrives at the following equation:

$$\begin{aligned} \tau \frac{\partial^2 c(x, t)}{\partial t^2} + \frac{\partial c(x, t)}{\partial t} = D \frac{\partial^2 c(x, t)}{\partial x^2} + (\beta - v_0) \frac{\partial c(x, t)}{\partial x} \\ - 2\alpha\beta c(x, t) \frac{\partial c(x, t)}{\partial x}. \end{aligned} \quad (20)$$

Equation (20) is a hyperbolic variant of the nonlinear diffusion equation. It reduces to the spatially extended system described by the evolution equation (4) for $\tau = 0$. We now seek as before a traveling wave solution of Eq. (20) of the form $U(z) = c(x - v't)$, $z = x - v't$, where v' is the wave speed

to be determined. This results in the following equation:

$$-\frac{(v' + \beta - v_0)}{2\alpha\beta} \frac{\partial U}{\partial z} + U \frac{\partial U}{\partial z} - \frac{(D - \tau v'^2)}{2\alpha\beta} \frac{\partial^2 U}{\partial z^2} = 0, \quad (21)$$

where $P' = \frac{v' + \beta - v_0}{2\alpha\beta}$ and $K' = \frac{D - \tau v'^2}{2\alpha\beta}$.

To find a physically meaningful solution of Eq. (21), $U(z)$ must remain bounded for large values of z . For this, $U(z)$ represents the wave form with the property that it tends asymptotically to constant values u_1 as $z \rightarrow -\infty$ and u_2 as $z \rightarrow \infty$. We assume that $u_1 > u_2$. A direct integration of Eq. (21) gives

$$\frac{\partial U}{\partial z} = \frac{1}{2K'} (U^2 - 2P'U - 2Q'), \quad (22)$$

where Q' is the constant of integration. If u_1 and u_2 are two roots of the quadratic equation $(U^2 - 2P'U - 2Q') = 0$, then the constants P' and Q' can be obtained as

$$P' = \frac{v' + \beta - v_0}{2\alpha\beta} = \frac{1}{2}(u_1 + u_2), \quad (23)$$

which yields

$$v' = \beta[\alpha(u_1 + u_2) - 1] + v_0 \quad (24)$$

and

$$Q' = -\frac{1}{2}u_1u_2. \quad (25)$$

Proceeding exactly as before, we obtain

$$U(z) = \frac{1}{2}(u_1 + u_2) - \frac{1}{2}(u_1 - u_2) \tanh \left[\frac{z}{4\delta'} \right], \quad (26)$$

where the shock thickness is given by

$$\delta' = \frac{K'}{u_1 - u_2} = \frac{\rho(D - \tau v'^2)}{2D_T \nabla T (u_1 - u_2)}. \quad (27)$$

The above analysis shows that the wave speed v' [Eq. (24)] remains unaffected by the presence of a finite relaxation time τ of the flux. However, it is apparent that the shape of the wave form is not only affected by the mass diffusion coefficient D but also by an additional contribution $\tau v'^2$ due to the finite relaxation time τ of the flux such that $(D - \tau v'^2)$ behaves as effective kinematic viscosity. To show how a small but finite relaxation time of the diffusive flux modifies the shape of the traveling wave, we plot the wave profile $[U(z)]$ given in Eq. (26) as a function of the collective coordinate z as depicted in Fig. 5 for different values of τ . Subsequently, we observe that the balance between the steepening effect of the nonlinear convective term due to thermodiffusion as well as the smoothening effect of mass diffusion is modified by the presence of the relaxation time or the wave-speed-dependent term $\tau v'^2$. Moreover, a shock wave can appear when the mass diffusion coefficient D is exactly balanced by the product of the finite relaxation time of the flux and the wave velocity. This can be achieved by suitably adjusting the values of the applied constant temperature gradient and the values of the imposed boundary condition (u_1, u_2). To get a better insight into the development of shock waves in colloidal suspensions by adjusting the relevant parameters to obtain a shock thickness $\delta' \rightarrow 0$, we plot a two-parameter

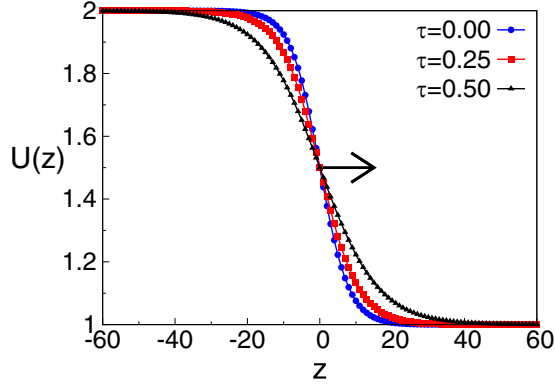


FIG. 5. A plot of the traveling waves $U(z)$ as a function of collective coordinate z for different values of the finite relaxation time of flux τ of the dispersing colloidal particles. The other parameters are $v_0 = 1.0$, $\alpha = 1.0$, $D = 1$, $D_T = 1.0$, $u_1 = 2.0$, $u_2 = 1.0$, and $\nabla T = 1.0$. The arrow refers to the direction of propagation.

phase diagram as shown in Fig. 6. We vary the externally applied temperature gradient ∇T as a function of the sum of the two boundary conditions such as $u_1 + u_2$ (provided $u_1 > u_2$ for a right-moving traveling wave), which gives us more flexibility to choose the boundary values. The bifurcation curve can provide the probable choice of parameters to obtain a transition from a traveling wave to a shock wave in the present setup. Thus, the occurrence of a shock wave can be realized in a colloidal mixture without neglecting the mass diffusion.

Finally, for a better perspective on our theoretical study of thermodiffusion-induced spatiotemporal instability in a colloidal suspension in the form of a traveling wave in our system within workable ranges of experimental constants, we propose a probable experimental setup as follows. We can carry out the experiment in a one-dimensional Soret cell of length $L = 50 \mu\text{m}$, mimicking a constantly fed unstirred reactor (CFUR) as used mostly in the study of spatial patterns and waves in reaction-diffusion systems. The CFUR consists of a gel layer whose two ends are in contact with CSTR maintained at two different temperatures. This enables us to

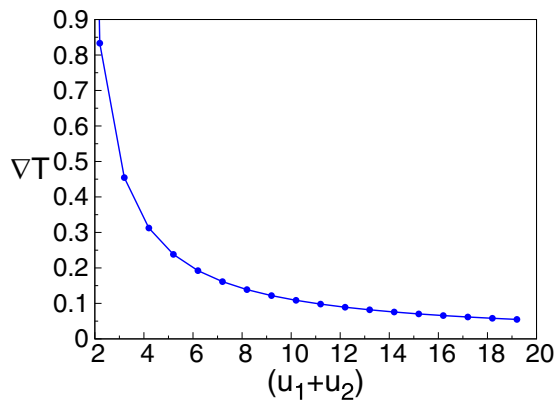


FIG. 6. Plot of externally applied constant thermal gradient vs boundary condition in the limit when shock thickness is zero. The other parameters are $v_0 = 1.0$, $\alpha = 1.0$, $D = 1$, $D_T = 1.0$, $\tau = 0.25$.

have a horizontal gradient of temperature across the Soret cell consisting of the colloidal suspension. We can maintain a temperature difference of 25 K between the two ends of the Soret cell, thus generating a gradient of $0.5 \text{ K } \mu\text{m}^{-1}$. If the thermal diffusion coefficient D_T is $\sim 10 \mu\text{m}^2 \text{K}^{-1} \text{s}^{-1}$, then the velocity obtained might be $\sim 5 \mu\text{m s}^{-1}$ in the case of a simple colloid with $v_0 = 0 \mu\text{m s}^{-1}$ [15,16]. The value of the transition layer thickness δ can be obtained as $\sim 3 \mu\text{m}$. The wavefront propagation can be measured by a camera and other necessary devices.

IV. CONCLUDING REMARKS

In conclusion, we have shown that the thermal diffusion equation for a constant thermal gradient in a self-propelled colloidal system can be solved exactly, which appears as a traveling wave or moving front. The velocity of the traveling wave depends on the applied temperature gradient, the thermal diffusion coefficient, the imposed boundary condition, and the self-propulsion velocity of the colloidal particles. By adjusting any of the above-mentioned parameters, one can control the velocity as well as the shape of the traveling wave. We have discussed the cases of both positive and negative thermal diffusion coefficients of self-propelling colloids. We have shown that, in the case of colloids having a positive self-propulsion velocity, a directional reversal of a moving front can occur with an increase of the applied constant temperature gradient if the colloidal particles move toward the hot end as a response to the applied temperature gradient. Moreover, a condition might occur when the concentration profile shows a wavelike inhomogeneity in space but is stationary with respect to time for a certain value of the applied constant temperature gradient. We furthermore extended our study by taking care of the finite inertia of the dispersing species, which at a microscopic level corresponds to the model of a correlated random walk. We have shown that while the wave speed remains unaffected by the presence of a finite inertia of the dispersing colloidal particles, the shock thickness gets significantly modified by it. Our result shows that the traveling wave solution reduces to a shock wave when the mass diffusion coefficient is exactly balanced by the product of the finite relaxation time of the flux and the square of the wave velocity. This provides a fundamental insight into developing a shock wave by treating mass diffusion and thermal diffusion on an equal footing by tuning the velocity of the traveling wave. Since the present work does not consider the effect of temperature and concentration dependence of the Soret coefficient of colloids, a reasonable extension of the current work would be a detailed investigation of these features to uncover complex spatiotemporal phenomena in one or higher dimensions.

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