Probabilistic characteristics of nonlinear waves in nondispersive media of the hydrodynamic type

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The paper considers the probability distributions of the nonlinear wave characteristics in nondispersive media that satisfy the Riemann- and the Kardar-Parisi-Zhang-type equations. By using the Lagrangian and Euler relations of statistical descriptions, expressions are obtained for the probability density of the Riemann wave (displacement) integral through the initial probability density of displacement, velocity, and acceleration. The case of Gaussian initial statistics is considered when multivalued sections in nonlinear waves arise at arbitrarily small distances from the entrance. The expressions obtained in this case should be interpreted as the relative residence time of the process in a certain displacement range. It is shown that, due to the Riemann equation locality, the appearance of ambiguity in the wave profile, which occurs mainly at negative values, does not affect the probability density form at positive bias values.

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I. INTRODUCTION

The most famous solution for nonlinear waves in the nondispersive media is the so-called simple, or the Riemann, wave of the form

$$u(x,t) = U[x - V(u)t], \tag{1.1}$$

where $u(x,t)$ is the wave variable, $U(x)$ is the initial perturbation, $V(u)$ is the nonlinear medium characteristic, x is the coordinate, and t is the time. Formula (1.1) is an implicit solution of the equation

$$\frac{\partial u}{\partial t} + V(u)\frac{\partial u}{\partial x} = 0, \tag{1.2}$$

Such waves are intensively studied in nonlinear acoustics [1,2], radiophysics [3], and oceanology [4]. Mathematically, the Riemann waves are particular solutions of partial differential hyperbolic systems, and their theory is fairly well developed [5].

From the point of view of physics, the Riemann waves, like unidirectional waves, arise at certain ratios between the various components of the wave fields (displacement, velocity, pressure, etc.). In addition to this, along with the proper Riemann waves of type (1.1), the Riemann waves integrals arise in applications, such as when using potentials in hydrodynamics

$$u(x,t) = \frac{\partial \Phi(x,t)}{\partial x}. \tag{1.3}$$

Depending on the meaning of the variable x, the value of Φ can describe the moving shoreline dynamics when the sea waves run up on the shore [6,7], the expanding surfaces and fire fronts [8], and some other characteristics. In cosmology, when describing the large-scale structure of the universe, the Zel'dovich approximation is widely known, which describes the initial nonlinear stage of gravitational instability. In this case, the particle motion is reduced in the corresponding variables to the Riemann equation, and the evolution of the velocity and potential fields is equivalent to the evolution of the optical wave behind the phase screen [9,10]). Partially, these equations are given in Sec. II.

If the initial distributions of the wave fields are random, there arise problems of the statistical description of the Riemann waves and the integrals of them. The dynamics of the proper random Riemann waves (1.1) is well known [11–14], but the probability distributions of the Riemann wave integrals have not been studied yet. Perhaps we should mention here our work [15], in which this problem was solved in the approximation of a narrow-band initial perturbation, as well as some previously published articles [16,17], where the wave field moments were studied.

One of the serious difficulties encountered in the random Riemann wave study is their breaking (gradient catastrophe). If in dynamic problems the Riemann wave exists on a finite time interval, in a random field, for example, with Gaussian statistics, the ambiguity in the wave profile arises at arbitrarily small times. In Refs. [18,19] the ambiguity effect on the Riemann wave spectra was studied. In particular, it was shown that the use of the double Fourier transform to solve Eq. (1.2) leads to some effective attenuation of the Riemann wave energy, despite the formal viscosity absence in this equation. The situation, where only unambiguous branches of the solution are taken into account, is realized, in particular, for acoustic turbulence, commonly called Burgers turbulence [11,13,14]. In this case, the generalized solutions of the Riemann equation are considered when the ambiguity is eliminated by introducing discontinuities. In this case, a fairly complete statistical description can be carried out at large times (long paths) when limit theorems of the emission theory of random processes can be used. In Burgers turbulence, either statistical characteristics of the velocity itself or its gradient is considered [20].

In this paper, we will consider multivalued solutions of the Riemann equation, and we will not be interested in velocity itself and its potential, the integral of velocity. For ergodic processes, the probability density is known to coincide with the relative residence time of the process at a certain interval. It is this property that will be used to interpret the probability distribution of the Riemann wave and its integral. After the occurrence of multivalued solutions, there arise sections in the wave profile where the points of a certain interval of the initial profile overtake each other. There are various options for taking into account multivalued solutions at the relative residence time: (1) the corresponding intervals are taken into account with the minus sign, (2) the breaking intervals are neglected, and (3) all the intervals with the plus sign are taken into account. The formally obtained probability distribution of the Riemann wave and its integral will reflect the way of taking into account multivalued solutions at the relative residence time: the loss of positivity of the probability distribution in the first case and the violation of normalization by one in the second and third ones. Nevertheless, the probability distribution with the positive values of the Riemann wave integral is expected to depend weakly on the method of taking into account the multivalued branches of the solution, and a certain part of this distribution remains physically significant.

A number of problems in which it becomes necessary to study the Riemann wave integrals are discussed in Sec. II. The probability Riemann wave characteristics and its integral in the Lagrangian and Euler representations are obtained in Sec. III. The important problem of taking into account the ambiguity of the wave profile (after the wave breaks) on probabilistic characteristics is studied in Sec. IV. The results obtained are summarized in the Conclusion.

II. BASIC EQUATIONS

As an important example, we consider the classical eikonal equation on the plane that arises in problems of geometry optics and acoustics:

$$(\nabla\varphi)^2 = \left(\frac{\partial\varphi}{\partial x}\right)^2 + \left(\frac{\partial\varphi}{\partial z}\right)^2 = 1. \tag{2.1}$$

Assuming that z is the axis of the wave beam, and x is the transverse coordinate, in the small-angle approximation it follows from (2.1)

$$\frac{\partial\varphi}{\partial z} = \sqrt{1 - \left(\frac{\partial\varphi}{\partial x}\right)^2} \approx 1 - \frac{1}{2}\left(\frac{\partial\varphi}{\partial x}\right)^2. \tag{2.2}$$

Excluding the regular phase shift along the z axis, Eq. (2.2) takes the form

$$\frac{\partial\varphi}{\partial z} + \frac{1}{2}\left(\frac{\partial\varphi}{\partial x}\right)^2 = 0. \tag{2.3}$$

Introducing the beam angle (the phase gradient)

$$u(x,z) = \frac{\partial\varphi}{\partial x}, \tag{2.4}$$

we arrive at the simple wave equation (1.1) with z replaced by t. This case was used in the review [10] to draw the analogy between the problems of optics and cosmology.

Let us note that after replacing z by t, Eq. (2.3) coincides with the homogeneous Kardar-Parisi-Zhang equation in the absence of the viscous term [8]. This equation is the simplest kinematic model for fire front evolution and other changing surfaces [21,22].

Another important practical problem, where the equation of a simple wave and its integral arises, is the problem of the sea wave run-up on the shore. In the dissipation absence, the nonlinear equations of hydrodynamics (shallow water) in the vicinity of the coast take the form

$$\frac{\partial H}{\partial t} + \frac{\partial}{\partial x}(Hu) = 0, \qquad \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + g\frac{\partial H}{\partial x} = \alpha, \tag{2.5}$$

where $H(x,t)$ is the water flow thickness, $u(x,t)$ is the depth-averaged flow velocity, g is the gravity acceleration, and α is the bottom slope near the coast. These equations can be linearized by using hodograph transformations, which made it possible to find the family of exact solutions [23–28]. In particular, the moving shoreline (edge) dynamics (the boundary between water and land) is described by formulas [6,7]

$$u(t) = u[x(t),t] = U_0\left(t + \frac{u}{\alpha g}\right), \tag{2.6}$$

$$r(t) = \alpha x(t) = R_0\left(t + \frac{u}{\alpha g}\right) - \frac{u^2}{2g}. \tag{2.7}$$

Here $r(t)$ and $u(t)$ are the vertical mixing and the moving shoreline (edge) speed, and R_0 and U_0 are the same characteristics calculated in the framework of the linear shallow water equations at $x = 0$. Naturally, the speed and the shoreline (edge) shift are connected by kinematic relations

$$u(t) = \frac{1}{\alpha}\frac{dr}{dt}, \quad U_0(t) = \frac{1}{\alpha}\frac{dR_0}{dt}. \tag{2.8}$$

Formally, the solutions given above are not the functions of two variables, like the Riemann waves. However, having replaced $(\alpha g)^{-1}$ with x, it is easy to see that function (2.6) coincides with the Riemann wave (when replacing the arguments with each other).

We have given a number of physical examples where the problems of studying the Riemann waves and their integrals arise. For generality, we will present these equations in a generalized dimensionless form

$$\frac{\partial u}{\partial \xi} - u\frac{\partial u}{\partial t} = 0, \tag{2.9}$$

$$\frac{\partial r}{\partial \xi} = \frac{1}{2}\left(\frac{\partial r}{\partial t}\right)^2, \qquad \frac{\partial r}{\partial \xi} - u\frac{\partial r}{\partial t} = -\frac{1}{2}u^2, \tag{2.10}$$

where for definiteness the quantity u will be called the velocity, and r the displacement.

As is known, the wave field description in the framework of partial differential equations in hydrodynamics is usually called the Euler equation. It describes the temporary field behavior at some fixed point. To solve Eqs. (2.9) and (2.10) it is convenient to pass over to the equations for the characteristics, the equations in ordinary derivatives

$$\frac{dU}{d\xi} = 0, \quad \frac{dT}{d\xi} = -U, \quad \frac{dR}{d\xi} = -\frac{1}{2}U^2. \tag{2.11}$$

The solutions of the characteristic system are trivial:

$$U(\tau, \xi) = U_0(\tau), \; T(\tau, \xi) = \tau - \xi U_0(\tau),$$
$$R(\tau, \xi) = R_0(\tau) - \frac{\xi}{2}U_0^2(\tau), \tag{2.12}$$

where $R_0(\tau)$ and $U_0(\tau)$ describe the wave field characteristics at the input of the medium ($\xi = 0$). This is the so-called Lagrangian description when we observe how the parameters of an individual point of the wave (particle) profile behave when the coordinate ξ changes. Moreover, the variable τ is a temporary Lagrangian coordinate.

To go from the Lagrangian description (2.11) and (2.12) to the Euler one it is necessary to find the function $\tau = \tau_*(t, \xi)$ from the equation

$$t = \tau - \xi U_0(\tau). \tag{2.13}$$

Then the wave field in the Euler representation will be expressed as follows through the Lagrangian field:

$$u(t, \xi) = U_0[\tau_*(t, \xi)],$$
$$r(t, \xi) = R_0[\tau_*(t, \xi)] - \frac{\xi}{2}U_0^2[\tau_*(t, \xi)]. \tag{2.14}$$

For the wave field to be unambiguous in the Euler representation, the unambiguity of the equation solution (2.13) is necessary, that is, equivalent to the positivity of the Jacobian transformation

$$J(\tau, \xi) = \frac{\partial T}{\partial \tau} = 1 - \xi\frac{dU_0}{d\tau} > 0. \tag{2.15}$$

The same condition can be easily obtained by differentiating the Riemannian wave with respect to any variable, and the solution is valid until a gradient catastrophe (collapse) sets in.

Here we will illustrate the displacement and velocity of the Riemann wave particles in the Lagrangian and Euler representations in the case of harmonic oscillations in the linear theory:

$$r(t, 0) = R_0(t) = \cos(t). \tag{2.16}$$

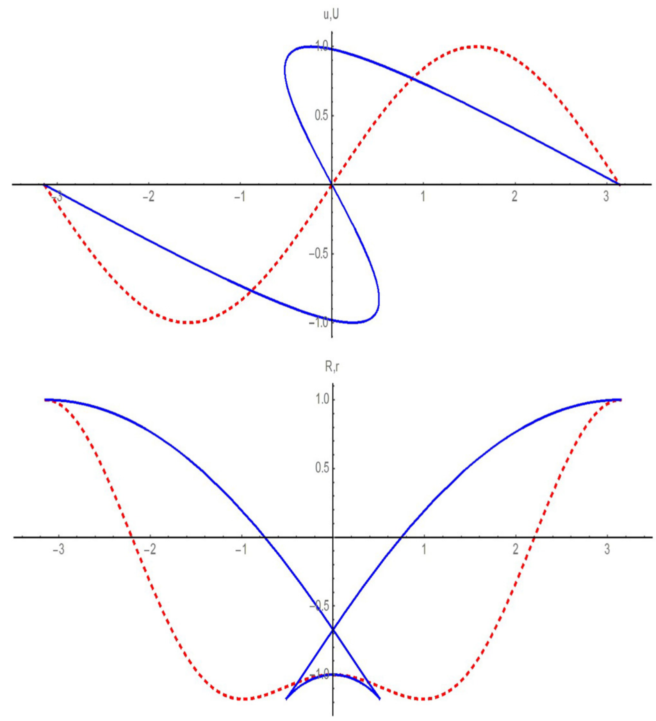

FIG. 1. The velocity profile (the upper figure) and the displacement (the lower figure) in the Lagrangian (the dashed line) and the Euler representations (the solid line) for $\xi = 1.8$.

In Fig. 1 the velocity and displacement profiles in the Lagrangian representation $U(t, \xi)$ and $R(t, \xi)$ are shown by the dashed line, and those in the Euler representation are shown by the solid line for $\xi = 1.8$. Let us note that it follows from (2.15) that the solution in the Euler representation is unambiguous under the condition $\xi < 1$. However, we have specifically chosen a larger value of the coordinate to illustrate the fundamental difference in the profiles in the Lagrangian and Euler representations, when the wave remains unambiguous in the Lagrangian representation and ambiguous in the Euler one.

It is the combination of the Euler and the Lagrangian approaches that allows us to find the statistical characteristics of the nonlinear wave at a fixed distance from the input [12,13].

III. PROBABILITY DISTRIBUTIONS OF THE RIEMANN WAVE AND ITS INTEGRAL

Let us suppose that the statistical properties of the wave field at the nonlinear medium input ($\xi = 0$) are known and consider the wave field at the input to be a stationary random process with a probability density

$$w_{0,r,u}(R, U) = \langle \delta[R - R_0(\tau)]\delta[U - U_0(\tau)]\rangle = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \delta(z - R)\delta(v - U)w_{0,r,u}(z, v)dzdv, \tag{3.1}$$

where $\langle\cdot\rangle$ mean the brackets of statistical averaging. Then in the Lagrangian representation the probability density of displacement and velocity is

$$w_{\mathrm{Lag}}(r; \xi) = \int w_{0,r,u}\left(r + \frac{\xi}{2}u^2, u\right)du, \tag{3.2}$$

$$w_{\mathrm{Lag}}(u; \xi) = \int w_{0,r,u}\left(r + \frac{\xi}{2}u^2, u\right)dr, \tag{3.3}$$

while the two-dimensional probability density of displacement and velocity is

$$w_{\text{lag}}(r, u; \tau, \xi) = \left\langle \delta\left[R_0(\tau) - \frac{1}{2}U_0^2(\tau)\xi - r\right]\delta[U_0(\tau) - u]\right\rangle = w_{0,r,u}\left(r + \frac{u^2\xi}{2}, u\right). \tag{3.4}$$

For the correct transition from the probability distribution in the Lagrangian representation (3.2)–(3.4) to the probabilistic distribution in the Euler one, we also need to know additionally the Jacobian behavior (2.15), namely, the Lagrangian probability density of four variables: t, r, u, j:

$$\begin{aligned} w_{\text{Lagr}}(t, r, u, j; \tau, \xi) &= \langle \delta[t - T(\tau, \xi)]\delta[r - R(\tau, \xi)]\delta[u - U(\tau, \xi)]\delta[j - J(\tau, \xi)]\rangle \\ &= \delta(\tau - t - \xi u)\left\langle \delta[r - R_0(\tau) + \frac{\xi}{2}U_0^2(\tau)]\delta[u - U_0(\tau)]\delta[j - 1 + \xi a_0(\tau)]\right\rangle, \end{aligned} \tag{3.5}$$

where we have introduced another random variable that has the acceleration meaning

$$a_0(\tau) = dU_0/d\tau. \tag{3.6}$$

It is also worth noting that in (3.5) we used the filtering property of the delta function $\delta[u - U_0(\tau)]$.

Let us assume that the probability distribution of wave characteristics at the input is known: the three-point density of the probability of displacement, velocity, and acceleration:

$$\begin{aligned} w_{0,r,u,a}(R, U, A) &= \langle \delta[R - R_0(\tau)]\delta[U - U_0(\tau)]\delta[A - a_0(\tau)]\rangle \\ &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \delta(\alpha - R)\delta(\beta - U)\delta(\chi - A)w_{0,r,u,a}(\alpha, \beta, \chi)d\alpha d\beta d\chi \end{aligned} \tag{3.7}$$

Using the filtering property of the delta function

$$\delta[j - 1 + \xi a_0(\tau)] = \frac{1}{\xi}\delta\left[a_0 - \frac{1 - j}{\xi}\right] \tag{3.8}$$

in (3.5), for the Lagrangian probability density we get

$$w_{\text{Lagr}}(t, r, u, j; \tau, \xi) = \frac{1}{\xi}\delta(\tau - t - \xi u)w_{0,r,u,a}\left(r + \frac{\xi}{2}u^2, u, \frac{1 - j}{\xi}\right). \tag{3.9}$$

To find the Euler probability density, it is necessary to integrate (3.9) over τ:

$$\begin{aligned} \int_{-\infty}^{+\infty} w_{\text{Lagr}}(t, u, j, r; \tau, \xi)d\tau &= = \left\langle \frac{1}{J[\tau_*(t, \xi), \xi]}\delta[u - U(\tau_*(t, \xi), \xi)]\delta[j - J(\tau_*(t, \xi), \xi)]\delta[r - R(\tau_*(t, \xi), \xi)]\right\rangle \\ &= \left\langle \frac{1}{j(t, \xi)}\delta[u - u(t, \xi)]\delta[j - j(t, \xi)]\delta[r - r(t, \xi)]\right\rangle = \frac{1}{j}w_{\text{eul}}(u, j, r; t, \xi). \end{aligned} \tag{3.10}$$

Here again we have used the filtering property of the delta function

$$\delta[t - T(\tau, \xi)] = \frac{1}{j(t, \xi)}\delta[\tau - \tau(t, \xi)] \tag{3.11}$$

and the formulas (2.14) for the transition from the Lagrangian description to the Euler one. The appearance of the Jacobian in (3.11) is due to the fact that the expanding profile sections make a larger contribution to Euler statistics than the compressing ones. Using the puncturing property of the delta function $\delta(\tau - t - \xi u)$ from (3.10) and (3.11) we get

$$w_{\text{eul}}(u, j, r; t, \xi) = \frac{j}{\xi}w_0\left(u, \frac{1 - j}{\xi}, r + \frac{\xi}{2}u^2, t + \xi u\right). \tag{3.12}$$

After integrating (3.12) over j for the joint displacement probability density and velocity we obtain

$$w_{\text{eul}}(r, u; t, \xi) = w_{0,r,u}\left(r + \frac{\xi}{2}u^2, u; t + \xi u\right) - \xi\int_{-\infty}^{+\infty} w_{0,r,u,a}\left(r + \frac{\xi}{2}u^2, u, a; t + \xi u\right)da. \tag{3.13}$$

Then after integrating (3.13) over the variable r, we obtain the probability distribution of the velocity field. For the stationary process, the probability velocity density is preserved:

$$w_{\text{eul}}(u; \xi) = w_0(u), \tag{3.14}$$

since the broadening of individual sections of the Riemann wave is compensated by the compression of the others [12,13]. For the probability displacement distribution from (3.13) after the integration over u, we get

$$w_{\text{eul}}(r; \xi) = \int_{-\infty}^{+\infty} w_{0,r,u}\left(r + \frac{\xi}{2}u^2, u\right)du - \xi\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} w_{0,r,u,a}\left(r + \frac{\xi}{2}u^2, u, a\right)adadu. \tag{3.15}$$

As can be seen from the comparison with (3.2), the first term coincides with the Lagrangian probability density $w_{\text{Lag}}(r;\xi)$. The second term describes the difference between the Lagrangian probability density and the Euler one, namely, a relatively larger contribution from the expanding profile sections to the Euler probability density.

So we have shown that when calculating the single-point densities of the velocity distribution (the actual Riemann wave) it is sufficient to know its distribution at the input, and the nonlinearity does not affect the probability velocity distribution. The displacement probability density (the integral of the Riemann wave) is determined by the two-point distribution of displacement, speed, and acceleration. Such distributions are always poorly known for random processes and are almost never measured in laboratory or field conditions. Therefore, for specific calculations of the probability distribution of the Riemann wave integral, one has to make additional assumptions about the form of the three-point distribution function at the input.

IV. CALCULATIONS OF PROBABILITY DISTRIBUTION OF BIAS CHARACTERISTICS WITH GAUSSIAN INPUT STATISTICS

The expressions for the displacement probability density and velocity obtained above are written under the assumption that the wave profile in all implementations is unambiguous everywhere, that is, condition (2.15) holds, and we reproduce it here:

$$J(\tau,\xi) = 1 - \xi a_0(\tau) > 0. \tag{4.1}$$

Thus, the strict condition for the general expressions applicability for the Euler probability densities is limited to

$$\xi \leqslant 1/\max(a_0). \tag{4.2}$$

For a random field it is possible only if the initial probability distribution of the coastal motion acceleration is finitary and identically equal to zero at $a > \max(a_0)$. In Ref. [15] such a distribution model was suggested and probability displacement distributions for a narrow-band initial disturbance were obtained. However, if the initial field has Gaussian statistics, formally the Riemann solution ambiguities arise at arbitrarily small distances.

In nonlinear acoustics it is customary to consider the Riemann equation solution generalization when discontinuities are introduced into the solution. This discontinuities position is determined from the integral conservation laws known for the Riemann wave as the Oleinik-Lax principle [13]. Moreover, from all branches of the multivalued solution (Fig. 1), that branch is selected on which the displacement takes a maximum value. That is, in Eq. (2.13) from the set of solutions $\tau = \tau_*(t,\xi)$ it is necessary to choose those where the function $r(t,\xi)$ in formula (2.14) reaches the absolutely maximum value. Let us note that this solution coincides with the Burgers equation asymptotic solution with vanishingly low viscosity. The evolution of this random wave type is called acoustic turbulence, or Burgers turbulence, and sometimes even Burgulence [29,30]. In this case, a fairly complete statistical description can be carried out at large times (long paths) when the statistical theory of large overshoots can be used. Let us note that for the one-dimensional Burgers turbulence (as well as the three-dimensional Burgers equation used to model the large-scale Universe structure) for a certain class of initial conditions it is possible to give an almost exhaustive statistical description [11–14,29–33]. In particular, single-point and two-point probability distributions of the velocity field and even N-point probability distributions, and, accordingly, multipoint moment functions, were found. Moreover, as a result of multiple merging of discontinuities, a self-similar evolutionary regime is realized.

We will be interested in the initial stage of a random field evolution when the ambiguity regions occupy a relatively small place in the random field duration. However, ambiguity areas are important for determining probability characteristics. It should be said that sometimes many-valued solutions are physically realized, for example, for the optical wave phase (2.1)–(2.3), and we have the right to consider ambiguous solutions. In other cases, for example, in the hydrodynamics of the sea wave run-up on the shore, the wave after breaking has a shape that does not coincide with the Riemann solution. In this case, the above given analysis is necessary to assess the Riemann wave applicability to describe the probability density of a random field.

That is why we did not supply the Jacobian module $j(t,\xi)$ in formula (3.11), as is required in the generalized delta function theory [13]. To understand the ambiguity nature better, it is convenient to recall that for ergodic processes the probability density, for example, of the displacement $w(r,\xi)$ coincides with the relative residence time of the process $r(t)$ in the interval $(r, r+dr)$ [13]:

$$w(r;\xi) = \frac{1}{L}\sum_{n=1}^{N}\frac{dt_n}{dr}. \tag{4.3}$$

Formally, formula (4.3) is also valid for multivalued processes. Naturally, the probability density notion cannot be introduced for such processes. Nevertheless, in what follows, the expressions obtained below for the probability density of the Gaussian process based on the previous section will be referred to as the probability density as before. Physically, they describe precisely the relative residence time (4.3). After the ambiguity appearance in the Riemann solution in areas where the Jacobian (4.1) is negative, some intervals are also negative. There are various options to take into account multivalued solutions (of the type shown in Fig. 1) in the relative residence time: (1) the corresponding intervals are taken into account with the minus sign, (2) the breaking intervals are neglected, and (3) all intervals with the plus sign are taken into account. In interpreting the formulas obtained below, we will use precisely formula (4.3). Every situation of the kind corresponds to a certain sign and the delta function magnitude in the form (3.11).

First, let us interpret the probability density preservation of the velocity field. Obviously, an expression similar to (3.14) can also be written for the velocity field. Moreover, from the residence time dt_n of the process $u(t)$ in the interval $(u, u+du)$ from (2.12) we have $dt_n = dt_{n,0} \pm \xi du$, and the relative residence time (probability density) of the velocity field is preserved: $W(u,\xi) = W(u,0)$.

We will now discuss the evolution of the displacement probability density. Let us suppose that the initial field has Gaussian statistics and let the correlation function $B_0(\tau)$ and the spectrum $S_0(\omega)$ of the input displacement are given. Then, for the three quantity dispersions (displacement, velocity, and acceleration), the following relations are valid [34,35]:

$$B_0(\tau) = \int_{-\infty}^{+\infty} S_0(\omega)\exp(i\omega\tau)d\omega \tag{4.4}$$

$$\sigma_{R_0}^2 = \langle R_0^2(t)\rangle = B_0(0) = \int_{-\infty}^{+\infty} S_0(\omega)d\omega, \tag{4.5}$$

$$\sigma_{U_0}^2 = \langle U_0^2(t)\rangle = -\left.\frac{d^2B_0}{d\tau^2}\right|_0 = \int_{-\infty}^{+\infty} \omega^2 S_0(\omega)d\omega, \tag{4.6}$$

$$\sigma_{a_0}^2 = \langle a_0^2(t)\rangle = \left.\frac{d^4B_0}{d\theta^4}\right|_0 = \int_{-\infty}^{+\infty} \omega^4 S_0(\omega)d\omega, \tag{4.7}$$

$$\langle R_0(t)a_0(t)\rangle = \left.\frac{d^2B_0}{d\tau^2}\right|_0 = -\int_{-\infty}^{+\infty} \omega^2 S_0(\omega)d\omega = -\sigma_{U_0}^2, \tag{4.8}$$

$$\langle R_0(t)U_0(t)\rangle = 0, \quad \langle U_0(t)a_0(t)\rangle = 0. \tag{4.9}$$

We denote the correlation coefficient of the initial displacement and acceleration as

$$q = \frac{\langle R_0(t)a_0(t)\rangle}{\sigma_{R_0}\sigma_{a_0}} = -\frac{\sigma_{U_0}^2}{\sigma_{R_0}\sigma_{a_0}}. \tag{4.10}$$

The distributions of the three input quantities (displacement, velocity and acceleration) are described by the Gaussian formulas

$$w_0(U) = \frac{1}{\sqrt{2\pi\sigma_{U_0}^2}}\exp\left(-\frac{U^2}{2\sigma_{U_0}^2}\right), \tag{4.11}$$

$$w_{0,R,a}(R,a) = \frac{1}{2\pi\sigma_{R_0}\sigma_{a_0}\sqrt{1-q^2}}\exp\left[-\frac{1}{2(1-q^2)}\left(\frac{R^2}{\sigma_{R_0}^2} + \frac{a^2}{\sigma_{a_0}^2} - 2q\frac{ra}{\sigma_{R_0}\sigma_{a_0}}\right)\right], \tag{4.12}$$

$$w_{0,R,U,a}(R,U,a) = w_{0,U}(U)w_{0,R,a}(R,a). \tag{4.13}$$

Substituting these formulas in (3.15), we find the Euler probability distribution of the displacement

$$w_{\rm eul}(r,\xi) = \frac{1}{2\pi\sigma_{R_0}\sigma_{U_0}}\int_{-\infty}^{+\infty}\exp\left(-\frac{U^2}{2\sigma_{U_0}^2}\right)\exp\left[-\frac{(R+\xi U^2/2)^2}{2\sigma_{R_0}^2}\right]dU$$
$$-\frac{\xi\sigma_{U_0}}{2\pi\sigma_{R_0}^3}\int_{-\infty}^{+\infty}\exp\left(-\frac{U^2}{2\sigma_{U_0}^2}\right)\exp\left[-\frac{(R+\xi U^2/2)^2}{2\sigma_{R_0}^2}\right]\left(R+\frac{\xi}{2}U^2\right)dU. \tag{4.14}$$

We note here that acceleration dispersion was not included in this formula. Let us move on to the dimensionless variables

$$\varphi = R/\sigma_{R_0}, \quad v = U/\sigma_{U_0}, \quad l = \xi/L_{\rm nel}, \tag{4.15}$$

where $L_{\rm nel} = \sigma_{R_0}/\sigma_{U_0}^2$ is the characteristic distance of the nonlinear effect manifestation. In these variables, the Euler displacement probability density takes the form

$$w_{\rm eul}(\varphi,l) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\exp\left(-\frac{v^2}{2}\right)\exp\left[-\frac{(\varphi+lv^2/2)^2}{2}\right]\left[1+l\left(\varphi+\frac{lv^2}{2}\right)\right]dv, \tag{4.16}$$

and is expressed in terms of the Lagrangian probability density as follows:

$$w_{\rm eul}(\varphi,l) = w_{\rm Lag}(\varphi,l) - l\frac{\partial w_{\rm Lag}}{\partial\varphi}, \tag{4.17}$$

$$w_{\rm Lag}(\varphi,l) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\exp\left(-\frac{v^2}{2}\right)\exp\left[-\frac{(\varphi+lv^2/2)^2}{2}\right]dv. \tag{4.18}$$

From these expressions it is obvious that the normalization per unit of both the Lagrangian and Euler probability densities is preserved. Moreover, it follows from (4.17) and (4.18) that the shifts of the average values of the Lagrangian and Euler probability density displacement have different signs:

$$\int_{-\infty}^{+\infty}\varphi w_{\rm Lag}(\varphi,l)d\varphi = -\frac{l}{2}, \quad \int_{-\infty}^{+\infty}\varphi w_{\rm eul}(\varphi,l)d\varphi = \frac{l}{2}. \tag{4.19}$$

Figure 2 shows the graphs of the Lagrange and Euler distributions constructed according to formulas (4.17) and (4.18) for $l = 0$, 0.5, and 1. To emphasize the difference between these variables along the axes, the Lagrangian (R) and Euler (r) variables are plotted. It can be seen from the figure that the Lagrangian and Euler probability distributions shift in different directions with the increasing parameter l, as predicted by the formulas (4.19). For the Euler distribution, there is a relative increase in the probability of the positive bias values r. The appearance of negative values $w_{\rm eul}(r)$ for the negative

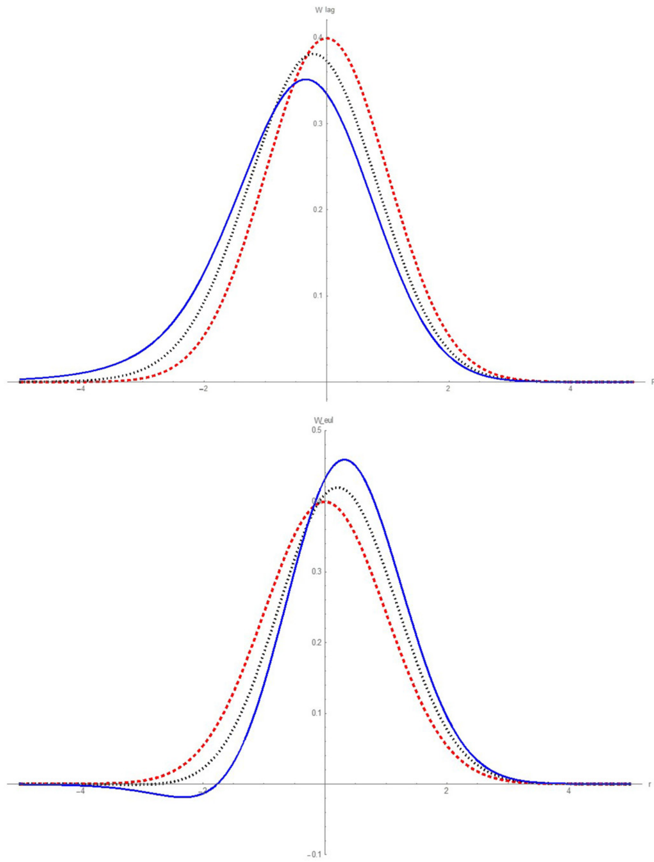

FIG. 2. The Lagrangian (the upper figure) and the Euler (the lower figure) displacement probability density at $l = 0$, (the dashed line), $l = 0.5$ (points), and $l = 1$ (the solid line).

r values is due to the fact that, in accordance with formula (3.11), we consider the relative residence process time (4.3), and in the multivalued solution, the breaking intervals are taken into account with the minus sign. As already mentioned, multivalued solutions for the Riemann wave characteristics do not have any physical meaning. Nevertheless, for the functions described by first-order equations of the type (2.9) and (2.10), the locality principle is valid: the function behavior outside the singularity zone does not depend on the function behavior in the singularity neighborhood zone. This allows us to hope that with positive argument values the probability density describes the evolution of the displacement probability distribution adequately enough. As applied to the problems of the sea wave run-up on the shore, the positive values of the argument r correspond to the wave flooding stage of the coast, which is of great practical importance.

As already noted, for multivalued solutions, there are three possible options to take multivalued sections into account in the relative residence time of the process in the given value range. In the first case, such intervals are taken into account with the minus sign, in the second one the run-up intervals are neglected, and in the third, only single-valued branches are taken into account. The first case was considered above. Below we will see how the probability density form changes in the last two cases. We first consider the case when we neglect all the intervals with a different sign (the second case). To do this, we integrate in the last integral (3.15) over a in the Jacobian positivity interval, that is, for $a < 1/\xi$. Then the probability density of the displacement has the form

$$w_{\text{eul}}^{+}(\varphi; l, q) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \exp\left(-\frac{v^2}{2}\right)\exp\left[-\frac{(\varphi + lv^2/2)^2}{2}\right] \times F(\varphi, l, q)dv, \tag{4.20}$$

where

$$F(\varphi, l, q) = 1 + \frac{l}{2}\left[\varphi + \frac{lv^2}{2}\right]\left[1 + Erf\left(\frac{\mu}{\sqrt{2}}\right) - \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{\mu}{2}\right)\right], \tag{4.21}$$

$$\mu = \frac{|q|}{\sqrt{1-q^2}}\left[\varphi + \frac{lv^2}{2} + \frac{1}{l}\right]. \tag{4.22}$$

Here $Erf(z) = \frac{2}{\sqrt{\pi}}\int_0^z \exp(-t^2)dt$ is the error integral. Expression (4.20) includes an additional parameter q, which we have already introduced earlier, as the correlation coefficient of the surface displacement and acceleration. It is worth noting that in addition to the nonlinear length L_{nel} (4.15), we have the characteristic breaking length of the velocity profile (the gradient catastrophe):

$$L_u = \frac{1}{\sigma_{a_0}} = |q|L_{\text{nel}}. \tag{4.23}$$

Thus, the dimensionless length included in the formulas for the probability distribution (4.20) is related to the variable ξ in the following way:

$$l = \frac{\xi}{L_{\text{nel}}} = \frac{\xi|q|}{L_u} \tag{4.24}$$

and it depends on the correlation coefficient of the displacement and acceleration q. Figure 3 shows the distribution (4.20) for $l = 1$ and various values of the correlation coefficient q. For relatively large values of the correlation coefficient modules $|q| \approx 1$ [Fig. 3(a)], the probability distributions for the positive bias values are almost identical in these models. This means that the wave field peculiarities (features) arise mainly at the negative r values and do not affect the probability distribution form for the positive r values. As we have underlined above, in the case of the sea wave run-up on the coast, the positive r values correspond to the coastal flooding stage, therefore, we can correctly predict the flooding characteristics. In the case of small coefficients q [Fig. 3(b)], the appearance of singularities is little correlated with the displacement r; singularities arise both for the positive and negative bias values. Let us note that, in accordance with (4.24), Fig. 3(b) corresponds to the stage of developed ambiguities.

To conclude this section, we give the formula for the displacement probability density when we consider all the module intervals in the polysemy domain. In this case, the integration over the variable a is divided into two intervals $(-\infty, 1/\xi)$ and $(\xi, +\infty)$, and on the right interval the sign of the Jacobian changes. As a result, the potential probabilistic distribution considered as the relative residence time of the

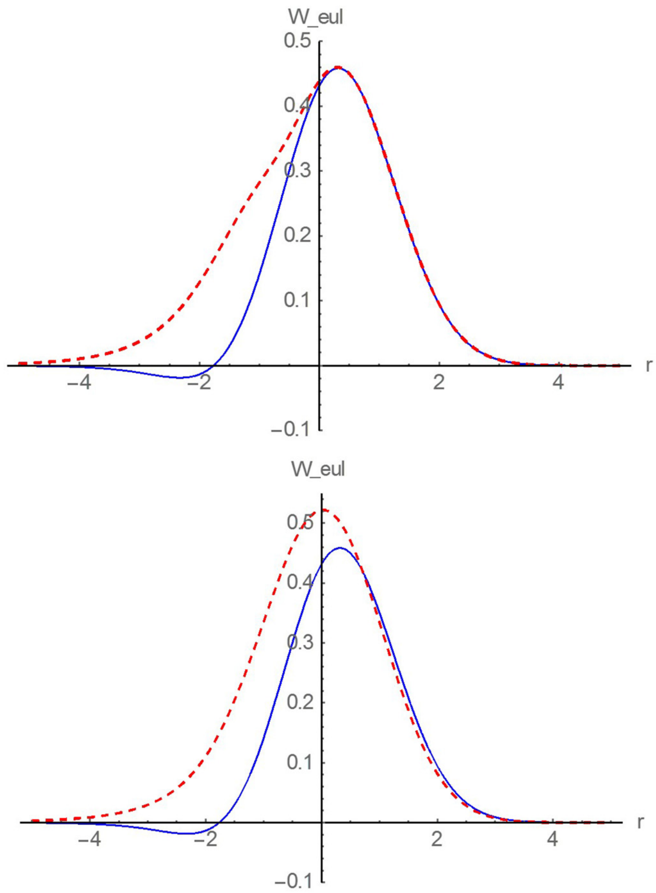

FIG. 3. The Euler displacement probability density (relative residence time) at $l = 1$ for two models when ambiguities are taken into account [(a) $|q| = 0.9$, (b) $|q| = 0.1$]. The solid line shows the displacement probability distribution when the intervals after breaking are taken into account with the minus sign. The dotted line shows the distribution when these intervals are not taken into consideration.

process in the given interval, takes the form

$$w_{\rm eul}^{\rm all}(\varphi; l, q) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \exp\left(-\frac{v^2}{2}\right) \exp\left[-\frac{(\varphi + lv^2/2)^2}{2}\right] \times F_1(\varphi, l, q) dv, \tag{4.25}$$

where

$$F_1(\varphi, l, q) = 1 + \xi l \sqrt{1 - q^2}\left[\varphi + \frac{lv^2}{2}\right]\left[1 + Erf\left(\frac{\mu}{\sqrt{2}}\right) - \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{\mu}{2}\right)\right]. \tag{4.26}$$

Figure 4 shows the Euler probability density (relative residence time) for two models of accounting ambiguities. As in the previous case, this figure shows that if the correlation value coefficient is not too small, the probability displacement distribution for the positive argument values is almost the same in these models.

So the main result of this section is the assertion that the effects of the wave breaking with a strong correlation between the displacement and acceleration have little effect on the distribution form in the area of the positive displacement values, namely, as already mentioned, they are important in the problems of the sea wave run-up on the shore.

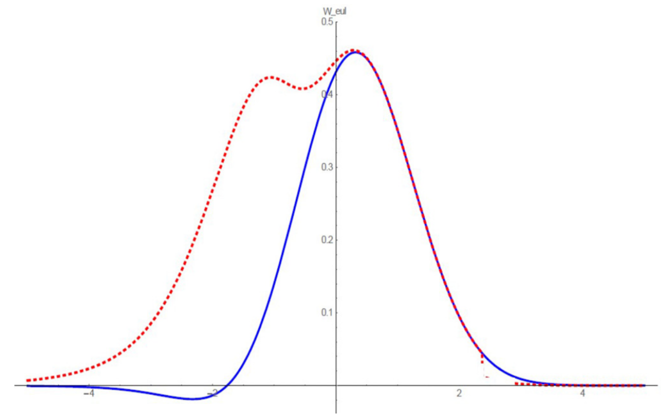

FIG. 4. The Euler potential probability density (relative residence time) for $l = 1$ and $|q| = 0.9$ for two models when ambiguities are taken into account. The solid line shows the probability distribution (4.16), when the intervals after breaking are taken into account with the minus sign. The dashed line shows the distribution (4.25) when all the intervals in the polysemy region are taken into account with the positive sign.

V. CONCLUSION

In this paper we consider the evolution of the probability distributions of nonlinear random waves in a nondispersive medium described by an equation of the Riemann type (a simple wave equation).

This equation and the corresponding equation for the integral from it (displacement) describe a wide class of physical phenomena: the phase front evolution in geometry optics, fire front evolution, intense acoustic waves, the moving shoreline (edge) dynamics when sea waves run-up on the shore, and some others. In the simplest case, the Riemann equation describes the evolution of gas of noninteracting particles. Moreover, the behavior of the particle itself is trivial: we have a simple uniform and rectilinear motion of an individual particle. The representation of the kind when we follow the individual particle motion (a fixed point in the wave profile) is usually called the Lagrangian description. However, as have already mentioned in the review [10], the consideration of a continuous medium, that is, not just one, but a whole ensemble of particles, leads to interesting and nontrivial results even in this simple case. This is a nonlinear profile distortion, the generation of harmonics. The field description at a fixed point and a fixed time point is usually called the Euler description. Naturally, we have the same thing for the evolution of the statistical characteristics of random fields. Since each particle moves at a constant speed, its Lagrangian probability density does not change, while in the Euler representation there is a significant distortion of the statistical field characteristics.

The dynamics of actually random Riemann waves is well known, but the probability distributions of the Riemann wave integrals have not been studied yet. In this paper we have obtained general expressions for the probability distributions of velocity and displacement using the Lagrange and Euler connection of the statistical description. It is shown in the paper that in order to find the probability displacement distribution it is necessary to know the joint probability distribution of displacement, velocity, and acceleration at the input. In the expression for the Euler probability density there appears a term describing the difference between the Lagrangian probability density and the Euler probability density, namely, the relatively larger contribution obtained from the expanding profile sections to the Euler probability density. If the Lagrangian probability density shifts toward negative biases the Euler probability density shifts toward positive biases. The applicability condition for these expressions is limited to the case of single-stream solutions of the Riemann equation.

Nevertheless, the case where the wave field at the input has Gaussian statistics, is discussed in the paper as well. In this case, it is suggested that the obtained formulas should be interpreted as the time of the relative process stay in a certain range of velocity or bias values. It is true for ergodic processes when averaging over an ensemble of realizations can be replaced by averaging over a separate rather long process implementation. Then the obtained formulas for the probability distribution describe the relative residence time of the process of the multivalued Riemann equation solutions. At the same time, there are various options to take into account the areas where profile points are overtaken, and so the Jacobian of the transformation from the Lagrangian variables to the Euler ones becomes negative. Depending on the choice procedure of taking these intervals into account, the relative residence time can have negative values while maintaining normalization by one, or remaining positive to have the normalization greater than one. This is due to the fact that for the multivalued solutions, the total duration of the elementary process intervals in a certain displacement interval becomes longer than the interval duration itself.

The paper presents graphs of the probability displacement distribution at large distances, comparable with the characteristic value of the nonlinear length. From these graphs it follows that, due to the locality of the Riemann equation, the appearance of ambiguity occurring mainly at negative values does not affect the probability density form at positive bias values. In the problems of the sea wave run-up on the shore it means that the effects of the wave collapse with a strong correlation between displacement and acceleration have little effect on the distribution in the area of positive displacement values corresponding to the shore flooding stage by the wave.

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