Jump intermittency as a second type of transition to and from generalized synchronization

Alexey A. Koronovskii, Olga I. Moskalenko,* Anatolii A. Pivovarov, and Vladislav A. Khanadeev
Saratov State University, 83 Astrakhanskaya Strasse, 410012 Saratov, Russia
and Regional Scientific and Educational Mathematical Center "Mathematics of Future Technologies," 410012, Saratov, Russia

Alexander E. Hramov
Innopolis University, 1 Universitetskaya Strasse, 420500 Innopolis, Russia

Alexander N. Pisarchik
Center for Biomedical Technology, Technical University of Madrid, Campus Montegancedo, 28223 Pozuelo de Alarcón, Madrid, Spain

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The transition from asynchronous dynamics to generalized chaotic synchronization and then to completely synchronous dynamics is known to be accompanied by on-off intermittency. We show that there is another (second) type of the transition called *jump intermittency* which occurs near the boundary of generalized synchronization in chaotic systems with complex two-sheeted attractors. Although this transient behavior also exhibits intermittent dynamics, it differs sufficiently from on-off intermittency supposed hitherto to be the only type of motion corresponding to the transition to generalized synchronization. This type of transition has been revealed and the underling mechanism has been explained in both unidirectionally and mutually coupled chaotic Lorenz and Chen oscillators. To detect the epochs of synchronous and asynchronous motion in mutually coupled oscillators with complex topology of an attractor a technique based on finding time intervals when the phase trajectories are located on equal or different sheets of chaotic attractors of coupled oscillators has been developed. We have also shown that in the unidirectionally coupled systems the proposed technique gives the same results that may obtained with the help of the traditional method using the auxiliary system approach.

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I. INTRODUCTION

Chaotic synchronization is a very important fundamental phenomenon in dynamical systems [1,2] determining the collective behavior of interacting nonlinear oscillators. Among plenty of issues related to this phenomenon, a significant role belongs to transitions from asynchronous dynamics to synchronous ones in coupled systems. Despite the existence of different types of chaotic synchronization, such as phase synchronization [3] (PS), generalized synchronization [4] (GS), time-scale synchronization [5] (TSS), lag synchronization [6] (LS), complete synchronization [7] (CS), etc., the routes to and from many kinds of synchronous regimes exhibit certain universal properties, in particular, an intermittent behavior. At the same time, due to different fundamental mechanisms underlying various synchronization regimes, the types of intermittency on these routes are also different.

The intermittent transitional types of behavior of coupled chaotic oscillators in the vicinity of synchronous regimes have attracted steadfast attention of the scientific community in the last two decades [8–12]. Nowadays, it is well known that the type of intermittency is mainly determined by the conditions (e.g., the value of the control parameter mismatch, the presence of noise, etc.) for which the transition is considered, and not only by the type of synchronization. For example, near the boundary of phase synchronization three different types of intermittency (namely, type-I, eyelet, and ring intermittencies) were detected. Specifically, in coupled systems with parameter mismatch, if the detuning between control parameters is relatively small, two different types of intermittency sequentially take place [8], i.e., type-I intermittency [13–15] and eyelet intermittency [16,17]. The latter occurs in the close vicinity of the boundary separating in the parameter space the areas of PS and the regime of type-I intermittency. Alternatively, if the mismatch between the control parameters of the interacting systems results in a significant difference between fundamental frequencies of the coupled chaotic oscillators, the transition to and from PS is accompanied by ring intermittency [10,11]. Next, the route to and from TSS is also characterized by the same types of intermittency, more precisely, eyelet and ring intermittencies [18]. Furthermore, under certain conditions the simultaneous coexistence of both these types of intermittent dynamics (so-called intermittency of intermittencies) is observed [19].

As far as GS is concerned, coupled chaotic oscillators have been believed for a long time to go from asynchronous oscillations to a synchronous regime only through on-off intermittency [20,21] (in the same way as in the case of LS [6,22] and CS [23]). In the present paper, we report on the *second* type of intermittent behavior revealed in the vicinity of the GS threshold. This type of intermittency takes place

*o.i.moskalenko@gmail.com

for chaotic systems with complex (in our case, two-sheeted) topology of a chaotic attractor, with statistical characteristics of reported intermittent dynamics, such as the dependence of the mean length of laminar phases (epochs of a synchronous motion) on the criticality parameter and laminar phase length distribution, differing sufficiently from characteristics of on-off intermittency. Taking into account the mechanism resulting in intermittent dynamics, we call such type of intermittent behavior *jump intermittency*. For reference purposes we shall also use the term "the first type of transition to generalized synchronization" for the classical scenario of transition to and from GS accompanied by on-off intermittency.

II. MATERIALS AND METHODS

As a model system for the study of a type of intermittent behavior near the boundary of GS, we use two coupled nonlinear oscillators with a complex (two-sheeted) topology of a chaotic attractor. The oscillators are modeled by either the three-dimensional Lorenz system

$$\begin{aligned}\dot{x} &= \sigma(y-x),\\ \dot{y} &= rx - y - xz,\\ \dot{z} &= -bz + xy,\end{aligned} \tag{1}$$

where $(x, y, z)^T$ is the state vector and $\sigma = 10$, $b = 8/3$, and $r = 35\text{-}40$ are the control parameters, or the four-dimensional chaotic Chen system [24]

$$\begin{aligned}\dot{x} &= a(y-x) + eyz,\\ \dot{y} &= cx - dxz + y + u,\\ \dot{z} &= xy - bz,\\ \dot{u} &= -ky,\end{aligned} \tag{2}$$

where $(x, y, z, u)^T$ is the state vector and $a = 35$, $b = 4.9$, $c = 25$, $d = 5$, $e = 35$, and $k = 110\text{-}190$ are the control parameters. Depending on the values of the control parameter, the system in Eq. (2) is known to exhibit chaotic dynamics characterized by both one and two largest positive Lyapunov exponents [24].

Both systems (1) and (2) belong to chaotic oscillators with a complex topology of a chaotic attractor. The phase spaces of the systems under study consist of two different subspaces $\mathbb{W}^{1,2}$ having the small common region $\mathbb{W}^0$ and only within $\mathbb{W}^0$ the phase trajectory can transit from subspace $\mathbb{W}^1$ to $\mathbb{W}^2$ and vice versa. Since each of the subspaces $\mathbb{W}^{1,2}$ looks like a flat sheet [see, e.g., Figs. 3(c) and 3(d)], within this paper we shall name these oscillators as "systems with two-sheeted attractors," whereas each of the subspaces $\mathbb{W}^{1,2}$ we shall mention as "sheet."

We start our consideration with two unidirectionally coupled Lorenz systems, each given by Eq. (1), the evolution operator of which is described as follows:

$$\begin{aligned}&\dot{x}_1 = \sigma(y_1 - x_1), \quad \dot{x}_2 = \sigma(y_2 - x_2) + \varepsilon(x_1 - x_2),\\ &\dot{y}_{1,2} = r_{1,2}x_{1,2} - y_{1,2} - x_{1,2}z_{1,2},\\ &\dot{z}_{1,2} = -bz_{1,2} + x_{1,2}y_{1,2},\end{aligned} \tag{3}$$

where ε is the coupling strength governing the transition from asynchronous dynamics to GS, $r_1 = 40$, and $r_2 = 35$.

To detect the boundary of GS, we use two approaches: the auxiliary system approach [25] and the calculation of the dependence of the largest conditional Lyapunov exponent on the coupling strength [26]. By calculating the largest Lyapunov exponent, we find critical coupling strength $\varepsilon_c = 9.95$, which we verify with the auxiliary system approach and the nearest neighbor method [4,27].

The auxiliary system approach consists in the introduction of another (auxiliary) system the evolution operator of which coincides with the equations of the response oscillator, while the initial conditions of the auxiliary system differ from the ones of the response oscillator, but belong to the same basin of attractor. If the drive and response systems are in GS, the states of the response and auxiliary systems coincide after transients, whereas for asynchronous dynamics the state of the auxiliary oscillator differs from the response one.

To detect intermittency below the threshold of GS, also known as the *intermittent generalized synchronization regime*, the difference between x variables of the response and auxiliary systems, $\xi(t) = x_2(t) - x_a(t)$, is analyzed. The time intervals, where the absolute value of this difference exceeds a predefined value of $\Delta = 0.005$, correspond to the epochs of asynchronous motion (in the sense of GS) referred to as *turbulent phases*, while the parts of time series with this difference being close to zero ($|\xi(t)| < \Delta$) are called *the laminar phases* where two coupled systems demonstrate GS. To study the properties of intermittent dynamics, we calculate statistical characteristics, namely, the normalized distributions of the laminar phase lengths for fixed values of the coupling strength, $N(\tau)$, and the dependence of the mean laminar phase $\langle\tau\rangle$ on the criticality parameter, ε. Unfortunately, the auxiliary system approach is not applicable to detect and study the generalized synchronization phenomenon in bidirectionally coupled chaotic oscillators [28] and, as a consequence, a radically different technique is required to separate the phases of the synchronous and asynchronous motion (in a sense of generalized synchronization) for oscillators with the mutual type of coupling.

Since the observed second type of transition to and from GS (jump intermittency) is directly related to the locations of representation points of phase trajectories on equal or different sheets of chaotic attractors of coupled oscillators (see Sec. III), we propose a technique to find such intervals and calculate their lengths l. To detect whether or not the representation points, corresponding to current states of the interacting systems, are on equal sheets of chaotic attractors, the time series of the systems should be shifted with respect to each other on the time delay $\Delta\tau$ found by means of the similarity function minimum [6] with a further comparison of the $x_{1,2}$ values. The moment when $x_1(t - \Delta\tau) > \Delta x$ and $x_2(t) < -\Delta x$ (where $\Delta x = 10$ for Lorenz oscillators) or vice versa corresponds to the divergence of trajectories of interacting systems into the different sheets of chaotic attractors, which come again together when $|x_1(t - \Delta\tau) - x_2(t)| < \Delta x/2$. The described methodology allows us to find the residence time lengths and obtain reasonable statistics for the analysis of intermittency characteristics. The normalized distributions of the residence times of phase trajectories on equal sheets of chaotic attractors, $N(l)$, and the dependence of the mean residence time of the phase trajectories located on the equal attractors' sheets, $\langle l\rangle$, on the coupling strength ε are also calculated. As we shall show below (see Sec. III A), in the unidirectionally coupled systems the time intervals when the drive and response oscillators are characterized by the representation points diverging from equal to nonequal sheets of chaotic attractors are responsible for the destruction of the epochs of the synchronized motion (in a sense of the generalized synchronization phenomenon), and, therefore, the laminar (synchronous) phases, τ, and time intervals when both the drive and response oscillators are characterized by the representation points being on equal sheets of chaotic attractors, l, are highly correlated with each other. As a consequence, the statistical characteristics obtained with the help of both the proposed (based on the consideration of the representation point locations) and usual (based on the auxiliary system approach) methods, i.e., $N(\tau)$ and $N(l)$ (as well as $\langle\tau\rangle$ and $\langle l\rangle$), in fact, coincide with each other for oscillators coupled unidirectionally, which means the developed technique may be used to detect the laminar and turbulent phases of motion for the oscillators coupled mutually.

To extend the obtained results to mutually coupled chaotic systems, we also consider two bidirectionally coupled Lorenz oscillators

$$\begin{aligned}\dot{x}_{1,2} &= \sigma(y_{1,2} - x_{1,2}) + \varepsilon(x_{2,1} - x_{1,2}),\\ \dot{y}_{1,2} &= r_{1,2}x_{1,2} - y_{1,2} - x_{1,2}z_{1,2},\\ \dot{z}_{1,2} &= -bz_{1,2} + x_{1,2}y_{1,2},\end{aligned} \tag{4}$$

with $r_1 = 40$ and $r_2 = 35$, and two coupled Chen oscillators

$$\begin{aligned}\dot{x}_{1,2} &= a(y_{1,2} - x_{1,2}) + ey_{1,2}z_{1,2},\\ \dot{y}_{1,2} &= cx_{1,2} - dx_{1,2}z_{1,2} + y_{1,2} + u_{1,2},\\ \dot{z}_{1,2} &= x_{1,2}y_{1,2} - bz_{1,2},\\ \dot{u}_{1,2} &= -k_{1,2}y_{1,2} + \varepsilon(x_{2,1} - x_{1,2}),\end{aligned} \tag{5}$$

with $k_1 = 110$ and $k_2 = 190$.

To explain theoretically the mechanism of transition to and from GS for coupled oscillators with two-sheeted topology of attractors and obtain analytical expressions for statistical characteristics of intermittency, we consider the simplest theoretical model in the form of one-dimensional stochastic differential equation

$$\frac{dx}{dt} = -\frac{dU(x)}{dx} + \eta(t), \tag{6}$$

with the asymmetric bistable potential

$$U(x) = \frac{x^4}{4} - \frac{x^2}{2} + bx, \tag{7}$$

where $\eta(t)$ is supposed to be zero mean δ-correlated Gaussian noise [$\langle\eta(t)\rangle = 0$, $\langle\eta(t)\eta(\tau)\rangle = D\delta(t-\tau)$], b is the asymmetry parameter, and D is the noise intensity.

III. RESULTS AND DISCUSSION

A. Unidirectionally coupled Lorenz oscillators

The dependencies of the four largest Lyapunov exponents [29] λ_i, $i = \overline{1,4}$ on the coupling strength ε for two unidirectionally coupled Lorenz systems Eq. (3) are given in Fig. 1. The point ε_c where the second Lyapunov exponent λ_2, being the largest conditional Lyapunov exponent, changes its sign from positive to negative corresponds to the onset of GS.

Just below the generalized synchronization onset the intermittent generalized synchronization regime takes place. The statistical characteristics of this kind of dynamical behavior obtained by means of the auxiliary system approach are present in Fig. 2. The laminar phase length distributions for several fixed values of the coupling strength are shown in Fig. 2(a), while the dependence of the mean length of the laminar phases on the coupling parameter ε is illustrated in Fig. 2(b). For reference, theoretical dependences

$$T = \langle\tau\rangle \sim (\varepsilon_c - \varepsilon)^{-1} \tag{8}$$

and

$$N(\tau) \sim \tau^{-3/2} \tag{9}$$

corresponding to on-off intermittency are known to be observed in the vicinity of GS, as shown in Fig. 2 by dashed lines. One can easily see that for unidirectionally coupled Lorenz systems (3) the observed statistical characteristics differ radically from ones prescribed for on-off intermittency. Therefore, one can conclude that in the considered case, namely, for the Lorenz oscillators which are characterized by the complex two-sheeted chaotic attractor, another type of intermittent behavior (namely, jump intermittency) accompanies the transition to and from GS.

To explain and confirm our findings, we now consider the mechanism of the laminar (synchronous) phase interruption in more detail. The fragment of typical time series corresponding to the intermittent behavior near GS observed in unidirectionally coupled Lorenz systems Eq. (3) below the critical

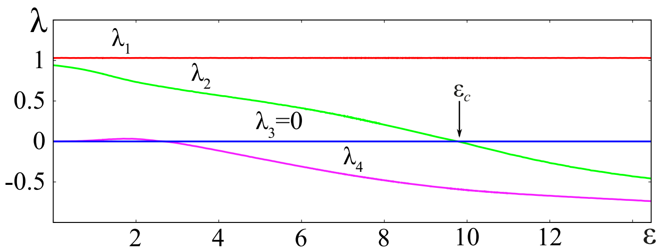

FIG. 1. Four largest Lyapunov exponents vs coupling strength ε for two unidirectionally coupled Lorenz oscillators (3). The arrow indicates the onset of GS.

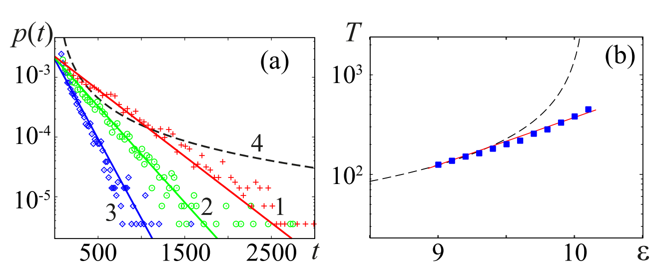

FIG. 2. (a) Normalized laminar phase length distributions and their approximations by the exponential law for fixed values of the coupling strength [plus signs (curve 1) correspond to $\varepsilon = 9.9$, circles (curve 2) correspond to $\varepsilon = 9.7$, and diamonds (curve 3) correspond to $\varepsilon = 9.3$]. (b) Mean length of laminar phases vs coupling parameter ε with its exponential approximation (solid line). The dashed lines represent theoretical curves defined by Eqs. (8) and (9) corresponding to on-off intermittency.

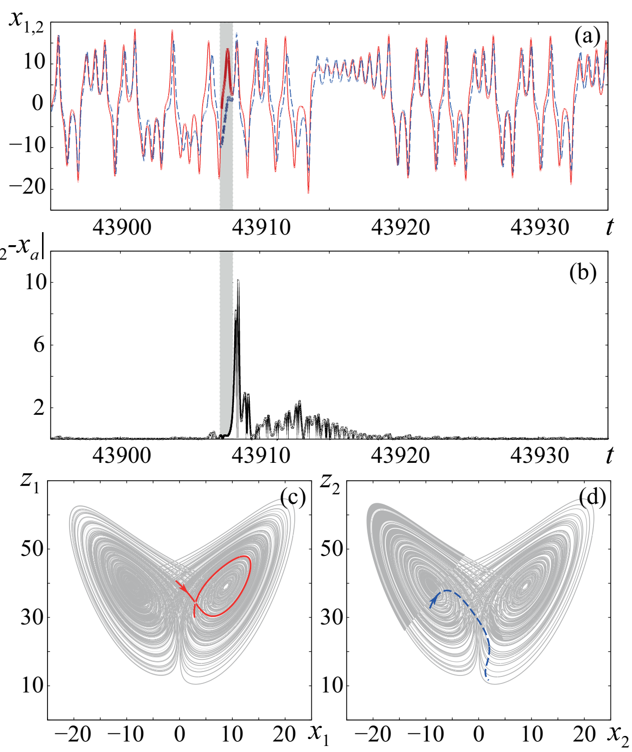

FIG. 3. (a) Fragment of time series of drive $x_1(t)$ (solid line) and response $x_2(t)$ (dashed line) unidirectionally coupled Lorenz oscillators given by Eq. (3). (b) Time evolution of the absolute value of the difference between x coordinates of the response and auxiliary systems, $|\xi(t)| = |x_2(t) - x_a(t)|$. (c) Drive and (d) response chaotic attractors of interacting Lorenz oscillators (light gray) and parts of trajectories in phase spaces corresponding to the beginning of turbulent (asynchronous) phases in drive (solid line) and response (dashed line) oscillators. $\varepsilon = 9.5$.

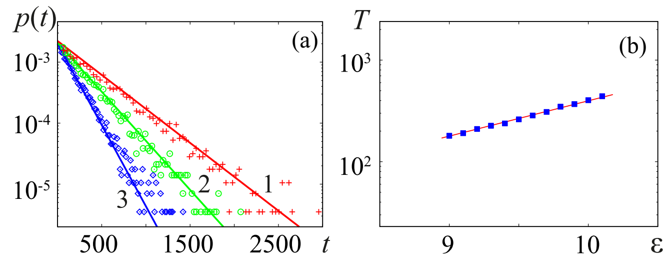

FIG. 4. (a) Normalized distributions of time interval lengths, when both Lorenz oscillators given by Eq. (3) are characterized by representation points being on equal sheets of attractors with their approximations by the exponential law obtained for fixed values of the coupling strength [plus signs (curve 1) correspond to $\varepsilon = 9.9$, circles (curve 2) correspond to $\varepsilon = 9.7$, and diamonds (curve 3) correspond to $\varepsilon = 9.3$]. (b) Mean duration of time intervals when phase trajectories of coupled systems are within equal sheets of chaotic attractors vs coupling parameter ε and its exponential approximation.

coupling strength ε_c is given in Fig. 3. Figure 3(a) illustrates the behavior of x coordinates of the drive $[x_1(t)]$ and response $[x_2(t)]$ systems. The dependence of the absolute value of the difference between x coordinates of the response and auxiliary systems, $\xi(t) = x_2(t) - x_a(t)$, is shown in Fig. 3(b).

According to the definition of GS and the concept of the auxiliary system approach, the synchronous behavior of the interacting unidirectionally coupled chaotic oscillators is characterized by the coincidence of the states of the response and auxiliary systems and, as a consequence, by the condition $\xi(t) \approx 0$. Therefore, a rapid growth of the difference between the states of the response and auxiliary systems, indicated in Fig. 3(b) by the vertical shadow rectangle, means the interruption of the laminar phase and the start of the turbulent (asynchronous) phase of motion for which $|\xi(t)|$ differs from zero.

One can see from Fig. 3 that the growing difference between the states of the response and auxiliary systems [Fig. 3(b)] is preceded by the divergence (jump) of the phase trajectories of the drive and response oscillators into different sheets of chaotic attractors [see Figs. 3(a), 3(c), and 3(d)]. This alteration of system dynamics leads to a dramatic sharp increase in the coupling term $\varepsilon(x_1 - x_2)$ which is responsible for the oscillator interaction. The value of the coupling term is rapidly increasing many times (in the case under consideration approximately by an order of magnitude), which may be considered and interpreted as a sudden powerful impulse leading the slave oscillator out of the steady synchronous (in the sense of GS) mode. Therefore, the divergence of the phase trajectories of interacting systems results in the destruction of the synchronous regime, and, as a consequence, the laminar phase is interrupted and the turbulent phase corresponding to the asynchronous motion starts. Due to the sufficiently large coupling strength, ε, the interacting systems tend to return to the synchronous state. However, since some time is needed for the oscillators to reach synchronization, the response and drive systems become synchronized again after some transients [see Figs. 3(a) and 3(b)]. Since the condition of the laminar phase defines a region on the full chaotic attractor and the laminar length corresponds to the residence time in this region, it can explain qualitatively the exponential character of the laminar phase length distribution.

Thus, we can conclude that the lengths of the laminar phases and time intervals when both the drive and response oscillators are characterized by the representation points being on equal sheets of chaotic attractors must not coincide (but should correlate) with each other. The divergence of the system phase trajectories to different attractor sheets, in turn, due to the dramatic increase of the coupling term magnitude gives rise to the asynchronous motion stage corresponding to the turbulent phase of the intermittent generalized synchronization regime. This fact results in a good agreement between distributions of the laminar phase lengths and durations of the time intervals when phase trajectories of the coupled systems are within equal sheets of chaotic attractors, as well as the dependencies of the mean values of these quantities on the coupling parameter ε (compare Figs. 2 and 4). Obviously, this finding allows us to use the proposed technique based on the consideration of the representation point locations to estimate carefully the statistical characteristics of the intermittent behavior below the generalized synchronization onset.

Thus, based on the results presented above, we can conclude that in the coupled nonlinear oscillators not only on-off intermittency can occur on the route to and from the

GS, but also another radically different mechanism called *jump intermittency* can take place. The former mechanism usually occurs for relatively small coupling strengths, whereas for stronger coupling the second type of transition takes place [30]. More precisely, for the systems with the complex topology of a chaotic attractor the "classical" mechanism of transition to and from generalized chaotic synchronization also exists, but it takes place for the smallest values of the coupling strength and is masked and suppressed by the second mechanism considered above and caused by two-sheeted attractor structure.

As mentioned above, the statistical characteristics of the revealed type of intermittency (jump intermittency) differ significantly from on-off intermittency. One can see from Figs. 2 and 4 that both the laminar phase distributions for the fixed value of the coupling strength ($N(\tau)$, $N(l)$) and the dependencies of the mean length of the laminar phase on the coupling parameter ε for the regime of jump intermittency are governed by the exponential law.

B. Theoretical background of the second type of intermittency near GS onset

The theoretical explanation of the intermittent behavior of coupled systems with two-sheeted topology of attractors consists in the consideration of the bistable system behavior in the presence of noise. Since laminar (synchronous) phases of motion in the vicinity of GS in this kind of systems are interrupted by the divergence of phase trajectories of the coupled oscillators on different sheets of attractors (see Sec. III A and Fig. 3), the observed dynamics can be treated as switches between two possible states (synchronous, when both phase trajectories are on equal sheets of attractors, and asynchronous, when the phase trajectories are on different sheets) induced by deterministic instability of dynamical chaos playing the same role as noise.

In this paper we suppose that such kind of behavior can be described with the help of the simplest model [Eqs. (6) and (7)]. This working hypothesis is grounded on the empirical thoughts so far, but will be proven below by the excellent agreement between the theoretical predictions deduced within the framework of the proposed model and data obtained with the help of numerical calculations.

Within the framework of the proposed model one of the minima of the potential function (7) is supposed to correspond to a synchronous (in the sense of GS) state, x_s, of the interacting systems, whereas the other one, x_a, refers to the asynchronous motion. Since an increase in the coupling strength enlarges epochs of the synchronous behavior, it is reasonable to assume that the coupling strength plays the role of asymmetry parameter b in the model system given by Eqs. (6) and (7). Having analyzed the model behavior (see [31]), one can find the residence time distribution (being the analog for the distribution of time interval lengths when coupled chaotic oscillators are characterized by representation points being on equal sheets of attractors) in the form

$$p(l) = \frac{1}{L}\exp\left(-\frac{l}{L}\right), \tag{10}$$

where $L = \langle l\rangle$ is the mean length of the residence times for the fixed values of the control parameters which can be estimated as

$$L \sim \exp\left(\frac{2U(x^*)}{D}\right)\int_{-\infty}^{x^*}\exp\left(-\frac{2U(\xi)}{D}\right)d\xi, \tag{11}$$

where x^* is the unstable equilibrium point separating two minima of the potential function given by Eq. (7) which, in turn, may be approximately simplified to the exponential law [31]

$$L \approx K\exp(\alpha b), \tag{12}$$

where K and α are constants and the asymmetry parameter b is supposed to be proportional to the coupling strength $b \sim \varepsilon$. Thus, one can see that the theoretical Eqs. (10) and (12) are in a good agreement with the numerically calculated data for the intermittent behavior of the unidirectionally coupled oscillators with two-sheeted chaotic attractors in the vicinity of GS (see Figs. 2 and 4).

Nevertheless, there is one important point to pay attention to. The exponential approximation of the dependence of the mean length L of the residence times on the coupling parameter ε (12) has been deduced in [31] under assumption that unstable equilibrium point x^* is approximately equal to the value of the asymmetry parameter b, i.e., $x^* \sim b$. It means that in immediate proximity to the catastrophe point $b_c = 2/(3\sqrt{3})$ corresponding to the boundary of generalized synchronization Eq. (12) may become less accurate. Although in the above considered case of the unidirectionally coupled Lorenz oscillators (3) this kind of accuracy is quite enough (see Figs. 2 and 4), for the high precision consideration of the system behavior in the very closest vicinity of the GS onset more careful analysis of model system (6) and (7) is required.

The coordinate of the unstable equilibrium point x^* separating two minima of the potential function $U(x)$ depends on the asymmetry parameter b as

$$x^*(b) = \frac{\cos\left(\frac{\varphi(b)}{3}\right) - \sqrt{3}\sin\left(\frac{\varphi(b)}{3}\right)}{\sqrt{3}}, \tag{13}$$

where $= \varphi(b)$ is connected with parameter b by

$$\tan\varphi = \frac{\sqrt{12-81b^2}}{9b}, \quad \text{with} \quad 0 < b \leqslant b_c = \frac{2}{3\sqrt{3}}. \tag{14}$$

To describe the system behavior in the closest vicinity of the catastrophe point b_c which corresponds to the disappearance of the potential function minimum x_a describing the asynchronous behavior of interacting oscillators (in the sense of generalized synchronization), and, as a consequence, to the onset of the generalized synchronization regime in the system with two-sheeted topology of attractors, one can decompose the relation (13) for the unstable point x^* in a Taylor series in the vicinity of the critical point b_c for $b = b_c + \delta$, where $\delta \leqslant 0$ and $|\delta| \ll b_c$:

$$x^* \approx \frac{1}{\sqrt{3}} - \frac{\sqrt{-\delta}}{\sqrt[4]{3}} + \frac{\delta}{6} - \frac{5(-\delta)^{3/2}}{24\sqrt[4]{27}} - \frac{\delta^2}{9\sqrt{3}} + O[-\delta]^{5/2}. \tag{15}$$

The potential function $U(x)$ calculated just below the catastrophe point b_c in the coordinate x^* corresponding to the local maximum, in turn, may be found as

$$U(x^*) \approx \frac{1}{12} + \frac{\delta}{\sqrt{3}} + \frac{2(-\delta)^{3/2}}{3\sqrt[4]{3}} + \frac{\delta^2}{12}. \tag{16}$$

Now, taking into account that the integral in Eq. (11) may be described adequately by the exponential approximation

$$\int_{-\infty}^{x^*} \exp\left(-\frac{2U(\xi)}{D}\right) d\xi \sim \exp(kb) \tag{17}$$

(see [31] for details), the dependence of the mean length of the residence times L on the deviation δ of the asymmetry parameter, b, from the critical point b_c may be written in the form

$$L \approx C\exp(k\delta) \times \exp\left[\frac{2}{D}\left(\frac{1}{12} + \frac{\delta}{\sqrt{3}} + \frac{2(-\delta)^{3/2}}{3\sqrt[4]{3}} + \frac{\delta^2}{12}\right)\right]. \tag{18}$$

The increase of the coupling strength ε between oscillators reduces the epochs of the asynchronous motion that in the considered model (6) and (7) is consistent with the extinction of the potential function minimum x_a (representing the stages of the asynchronous dynamics) with the growth of the asymmetry parameter b. The merging and disappearance of the extremum points x_a and x^* for $b = b_c$ in the model (6) and (7) corresponds to vanishing of the asynchronous stages of coupled oscillator dynamics at $\varepsilon = \varepsilon_c$. Having supposed that $(\varepsilon - \varepsilon_c) \sim (b - b_c)$, we can conclude that to apply the obtained theoretical prediction to coupled oscillators with two-sheeted topology of chaotic attractors one has to substitute $\beta(\varepsilon - \varepsilon_c)$ for δ in (18) where β is some constant.

Finally, one more important thing needs to be stressed. From the proposed theory one can see that the mean laminar phase length does not diverge at the onset of GS contrary to the well-studied cases of all known types of intermittency. Indeed, according to Eq. (18) the mean residence time length L does not tend to become infinite, but is limited by the value of $C\exp(1/6D)$ at critical point $\delta = 0$. This phenomenon can be explained by the fact that the transition to the generalized synchronization regime in the case of the oscillators with two-sheeted topology of chaotic attractors is determined by the collapse of the asynchronous epochs of the motion, whereas for all other types of intermittent dynamics the transition from intermittent to regular behavior is caused mainly by the unlimited increase of the laminar phase lengths.

C. Mutually coupled oscillators

To confirm and prove the generality of jump intermittency near the GS onset as well as to verify the theoretical predictions obtained above, let us consider two additional cases of the behavior of mutually coupled systems. The key feature of GS in bidirectionally coupled chaotic oscillators is the fundamental inapplicability of the auxiliary system approach to detect and study GS [28]. It was hitherto impossible to characterize intermittency near the GS threshold because there were no ways to detect and separate intervals of synchronous (laminar phase) and asynchronous (turbulent phase) behaviors. At the same time, the above-mentioned mechanism of the large rapid increase in the magnitude of the coupling term at the moments of the divergence of the phase trajectories in the different sheets of chaotic attractors resulting in the destruction of the synchronous motion seems to be actual also for the mutually coupled oscillators. Therefore, it can be used to estimate the synchronous and asynchronous phases of motion in the close vicinity of the generalized synchronization threshold, provided that this threshold is confirmed with the help of methods that remain operable in the case of mutual types of coupling.

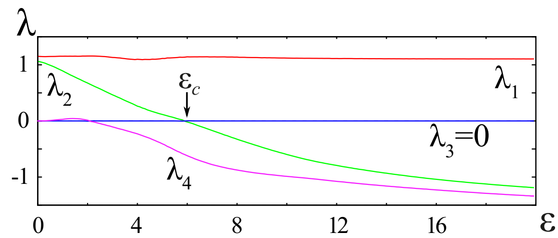

FIG. 5. Four largest Lyapunov exponents vs coupling strength ε in two mutually coupled Lorenz oscillators (4). The arrow indicates the onset of GS.

Thanks to the revealed mechanism of the destruction of the synchronous motion (due to the dramatic increase of the coupling term magnitude) and the scenario of the transition to and from GS in oscillators with the two-sheeted structure of a chaotic attractor and, as a consequence, the strong correlation between laminar phases of the coupled oscillators' motion and time intervals when phase trajectories of the interacting systems are within equal sheets of chaotic attractors, one can now accurately estimate statistical characteristics of the transitional behavior in the vicinity of GS, whereas the onset of GS can be found with the help of the Lyapunov exponent spectrum [32] and verified by means of the nearest neighbor method [4,27].

1. Lorenz oscillators with bidirectional coupling

Consider now intermittent GS in two mutually coupled Lorenz oscillators given by Eq. (4). The dependencies of the four largest Lyapunov exponents on the coupling strength ε are present in Fig. 5. Since the Lyapunov exponent spectrum of each Lorenz oscillator consists of one positive, one zero, and one negative Lyapunov exponents, for mutually coupled Lorenz oscillators the second positive Lyapunov exponent crosses zero and changes its sign from positive to negative at critical coupling strength point $\varepsilon_c = 5.9$ corresponding to the GS threshold. We also test both the presence of GS above ε_c and its absence below the critical point with the help of the nearest neighbor method. Therefore, the statistical characteristics for the residence times of the phase trajectory location on equal sheets of chaotic attractors are examined below this critical value of the coupling strength, ε_c. We select the coupling strength range $\varepsilon \in [4, 6]$ to study the system dynamics through the transition to GS.

One can easily see from Fig. 6(a) that for the mutually coupled Lorenz oscillators given by Eq. (4) the distributions of time interval lengths when both systems are characterized

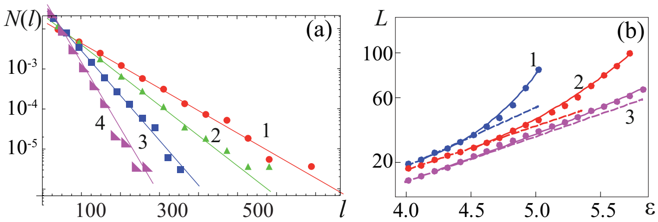

FIG. 6. (a) Normalized distributions of time interval lengths when both Lorenz oscillators given by Eq. (4) are characterized by representation points being on equal sheets of attractors with their exponential approximations obtained for fixed values of the coupling strength [circles (curve 1) correspond to $\varepsilon = 5.8$, up triangles (curve 2) correspond to $\varepsilon = 5.5$, squares (curve 3) correspond to $\varepsilon = 5.0$, and down triangles (curve 4) correspond to $\varepsilon = 4.5$]. (b) Mean durations of time intervals when phase trajectories of coupled systems are within equal sheets of chaotic attractors vs coupling parameter ε and their theoretical approximations (12) (dashed lines) and (18) (solid lines) obtained for three different values of control parameter r_1: $r_1 = 35$ (line 1), $r_1 = 37$ (line 2), and $r_1 = 45$ (line 3).

by representation points being on equal sheets of attractors are qualitatively the same as for the unidirectionally coupled systems given by Eq. (3) [compare with Fig. 4(a)].

We have also calculated the dependencies of the mean residence time of the phase trajectories located on the equal attractors' sheets, $L = \langle l \rangle$, on the coupling strength ε, for three different values of the control parameter r_1 to consider how the jump intermittency depends on the parameter mismatch between the coupled systems. One can see from Fig. 6(b) when the parameter mismatch is sufficient, i.e., the control parameter values (in our paper we have varied the control parameter r_1) differ noticeably from each other, that the dependence of the mean residence times L on the coupling strength ε is very close to the exponential law [see curve 3 in Fig. 6(b)]. This finding agrees well with the results obtained for two unidirectionally coupled Lorenz oscillators (3) considered above in Sec. III A.

With the decrease of the difference between control parameter values r_1 and r_2 of interacting systems (when, as a consequence, both oscillators become more and more equal) the dependence $L(\varepsilon)$ more and more deviates from the exponential law. Obviously, Eq. (12) becomes inapplicable to describe the intermittent behavior of the coupled systems with two-sheeted topology of chaotic attractors in the closest vicinity of the generalized synchronization regime onset, exactly in the same way as it was described above in Sec. III B, and we have to refine Eq. (18) obtained within the framework of the developed theory. Remarkably, in all cases (both completely identical oscillators and systems with different parameter mismatch) the theoretical relation Eq. (18) fits perfectly the calculated numerical data characterizing the intermittent behavior of mutually coupled Lorenz systems (4).

Therefore, we can conclude that both unidirectionally and mutually coupled Lorenz oscillators with two-sheeted topology of chaotic attractors exhibit (both for the identical and mismatched control parameters) the same type of transition to and from GS accompanied by the jump intermittency the statistical characteristics of which radically differ from on-off

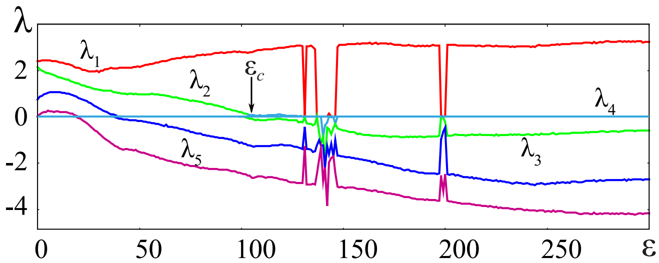

FIG. 7. Five largest Lyapunov exponents vs coupling strength ε for two mutually coupled four-dimensional chaotic oscillators given by Eq. (5). The arrow indicates the onset of GS.

intermittency observed in the vicinity of the onset of GS in the classical case.

2. Four-dimensional oscillators coupled mutually

In the next step in our paper, we consider the transitional behavior of the chaotic system with four-dimensional phase space and two-sheeted topology of its chaotic attractor [Eq. (2)] in the vicinity of GS. Again, the threshold of GS, $\varepsilon_c \approx 105$, is found with the help of the Lyapunov exponents spectrum (see Fig. 7) and verified by the nearest neighbor method. Below the critical coupling strength ε_c, we consider statistical characteristics for residence times of phase trajectory locations on equal sheets of chaotic attractors for the Chen coupled oscillators given by Eq. (5).

One can easily see from Fig. 8 that the statistical characteristics for two mutually coupled four-dimensional chaotic oscillators in the vicinity of the GS onset are exactly the same as for two Lorenz systems coupled either unidirectionally or bidirectionally. So, the distributions of time interval lengths, when two coupled chaotic oscillators given by Eq. (5) have representation points on equal sheets of attractors, obey an exponential law [see Fig. 8(a)]. Similarly, since the control parameters k_1 and k_2 of the coupled oscillators differ from each other sufficiently and one deals with the case of the large parameter mismatch, the dependence of the mean duration of

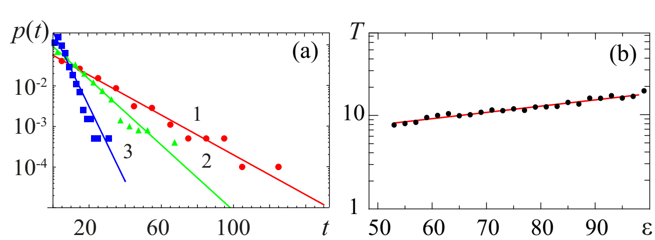

FIG. 8. (a) Normalized distributions of time interval lengths when two coupled chaotic oscillators each given by Eq. (5) are characterized by representation points being on equal sheets of attractors with their exponential approximations obtained for fixed values of coupling strength [circles (curve 1) correspond to $\varepsilon = 100$, up triangles (curve 2) correspond to $\varepsilon = 70$, and squares (curve 3) correspond to $\varepsilon = 50$]. (b) Mean duration of time intervals when phase trajectories of coupled systems are within equal sheets of chaotic attractors vs coupling parameter ε (circles) and its exponential approximation (solid line).

the residence times of phase trajectories can also be approximated by an exponential curve [Fig. 8(b)].

Thus, having examined the transitions to and from GS in both unidirectionally and mutually coupled oscillators with the complex two-sheeted structure of chaotic attractors, we conclude that we deal with the jump intermittency the statistical characteristics of which completely differ from ones revealed for well-known on-off intermittency taking place near the onset of chaotic GS in the case of transition for classical chaotic systems, like Rössler oscillators. The core mechanism resulting in the establishment of the synchronous regime (in the sense of GS) is explained by two-sheeted topology of chaotic attractors of interacting systems.

IV. CONCLUSION

In our paper, we have revealed the second type of intermittent generalized synchronization differing greatly from on-off intermittency previously observed in the vicinity of the GS onset. This type of intermittent behavior, called *jump intermittency*, accompanies the transition to and from GS in coupled oscillators with complex two-sheeted topology of chaotic attractors. Due to the complex attractor topology, the mechanism responsible for switches of phase trajectories of the coupled oscillators between locations at equal and different sheets of attractors suppresses the "traditional" scenario of the transition to and from GS (accompanied by on-off intermittency), which results in an increase in the critical coupling strength (corresponding to the GS onset) and the presently studied type of intermittent behavior. In our paper we have shown that jump intermittency takes place for the different mismatches of the control parameters of interacting systems, whereas the statistical characteristics of intermittent behavior below the onset of the generalized synchronization regime for the systems with two-sheeted topology of a chaotic attractor differ radically from the characteristics that are known for the classical case of intermittent generalized synchronization. Moreover, in this paper we have reported on the type of intermittency (namely, jump intermittency) for which the mean length of the laminar phases does not diverge at the critical point (contrary to all other known types of intermittent behavior) due to the different mechanism of transition connected with the collapse of the stable state being responsible for the asynchronous (from the point of view of GS) stage of motion.

The revealed mechanism governing the system dynamics in the vicinity of the generalized synchronization onset allows us to develop a technique based on the consideration of the representation point locations for estimation of the statistical characteristics of intermittent behavior of coupled oscillators with the two-sheeted topology of attractors. The developed technique gives results that are equal to ones obtained with the help of the usual method based on the auxiliary system approach in the case of the unidirectionally coupled oscillators and may be used efficiently for mutually coupled systems with complex topology of attractors for which up to now there were no known methods for laminar and turbulent epoch detection.

The theoretical expressions describing statistical characteristics of the revealed second type of intermittent GS are also given in this paper. The obtained theoretical predictions agree well with the results of numerical simulations of intermittent behavior near the generalized synchronization onset for both the unidirectionally and mutually coupled chaotic oscillators with two-sheeted topology of attractors. The fact that the phenomenon in chaotic deterministic systems is described by the stochastic model is not surprising since, on the one hand, the proposed stochastic model represents the bistable essence of the observed effect and, on the other hand, there is a known close relationship between the chaotic deterministic dynamics and the stochastic system behavior (see, e.g., [33–37]). Moreover, it is possible to explain the absence of the divergence of the mean length of the laminar phases through the transition to the GS regime in bistable systems by means of the notion of deterministic nonlinear dynamics. Indeed, the stable state, describing in the framework of the stochastic model the asynchronous regime, corresponds to a certain region in the phase space of coupled chaotic oscillators. Around the GS transition point the attractor transforms so it gets out of this region (that is clearly indicated by the sign change of the second positive Lyapunov exponent; see Figs. 1, 5, and 7) and no divergence will be observed.

We firmly believe that our results significantly expand the existing theoretical understanding of the mechanisms and properties of GS and the transitional behavior in the interacting systems with complex chaotic dynamics.

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