Correlations and responses for a system of n coupled linear oscillators with asymmetric interactions

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We focus on the asymmetry of the interaction in the optimal velocity (OV) model, which is a model of self-driven particles, and analytically investigate the effects of the asymmetry on the fluctuation-response relation, which is one of the remarkable relationships in statistical physics. By linearizing a modified OV model, i.e., the backward-looking optimal velocity model, which can easily control the magnitude of asymmetry in the interaction, we derive n coupled linear oscillators with asymmetric interactions. We analytically solve the equations of the n coupled linear oscillators and calculate the response and correlation functions. We find that the fluctuation response relation does not hold in the n coupled linear oscillators with asymmetric interactions. Moreover, as the magnitude of the asymmetry increases, the difference between the response and correlation functions increases .

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I. INTRODUCTION

A myriad of collective motions arise from groups of living organisms such as cells, animals, and human beings. Many of such collective motions are produced by self-organization, which is governed by local interactions between parts of a system. The topic of collective motions is important in nonequilibrium physics, and the properties of collective dynamics have been studied from the statistical physics perspective [1].

A mathematical model reproducing a moving cluster of many particles was introduced as a model for traffic flow in 1994 and called the optimal velocity (OV) model [2–4]. The OV model is a model of a nonequilibrium dissipative system with an asymmetric interaction. The asymmetric interaction means that a particle only interacts with the particle in front of it in the direction of motion. In the OV model, the asymmetry of the interaction is explicitly represented in the equation of motion. Varying the magnitude of the asymmetry, Nakayama *et al.* studied how this asymmetry affects the condition of generating a moving cluster [5]. Moreover, asymmetric interactions have been elucidated to be the central cause of traffic jams [6,7].

We address the following question: How does the asymmetry in the interaction of the OV model affect the physical nature? To answer this question, we focus on the fluctuation-response relation (FRR) [8,9]. We note that the FRR is the relation between the response function and the correlation function. When a system is in a steady-state equilibrium, where the detailed-balance condition is satisfied, we can define the temperature as the coefficient connecting the response and correlation functions [9,10]. In contrast, when a system is far from equilibrium, such a system trivially does not satisfy the FRR.

The FRR must be broken in a dissipative system with an asymmetric interaction when the force produced by the asymmetric interaction is not a conservative force. We are interested in studying how the asymmetry in the interaction of the OV model leads to the FRR violation. Thus, we use n coupled linear oscillators with asymmetric interactions as in the OV model. To elucidate the relation, we analytically calculate the linear response and correlation functions in this model. Moreover, we evaluate both functions in several cases and investigate the effects of asymmetry on the FRR violation.

In Sec. II, we introduce the n coupled oscillators by linearizing the OV model. In Sec. III, we solve the equations of motion of the n coupled oscillators and calculate the response function and correlation function. In Sec. IV, to discuss the effect of the asymmetry on the FRR violation, we compare the response and correlation functions. In Sec. V, we provide conclusions regarding the results and some perspectives.

II. MODEL

In this section, we introduce the n coupled oscillators. In Sec. II A, we review the extended OV model, i.e., the backward-looking OV (BL-OV) model [5]. In Sec. II B, we

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derive the n coupled linear oscillators with asymmetric interactions from the BL-OV model.

A. BL-OV model

We briefly review the BL-OV model [5], which can easily control the magnitude of asymmetry in the interaction rather than other many-particle models. In the original OV model, each particle interacts with only the particle in front of it. In the BL-OV model, each particle interacts with two particles in front of and behind it. The model with n particles is represented by a system of n equations of motion; the motion of the jth particle is governed by the following equation:

$$\frac{d^2x_j}{dt^2} = a\left\{[V_{\mathrm{F}}(\Delta x_{j+1}) + V_{\mathrm{B}}(\Delta x_j)] - \frac{dx_j}{dt}\right\}, \quad (j = 1, \dots, n), \tag{1}$$

where a is the "sensitivity," which represents the response speed. $\Delta x_{j+1} := x_{j+1} - x_j$ and $\Delta x_j := x_j - x_{j-1}$ are the distances to the particles in the front and the rear, respectively. V_{F} and V_{B} are called OV functions, which are expressed by $V_{\mathrm{F}}(x) = \alpha_{\mathrm{F}}[\tanh(x-\beta) + \Gamma]$ and $V_{\mathrm{B}}(x) = -\alpha_{\mathrm{B}}[\tanh(x-\beta) + \Gamma]$, respectively. α_{F} and α_{B} are positive constants that determine the interaction strengths; β determines the inflection point of tanh functions. Γ is a positive constant, with $\Gamma \equiv \tanh\beta$. In this study, the model employs periodic boundary conditions, $x_0 = x_n$ and $x_{n+1} = x_1$.

The BL-OV model has a homogeneous solution, where particles are uniformly distributed, $x_{j+1} - x_j = b$, $j = 1 \dots n$, and moving with the same velocity, $V_{\mathrm{F}}(b) + V_{\mathrm{B}}(b)$. We set the left-hand side of Eq. (1) to 0 and set the distance between neighbors to b. Then, the homogeneous solution of the jth particle is written as

$$\tilde{x}_j(t) = bj + [V_{\mathrm{F}}(b) + V_{\mathrm{B}}(b)]t. \tag{2}$$

B. n coupled linear oscillators with asymmetric interactions

We briefly review the derivation of the linearized BL-OV model. We write the position of each particle as the sum of the homogeneous solution Eq. (2) and a small deviation y_j, $x_j = \tilde{x}_j + y_j$. Substituting this equation into Eq. (1) and ignoring terms in $y_{j+1} - y_j$ or $y_j - y_{j-1}$ whose order is greater than one, we rewrite y_j as x_j. We obtain the n coupled linear oscillators with asymmetric interactions:

$$\frac{d^2x_j}{dt^2} = a\left\{[V'_{\mathrm{F}}(b)\Delta x_{j+1} + V'_{\mathrm{B}}(b)\Delta x_j] - \frac{dx_j}{dt}\right\}, \tag{3}$$

where x_j is the small deviation; $V'_{\mathrm{F}}(b) = \frac{dV_{\mathrm{F}}(x)}{dx}|_{x=b}$, $V'_{\mathrm{B}}(b) = \frac{dV_{\mathrm{B}}(x)}{dx}|_{x=b}$, $\Delta x_{j+1} \equiv x_{j+1} - x_j$, $\Delta x_j \equiv x_j - x_{j-1}$, and $x_n = x_0$. The homogeneous flow is stable for the small deviation under the following condition (the detailed process is shown in Appendix A):

$$a > 2\frac{[V'_{\mathrm{F}}(b) + V'_{\mathrm{B}}(b)]^2}{V'_{\mathrm{F}}(b) - V'_{\mathrm{B}}(b)}. \tag{4}$$

For convenience, we rewrite the constants as $k_{\mathrm{L}}/m = aV'_{\mathrm{F}}(b)$, $-k_{\mathrm{R}}/m = aV'_{\mathrm{B}}(b)$, and $\gamma/m = a$. We obtain the coupled oscillatorlike equation as follows:

$$m\frac{d^2x_j}{dt^2} = k_{\mathrm{L}}(x_{j+1} - x_j) - k_{\mathrm{R}}(x_j - x_{j-1}) - \gamma\frac{dx_j}{dt}. \tag{5}$$

In this equation, we control the magnitude of the asymmetry by changing the values of k_{L} and k_{R}. When we set $k_{\mathrm{L}} = k_{\mathrm{R}}$, we obtain the equation of damped coupled oscillators. Substituting $a = \frac{\gamma}{m}$, $V'_{\mathrm{F}}(b) = k_{\mathrm{L}}/\gamma$, and $V'_{\mathrm{B}}(b) = -k_{\mathrm{R}}/\gamma$ into Eq. (4), we transform the stability condition into the following:

$$\frac{\gamma^2}{m} > 2\frac{(k_{\mathrm{L}} - k_{\mathrm{R}})^2}{k_{\mathrm{L}} + k_{\mathrm{R}}}. \tag{6}$$

Hereafter, in this paper, we set m, γ, k_{L}, and k_{L} to satisfy Eq. (6).

In the next section, we calculate the response and correlation functions.

III. RESPONSE FUNCTION AND CORRELATION FUNCTION

To investigate the response and correlation functions, we add a small external force, f_j, and Gaussian white noise, R_j. In short, we consider the response and correlation function of following system of equations:

$$m\frac{d^2x_j}{dt^2} = k_{\mathrm{L}}(x_{j+1} - x_j) - k_{\mathrm{R}}(x_j - x_{j-1}) - \gamma\frac{dx_j}{dt} + R_j(t) + f_j(t), \tag{7}$$

where $R_j(t)$, $j = 1, \dots, n$, satisfies $\langle R_j(t)\rangle_0 = 0$ and $\langle R_j(t)R_k(s)\rangle_0 = 2\gamma k_{\mathrm{B}}T\delta_{j,k}\delta(t-s)$, with $\langle\cdot\rangle_0$ indicating the ensemble average without applying the external force. We suppose that each external force f_j is sufficiently small and expressed as

$$f_j(t) = \begin{cases} 0 & (t < 0), \\ f_j(t) & (t \geqslant 0). \end{cases}$$

We can solve Eq. (7) (the detailed process is given in Appendix B), and we obtain the solution for the given explicit number n by the following recursive calculations:

$$\begin{aligned} x_i(t) = &\frac{1}{n}\sum_{j=1}^{n}\sum_{k=1}^{n} c^{(j-1)(i-k)}\frac{\left[u_k(t_0) - \lambda_{\mathrm{B}}^{(j)}x_k(t_0)\right]}{\left(\lambda_{\mathrm{A}}^{(j)} - \lambda_{\mathrm{B}}^{(j)}\right)}e^{\lambda_{\mathrm{A}}^{(j)}(t-t_0)} \\ &- \frac{1}{n}\sum_{j=1}^{n}\sum_{k=1}^{n} c^{(j-1)(i-k)}\frac{\left[u_k(t_0) - \lambda_{\mathrm{A}}^{(j)}x_k(t_0)\right]}{\left(\lambda_{\mathrm{A}}^{(j)} - \lambda_{\mathrm{B}}^{(j)}\right)}e^{\lambda_{\mathrm{B}}^{(j)}(t-t_0)} \\ &+ \frac{1}{n}\sum_{j=1}^{n}\sum_{k=1}^{n} c^{(j-1)(i-k)}\int_{t_0}^{t} ds\frac{\left[R_k(s) + f_k(s)\right]e^{\lambda_{\mathrm{A}}^{(j)}(t-s)}}{m\left(\lambda_{\mathrm{A}}^{(j)} - \lambda_{\mathrm{B}}^{(j)}\right)} \\ &- \frac{1}{n}\sum_{j=1}^{n}\sum_{k=1}^{n} c^{(j-1)(i-k)}\int_{t_0}^{t} ds\frac{\left[R_k(s) + f_k(s)\right]e^{\lambda_{\mathrm{B}}^{(j)}(t-s)}}{m\left(\lambda_{\mathrm{A}}^{(j)} - \lambda_{\mathrm{B}}^{(j)}\right)}, \end{aligned} \tag{8}$$

where

$$\lambda_{\mathrm{A}}^{(j)} = \frac{-\gamma + \sqrt{\gamma^2 + 4mg(c^{j-1})}}{2m}, \tag{9}$$

$$\lambda_{\mathrm{B}}^{(j)} = \frac{-\gamma - \sqrt{\gamma^2 + 4mg(c^{j-1})}}{2m}. \tag{10}$$

The complex constant c and function $g(c^j)$, $j = 1, \dots, n-1$ are given by the following:

$$\begin{aligned}
c &:= e^{i\frac{2\pi}{n}},\\
g(c^0) &= g(1) = 0,\\
g(c) &= -(k_{\mathrm{L}} + k_{\mathrm{R}}) + k_{\mathrm{L}}c + k_{\mathrm{R}}c^{-1},\\
g(c^2) &= -(k_{\mathrm{L}} + k_{\mathrm{R}}) + k_{\mathrm{L}}c^2 + k_{\mathrm{R}}c^{-2},\\
&\vdots\\
g(c^j) &= -(k_{\mathrm{L}} + k_{\mathrm{R}}) + k_{\mathrm{L}}c^j + k_{\mathrm{R}}c^{-j}\\
&\vdots\\
g(c^{n-1}) &= -(k_{\mathrm{L}} + k_{\mathrm{R}}) + k_{\mathrm{L}}c^{-1} + k_{\mathrm{R}}c.
\end{aligned}$$

A. Response function

Using the solution of Eq. (7), we derive the response function of the velocity $u_j(t) = \frac{dx_j}{dt}(t)$. When we apply the external forces to the system, we can expect that the velocity $u_j(t)$ will respond to the given forces as

$$\langle u_j(t)\rangle_{\vec{f}} - \langle u_j\rangle_0 = \sum_{k=1}^{n} \int_{-\infty}^{t} ds\, \phi_{jk}(t-s) f_k(s), \tag{11}$$

where $\langle\cdot\rangle_{\vec{f}}$ represents the ensemble average with applying the external force $\vec{f}$; $\langle u_j\rangle_0$ represents the steady current of the jth particle; and ϕ_{jk} is the response function for the velocity of the jth particle when we apply the external force to the kth particle.

When we set the external force to $\vec{f} = (0, \dots, 0, f_k(t), 0 \dots, 0)^{\top}$, we have the following:

$$\langle u_j(t)\rangle_{f_k} - \langle u_j\rangle_0 = \int_{t_0}^{t} ds f_k(s)\phi_{jk}(t-s),$$

where $\langle\cdot\rangle_{f_k}$ represents the ensemble average when applying the external force $\vec{f} = (0, \dots, 0, f_k(t), 0 \dots, 0)^{\top}$. With such an external force, the ensemble average of u_j is obtained as

$$\begin{aligned}
&\langle u_j(t)\rangle_{f_k} - \langle u_j\rangle_0\\
&= \frac{1}{n}\sum_{p=1}^{n}\sum_{q=1}^{n} \delta_{qk} c^{(p-1)(j-q)} \lambda_{\mathrm{A}}^{(p)} \int_{t_0}^{t} ds \frac{f_q(s) e^{\lambda_{\mathrm{A}}^{(p)}(t-s)}}{m\big(\lambda_{\mathrm{A}}^{(p)} - \lambda_{\mathrm{B}}^{(p)}\big)}\\
&\quad - \frac{1}{n}\sum_{p=1}^{n}\sum_{q=1}^{n} \delta_{qk} c^{(p-1)(j-q)} \lambda_{\mathrm{B}}^{(p)} \int_{t_0}^{t} ds \frac{f_q(s) e^{\lambda_{\mathrm{B}}^{(p)}(t-s)}}{m\big(\lambda_{\mathrm{A}}^{(p)} - \lambda_{\mathrm{B}}^{(p)}\big)}.
\end{aligned} \tag{12}$$

Comparing Eqs. (11) and (12), we obtain the following response function for u_j with f_k (we show the detailed calculation in Appendix C):

$$\phi_{jk}(t) = \sum_{p=1}^{n} \frac{c^{(p-1)(j-k)}}{nm}\left[\frac{\lambda_{\mathrm{A}}^{(p)} e^{\lambda_{\mathrm{A}}^{(p)} t} - \lambda_{\mathrm{B}}^{(p)} e^{\lambda_{\mathrm{B}}^{(p)} t}}{\lambda_{\mathrm{A}}^{(p)} - \lambda_{\mathrm{B}}^{(p)}}\right]. \tag{13}$$

B. Correlation function

We derive a correlation function for the velocity. The correlation function is defined as

$$C_{jk}(t-s) = \langle (u_j(t) - \bar{u}_j)(u_k(s) - \bar{u}_k)\rangle_0,$$

where $\bar{u}_j = \langle u_j\rangle_0$ and $\bar{u}_k = \langle u_k\rangle_0$. In the case of $s \leqslant t$, we obtain the explicit form of the correlation function for the linearized BL-OV model (we show the detailed calculation in Appendix D):

$$\begin{aligned}
C_{jk}(t-s) &= \frac{1}{2\gamma}\frac{2\gamma k_{\mathrm{B}}T}{n^2}\frac{1}{m}\sum_{p=1}^{n} c^{(p-1)(j-o)}\left[\frac{\lambda_{\mathrm{A}}^{(p)} e^{\lambda_{\mathrm{A}}^{(p)}(t-s)} - \lambda_{\mathrm{B}}^{(p)} e^{\lambda_{\mathrm{B}}^{(p)}(t-s)}}{\lambda_{\mathrm{A}}^{(p)} - \lambda_{\mathrm{B}}^{(p)}}\right] + \frac{2\gamma k_{\mathrm{B}}T}{n^2}\frac{1}{m^2}\sum_{p=1}^{n}\sum_{q\neq j}\frac{c^{(p-1)(j-k)}}{\big(\lambda_{\mathrm{A}}^{(p)} - \lambda_{\mathrm{B}}^{(p)}\big)}\\
&\quad\times\left[\frac{-\lambda_{\mathrm{A}}^{(p)}\lambda_{\mathrm{A}}^{(p)} e^{\lambda_{\mathrm{A}}^{(p)}(t-s)}}{\big(\lambda_{\mathrm{A}}^{(p)} + \lambda_{\mathrm{A}}^{(q)}\big)\big(\lambda_{\mathrm{A}}^{(p)} + \lambda_{\mathrm{B}}^{(q)}\big)} + \frac{\lambda_{\mathrm{B}}^{(p)}\lambda_{\mathrm{B}}^{(p)} e^{\lambda_{\mathrm{B}}^{(p)}(t-s)}}{\big(\lambda_{\mathrm{B}}^{(p)} + \lambda_{\mathrm{A}}^{(q)}\big)\big(\lambda_{\mathrm{B}}^{(p)} + \lambda_{\mathrm{B}}^{(q)}\big)}\right] + \frac{2\gamma k_{\mathrm{B}}T}{n^2}\frac{1}{m^2}\sum_{p=1}^{n}\sum_{q=1}^{n}\sum_{l\neq k}\frac{c^{(p-1)(j-l)+(q-1)(k-l)}}{\big(\lambda_{\mathrm{A}}^{(p)} - \lambda_{\mathrm{B}}^{(p)}\big)}\\
&\quad\times\left[\frac{-\lambda_{\mathrm{A}}^{(p)}\lambda_{\mathrm{A}}^{(p)} e^{\lambda_{\mathrm{A}}^{(p)}(t-s)}}{\big(\lambda_{\mathrm{A}}^{(p)} + \lambda_{\mathrm{A}}^{(q)}\big)\big(\lambda_{\mathrm{A}}^{(p)} + \lambda_{\mathrm{B}}^{(q)}\big)} + \frac{\lambda_{\mathrm{B}}^{(p)}\lambda_{\mathrm{B}}^{(p)} e^{\lambda_{\mathrm{B}}^{(p)}(t-s)}}{\big(\lambda_{\mathrm{B}}^{(p)} + \lambda_{\mathrm{A}}^{(q)}\big)\big(\lambda_{\mathrm{B}}^{(p)} + \lambda_{\mathrm{B}}^{(q)}\big)}\right].
\end{aligned} \tag{14}$$

For the case of $t \leqslant s$, we obtain the correlation function in the same form (we show the explicit form Eq. (D4) in Appendix D).

Comparing the response function and the first term of the right-hand side of the correlation function Eq. (14), we find that these terms are the same except for the constant multiplier. In other words, we can express the correlation function as

$$\begin{aligned}
C_{jk}(t-s) &= \frac{k_{\mathrm{B}}T}{n}\phi_{jk}(t-s) + \frac{2\gamma k_{\mathrm{B}}T}{n^2}\frac{1}{m^2}\sum_{p=1}^{n}\sum_{q\neq k}\frac{c^{(p-1)(j-k)}}{\big(\lambda_{\mathrm{A}}^{(p)} - \lambda_{\mathrm{B}}^{(p)}\big)}\left[\frac{-\lambda_{\mathrm{A}}^{(p)}\lambda_{\mathrm{A}}^{(p)} e^{\lambda_{\mathrm{A}}^{(p)}|t-s|}}{\big(\lambda_{\mathrm{A}}^{(p)} + \lambda_{\mathrm{A}}^{(q)}\big)\big(\lambda_{\mathrm{A}}^{(p)} + \lambda_{\mathrm{B}}^{(q)}\big)} + \frac{\lambda_{\mathrm{B}}^{(p)}\lambda_{\mathrm{B}}^{(p)} e^{\lambda_{\mathrm{B}}^{(p)}|t-s|}}{\big(\lambda_{\mathrm{B}}^{(p)} + \lambda_{\mathrm{A}}^{(q)}\big)\big(\lambda_{\mathrm{B}}^{(p)} + \lambda_{\mathrm{B}}^{(q)}\big)}\right]\\
&\quad + \frac{2\gamma k_{\mathrm{B}}T}{n^2}\frac{1}{m^2}\sum_{p=1}^{n}\sum_{q=1}^{n}\sum_{l\neq k}\frac{c^{(p-1)(j-l)+(q-1)(k-l)}}{\big(\lambda_{\mathrm{A}}^{(p)} - \lambda_{\mathrm{B}}^{(p)}\big)}\left[\frac{-\lambda_{\mathrm{A}}^{(p)}\lambda_{\mathrm{A}}^{(p)} e^{\lambda_{\mathrm{A}}^{(p)}|t-s|}}{\big(\lambda_{\mathrm{A}}^{(p)} + \lambda_{\mathrm{A}}^{(q)}\big)\big(\lambda_{\mathrm{A}}^{(p)} + \lambda_{\mathrm{B}}^{(q)}\big)} + \frac{\lambda_{\mathrm{B}}^{(p)}\lambda_{\mathrm{B}}^{(p)} e^{\lambda_{\mathrm{B}}^{(p)}|t-s|}}{\big(\lambda_{\mathrm{B}}^{(p)} + \lambda_{\mathrm{A}}^{(q)}\big)\big(\lambda_{\mathrm{B}}^{(p)} + \lambda_{\mathrm{B}}^{(q)}\big)}\right].
\end{aligned} \tag{15}$$

In the case of $k_L = k_R$, the correlation function is reduced to the FRR:

$$C_{jk}(t-s) = k_B T \sum_{p=1}^{n} \frac{c^{(p-1)(j-k)}}{nm} \left[\frac{\lambda_A^{(p)} e^{\lambda_A^{(p)}|t-s|} - \lambda_B^{(p)} e^{\lambda_B^{(p)}|t-s|}}{\lambda_A^{(p)} - \lambda_B^{(p)}} \right]$$
$$= k_B T \phi_{jk}(t-s).$$

When the interaction is symmetric, by comparing this correlation function and response function Eq. (13), we can confirm that the FRR is satisfied as a matter of course (we show the detailed calculation in Appendix E). From relation Eq. (15), we find that the FRR is violated in the asymmetric case. In Sec. IV, we show that the magnitude of the asymmetry increases the difference between correlations and responses.

IV. EFFECT OF ASYMMETRY ON THE FRR

From Eq. (15), the magnitude of the asymmetry and the number of particles affect the difference between correlations and responses. Thus, we evaluate these effects in this section.

Plotting the response and correlation functions, we show the time variation of the response function Eq. (13) and correlation function Eq. (14). We consider the case in which the observed particle is the same as a particle to which an external force is applied. Hereafter, we set $m = 1$, $\gamma = 1.5$, and $k_B T = 1$ for convenience. Moreover, because velocities u_i are real observables, we neglect the imaginary part of the response and correlation functions in this section.

In Fig. 1, we present the differences between the response and correlation functions considering the magnitude of the asymmetry. Comparing Figs. 1(a) and 1(b), we see that as the magnitude of the asymmetry increases, the difference between the response and correlation functions increases. In addition, we find oscillations in the response and correlation functions. These oscillatory behaviors should originate from the oscillation of the force caused by the interactions.

We show the effect of asymmetry in the frequency domain. We define the Fourier transform of the response and correlation functions as $\hat{\phi}_{11}(l) := \sum_{k=1}^{n} \exp[-ik\omega(l)]\phi_{11}(t)$ and $\hat{C}_{11}(l) := \sum_{k=1}^{n} \exp[-ik\omega(l)]C_{11}(t)$, where $t = kh$, $k = 1, \dots, N$, and $\omega(l) := \frac{2\pi l}{N}$ (h represents the time step). As seen in Figs. 2(a) and 2(b), the difference between correlations and responses is maximized in the low frequency range. Moreover, as the number of particles increases, the frequency at which the difference is maximized becomes lower.

V. DISCUSSION

In the present study, we investigated the effects of the asymmetry on the interaction of the OV model using n coupled linear oscillators with asymmetric interactions. We evaluated the response function and the correlation function in the n-body case. We found that the asymmetric interaction causes FRR violation. We also numerically compared the response and correlation functions for several numbers of particles and magnitudes of the asymmetry. Moreover, comparing the response and correlation functions in the frequency domain, we found that the difference between the correlation and response is maximized at low frequencies.

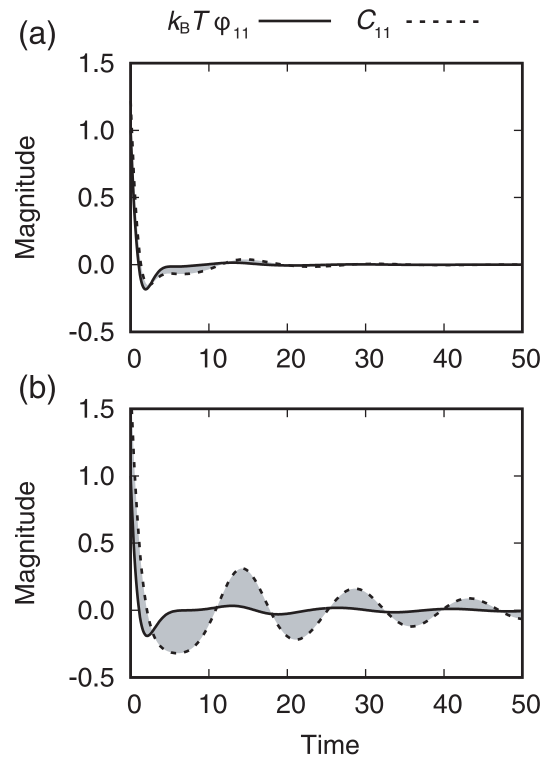

FIG. 1. Time evolution of response function $\Phi_{11}(t)$ and correlation function $C_{11}(t)$ for $k_L = 1.0$, $n = 9$, and $k_R = 0.25$ and 0.05. Solid and dashed curves represent the response function ϕ_{11} and correlation function C_{11}, respectively. Gray regions represent the difference between the response and correlation. (a) $k_R = 0.25$. (b) $k_R = 0.05$.

We should discuss the FRR violation from physical viewpoints. Harada and Sasa provided a physical justification for the FRR violation [10,11]. In Ref. [10], Harada and Sasa showed that the FRR violation is related to the energy dissipation J_i defined by Ref. [12]: $\langle J_i(t)\rangle_0 = \gamma_i\{\langle u_i\rangle_0^2 + \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}[\hat{C}_{ii}(\omega) - 2k_B T \hat{\phi}'_{ii}(\omega)]\}$. Such a relation between the FRR violation and energy dissipation is beneficial for understanding the nature of the asymmetric interaction. Even if applying the original definition of energy dissipation, $J_i(t)\Delta t \equiv \int_t^{t+\Delta t}[\gamma_i u_i(s) - R_i(s)] \circ dx_i(s)$, which is difficult for the dissipative system with the asymmetric interaction, we may be able to find another definition of energy dissipation that corresponds to the FRR violation. Moreover, we suppose that such a definition of energy dissipation will help us understand the physical mechanism of pattern formation in the dissipative system with the asymmetric interaction. Since a generalized relation connecting the response function due to a small perturbation and a suitable correlation exists [13], we may be able to find such a connecting relation.

In the present study, we focused only on the asymmetry and considered only the trivial steady-state solution. However, the equation of motion of the BL-OV model (and the OV model) has a jam flow solution that represents the dynamic state in which particles form a moving cluster. In previous studies [6,7,14–16], the analysis focused on the jam flow elucidated the relation between the pattern formation and the asymmetry

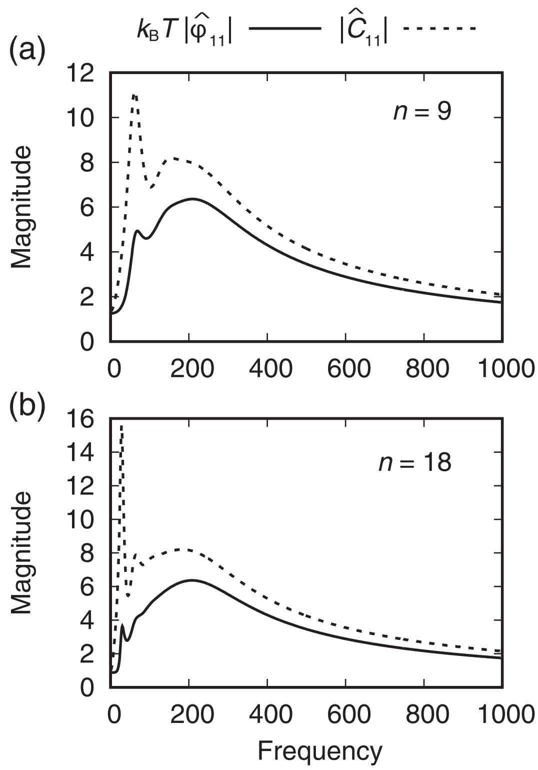

FIG. 2. Fourier transform of the response and correlation functions. Solid and dashed curves are amplitudes of the response and correlation functions, respectively. We set $k_L = 1$ and $k_R = 0.25$. (a) $n = 9$. (b) $n = 18$.

of the interaction. Therefore, investigating the FRR based on the jam flow solution may be worthwhile. Moreover, in the nonequilibrium states such as jam flow, we should also use the fluctuation theorem to clarify between the asymmetry and pattern [17,18].

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APPENDIX A: THE STABILITY CONDITION OF THE HOMOGENEOUS FLOW

We briefly review the stability condition of the homogeneous flow in the BL-OV model [5]. We obtained the linearized BL-OV model Eq. (3).

We use the Fourier expansion for calculating the stability condition of the variations y_j. Then, the variation is expressed as

$$\hat{x}_k(t) := \sum_{j=l}^{n} \exp[-ik\theta(l)]x_l(t),$$

$$x_l(t) := \frac{1}{n}\sum_{k=1}^{n} \exp[ik\theta(l)]\hat{x}_k(t),$$

where $k = 1, \dots, n$ and $\theta(l) := \frac{2\pi l}{n}$. We express the mode as $\hat{x}_k(t) = \exp[i\omega(k)t]$. Substituting these equations into Eq. (3), we obtain the following:

$$\frac{1}{n}\sum_{k=1}^{n} e^{i\theta(j)k+i\omega(k)t}[-\omega^2(k)]$$
$$= \sum_{k=1}^{n} e^{i\theta(j)k+i\omega(k)t}$$
$$\times a[V_F'(b)(e^{ik\theta(1)} - 1) + V_B'(b)(1 - e^{-ik\theta(1)}) - i\omega(k)].$$

Given the orthogonality of the Fourier basis, we find the equation for $\omega(k)$:

$$\omega^2 - ia\omega + a[V_F'(e^{i\theta} - 1) + V_B'(1 - e^{-i\theta})] = 0,$$

where $V_F' \equiv V_F'(b)$, $V_B' \equiv V_B'(b)$, $\theta \equiv k\theta(1)$, and $\omega := \omega(k)$. The solution of ω is obtained as

$$\omega = \frac{ia}{2}\left\{1 \pm \sqrt{1 + \frac{4}{a}[V_F'(e^{i\theta} - 1) + V_B'(1 - e^{-i\theta})]}\right\}. \tag{A1}$$

When the imaginary part of ω is negative, the mode $\hat{x}_k(t)$ increases with the lapse of time. Therefore, the stability condition is given as $\Im(\omega) > 0$.

We note that $\sqrt{a + ib}$, $(a, b \in \mathbf{R})$ can be transformed into $\sqrt{a + ib} = A + iB$, $(a, b, A, B \in \mathbf{R})$ as follows:

$$A = \pm\sqrt{\frac{a + \sqrt{a^2 + b^2}}{2}},$$
$$B = \pm\sqrt{\frac{-a + \sqrt{a^2 + b^2}}{2}}. \tag{A2}$$

Using Eq. (A1), we can rewrite the stability condition as $1 \pm A > 0$. From Eqs. (A1) and (A2), we can express A as:

$$A = \pm\Big[\frac{1}{2a}\Big(a + 4(\cos\theta - 1)(V_F' - V_B')$$
$$+\big\{a^2 + 8a(V_F' - V_B')(\cos\theta - 1)$$
$$- 32(\cos\theta - 1)(V_F' - V_B')^2$$
$$- 16\big[(V_F' + V_B')^2 - (V_F' - V_B')^2\big]$$
$$-64V_F'V_B'\cos^2\theta\big\}^{\frac{1}{2}}\Big)\Big]^{\frac{1}{2}}.$$

As a result, we find the stability condition as follows:

$$a > (1 + \cos\theta)\frac{[V_F'(b) + V_B'(b)]^2}{V_F'(b) - V_B'(b)}, \tag{A3}$$

Considering that $-1 \leqslant \cos\theta \leqslant 1$, we can rewrite the stability condition as

$$a > 2\frac{[V_F'(b) + V_B'(b)]^2}{V_F'(b) - V_B'(b)}. \tag{A4}$$

APPENDIX B: SOLUTION OF THE LINEARIZED BL-OV MODEL

We rewrite the equation of motion in vector form:

$$m\frac{d^2\vec{x}}{dt^2}(t) = \mathbf{M}\vec{x}(t) - \gamma\frac{d\vec{x}}{dt}(t) + \vec{R}(t), \tag{B1}$$

where

$$\vec{x}(t) := \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix}, \quad \vec{R}(t) := \begin{pmatrix} R_1(t) + f_1(t) \\ R_2(t) + f_2(t) \\ \vdots \\ R_n(t) + f_n(t) \end{pmatrix},$$

$$\mathbf{M} := \begin{pmatrix} -k_\mathrm{L} - k_\mathrm{R} & k_\mathrm{L} & 0 & 0 & k_\mathrm{R} \\ k_\mathrm{R} & -k_\mathrm{L} - k_\mathrm{R} & k_\mathrm{L} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \vdots & & \vdots \\ k_\mathrm{L} & 0 & \cdots & k_\mathrm{R} & -k_\mathrm{L} - k_\mathrm{R} \end{pmatrix}.$$

To solve Eq. (B1), we diagonalize the matrix $\mathbf{M}$ and derive the eigenvalues along with the corresponding eigenvectors. We note that $\mathbf{M}$ is a circulant matrix [19].

From the properties of the circulant matrix, we can easily find the eigenvalues,

$$g(1), g(c), g(c^2), \dots, g(c^{n-1}),$$

where $c, g(1), g(c), \dots, g(c^j), \dots, g(c^{n-1}), j = 1, \dots, n-1$, are given as

$$\begin{aligned}
c &:= e^{i\frac{2\pi}{n}}, \\
g(c^0) &= g(1) = 0, \\
g(c) &= -(k_\mathrm{L} + k_\mathrm{R}) + k_\mathrm{L} c + k_\mathrm{R} c^{n-1} \\
&= -(k_\mathrm{L} + k_\mathrm{R}) + k_\mathrm{L} c + k_\mathrm{R} c^{-1}, \\
g(c^2) &= -(k_\mathrm{L} + k_\mathrm{R}) + k_\mathrm{L} c^2 + k_\mathrm{R} c^{2(n-1)} \\
&= -(k_\mathrm{L} + k_\mathrm{R}) + k_\mathrm{L} c^2 + k_\mathrm{R} c^{-2}, \\
&\vdots \\
g(c^j) &= -(k_\mathrm{L} + k_\mathrm{R}) + k_\mathrm{L} c^j + k_\mathrm{R} c^{j(n-1)} \\
&= -(k_\mathrm{L} + k_\mathrm{R}) + k_\mathrm{L} c^j + k_\mathrm{R} c^{-j}, \\
&\vdots \\
g(c^{n-1}) &= -(k_\mathrm{L} + k_\mathrm{R}) + k_\mathrm{L} c^{(n-1)} + k_\mathrm{R} c^{(n-1)^2} \\
&= -(k_\mathrm{L} + k_\mathrm{R}) + k_\mathrm{L} c^{-1} + k_\mathrm{R} c.
\end{aligned}$$

A square matrix $\mathbf{P}$, whose columns are the n independent eigenvectors of $\mathbf{M}$, is obtained as

$$\mathbf{P} = \frac{1}{\sqrt{n}} \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & c & c^2 & \cdots & c^{(n-1)} \\ 1 & c^2 & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & \vdots \\ 1 & c^{(n-1)} & \cdots & \cdots & c^{(n-1)^2} \end{pmatrix}. \tag{B2}$$

Moreover, the inverse of matrix $\mathbf{P}$ can be expressed as

$$\begin{aligned}
\mathbf{P}^{-1} &= \frac{1}{\sqrt{n}} \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & c^{-1} & c^{-2} & \cdots & c^{-(n-1)} \\ 1 & c^{-2} & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & \vdots \\ 1 & c^{-(n-1)} & \cdots & \cdots & c^{-(n-1)^2} \end{pmatrix} \\
&= \frac{1}{\sqrt{n}} (\vec{p}_1 \quad \vec{p}_2 \quad \cdots \quad \cdots \quad \vec{p}_n).
\end{aligned}$$

Using these matrices, we can rewrite Eq. (B1) as

$$m\frac{d^2\mathbf{P}^{-1}\vec{x}}{dt^2}(t) = \mathbf{P}^{-1}\mathbf{M}\mathbf{P}\mathbf{P}^{-1}\vec{x}(t) - \gamma\frac{d\mathbf{P}^{-1}\vec{x}}{dt}(t) + \mathbf{P}^{-1}\mathbf{R}(t). \tag{B3}$$

1. Solution of the system of mode equations

Next, we define new variables as follows:

$$q_i(t) = \frac{1}{\sqrt{n}}\vec{p}_i \cdot \vec{x}(t), \quad i = 1, \dots, n. \tag{B4}$$

From Eq. (B3), the equation of $q_i(t)$ can be expressed as

$$m\frac{d^2 q_i}{dt}(t) = g(c^{i-1})q_i(t) - \gamma\frac{dq_i}{dt}(t) + R_i^q(t), \tag{B5}$$

where R_i^q is $R_i^q(t) = \vec{p}_i \cdot \vec{R}(t),\ i = 1, 2, \dots, n$.

We use variation of parameters and solve the equation for each mode. First, we consider the corresponding homogeneous equation of Eq. (B5):

$$m\frac{d^2 q_i}{dt}(t) = g(c^{i-1})q_i(t) - \gamma\frac{dq_i}{dt}(t). \tag{B6}$$

We obtain the homogeneous solution as follows:

$$\begin{aligned}
q_{i,1}(t) &= e^{\lambda_\mathrm{A}^{(i)} t}, \\
q_{i,2}(t) &= e^{\lambda_\mathrm{B}^{(i)} t}, \\
\lambda_\mathrm{A}^{(i)} &= \frac{-\gamma + \sqrt{\gamma^2 + 4mg(c^{i-1})}}{2m}, \qquad \text{(B7)} \\
\lambda_\mathrm{B}^{(i)} &= \frac{-\gamma - \sqrt{\gamma^2 + 4mg(c^{i-1})}}{2m}. \qquad \text{(B8)}
\end{aligned}$$

Next, we assume that a particular solution to the nonhomogeneous equation is given by

$$q_{i,p}(t) = A_i(t)q_{i,1}(t) + B_i(t)q_{i,2}(t), \tag{B9}$$

where $A_i(t)$ is a differential function satisfying

$$A_i'(t)q_{i,1}(t) + B_i'(t)q_{i,2}(t) = 0, \tag{B10}$$

where $A_i'(t) = \frac{dA_i}{dt}(t)$ and $B_i'(t) = \frac{dB_i}{dt}(t)$. Substituting Eq. (B9) into Eq. (B5) and applying Eq. (B10), we find

$$m\left[\lambda_\mathrm{A}^{(i)} A_i'(t) e^{\lambda_\mathrm{A}^{(i)} t} + \lambda_\mathrm{B}^{(i)} B_i'(t) e^{\lambda_\mathrm{B}^{(i)} t}\right] = R_i^q(t). \tag{B11}$$

Using condition Eq. (B10), we obtain the following system of equations:

$$A_i'(t) = \frac{R_i^q(t)e^{-\lambda_A^{(i)}t}}{m(\lambda_A^{(i)} - \lambda_B^{(i)})},$$
$$B_i'(t) = -\frac{R_i^q(t)e^{-\lambda_B^{(i)}t}}{m(\lambda_A^{(i)} - \lambda_B^{(i)})}.$$

We integrate these equations and find the solutions as follows:

$$A_i(t) = A_i(t_0) + \int_{t_0}^{t} ds \frac{R_i^q(s)e^{-\lambda_A^{(i)}s}}{m(\lambda_A^{(i)} - \lambda_B^{(i)})},$$
$$B_i(t) = B_i(t_0) - \int_{t_0}^{t} ds \frac{R_i^q(s)e^{-\lambda_B^{(i)}s}}{m(\lambda_A^{(i)} - \lambda_B^{(i)})}.$$

Substituting these solutions into Eq. (B9), we obtain the solution:

$$q_i(t) = A_i(t_0)e^{\lambda_A^{(i)}t} + B_i(t_0)e^{\lambda_B^{(i)}t} + \int_{t_0}^{t} ds \frac{R_i^q(s)e^{\lambda_A^{(i)}(t-s)}}{m(\lambda_A^{(i)} - \lambda_B^{(i)})} - \int_{t_0}^{t} ds \frac{R_i^q(s)e^{\lambda_B^{(i)}(t-s)}}{m(\lambda_A^{(i)} - \lambda_B^{(i)})}, \tag{B12}$$

where we rewrite $q_{i,p}$ as q_i for convenience. Moreover, we differentiate $q_i(t)$ to obtain

$$q_i'(t) = \lambda_A^{(i)}A_i(t_0)e^{\lambda_A^{(i)}t} + \lambda_B^{(i)}B_i(t_0)e^{\lambda_B^{(i)}t} + \frac{R_i^q(t)e^{\lambda_A^{(i)}(t-t)}}{m(\lambda_A^{(i)} - \lambda_B^{(i)})} + \lambda_A^{(i)}\int_{t_0}^{t} ds \frac{R_i^q(s)e^{\lambda_A^{(i)}(t-s)}}{m(\lambda_A^{(i)} - \lambda_B^{(i)})} - \frac{R_i^q(t)e^{\lambda_B^{(i)}(t-t)}}{m(\lambda_A^{(i)} - \lambda_B^{(i)})} - \lambda_B^{(i)}\int_{t_0}^{t} ds \frac{R_i^q(s)e^{\lambda_B^{(i)}(t-s)}}{m(\lambda_A^{(i)} - \lambda_B^{(i)})}. \tag{B13}$$

We set t to t_0 and transform Eqs. (B12) and (B13) into the following:

$$q_i(t_0) = A_i(t_0)e^{\lambda_A^{(i)}t_0} + B_i(t_0)e^{\lambda_B^{(i)}t_0},$$
$$q_i'(t_0) = \lambda_A^{(i)}A_i(t_0)e^{\lambda_A^{(i)}t_0} + \lambda_B^{(i)}B_i(t_0)e^{\lambda_B^{(i)}t_0}.$$

Using these equations, we express $A_i(t_0)$ and $B_i(t_0)$ as

$$A_i(t_0) = \frac{q_i'(t_0) - \lambda_B^{(i)}q_i(t_0)}{(\lambda_A^{(i)} - \lambda_B^{(i)})}e^{-\lambda_A^{(i)}t_0},$$
$$B_i(t_0) = -\frac{q_i'(t_0) - \lambda_A^{(i)}q_i(t_0)}{(\lambda_A^{(i)} - \lambda_B^{(i)})}e^{-\lambda_B^{(i)}t_0}.$$

We finally obtain

$$q_i(t) = \frac{q_i'(t_0) - \lambda_B^{(i)}q_i(t_0)}{(\lambda_A^{(i)} - \lambda_B^{(i)})}e^{\lambda_A^{(i)}(t-t_0)} - \frac{q_i'(t_0) - \lambda_A^{(i)}q_i(t_0)}{(\lambda_A^{(i)} - \lambda_B^{(i)})}e^{\lambda_B^{(i)}(t-t_0)} + \int_{t_0}^{t} ds \frac{R_i^q(s)e^{\lambda_A^{(i)}(t-s)}}{m(\lambda_A^{(i)} - \lambda_B^{(i)})} - \int_{t_0}^{t} ds \frac{R_i^q(s)e^{\lambda_B^{(i)}(t-s)}}{m(\lambda_A^{(i)} - \lambda_B^{(i)})}, \tag{B14}$$

$$q_i'(t) = \frac{q_i'(t_0) - \lambda_B^{(i)}q_i(t_0)}{(\lambda_A^{(i)} - \lambda_B^{(i)})}\lambda_A^{(i)}e^{\lambda_A^{(i)}(t-t_0)} - \frac{q_i'(t_0) - \lambda_A^{(i)}q_i(t_0)}{(\lambda_A^{(i)} - \lambda_B^{(i)})}\lambda_B^{(i)}e^{\lambda_B^{(i)}(t-t_0)} + \lambda_A^{(i)}\int_{t_0}^{t} ds \frac{R_i^q(s)e^{\lambda_A^{(i)}(t-s)}}{m(\lambda_A^{(i)} - \lambda_B^{(i)})} - \lambda_B^{(i)}\int_{t_0}^{t} ds \frac{R_i^q(s)e^{\lambda_B^{(i)}(t-s)}}{m(\lambda_A^{(i)} - \lambda_B^{(i)})}. \tag{B15}$$

2. Solution of the equation for each particle

The original variables and transformed variables have the relation

$$\vec{x}(t) = \mathbf{P}\vec{q}(t),$$
$$\vec{q}(t) = \mathbf{P}^{-1}\vec{x}(t).$$

We note that $\vec{x}(t)$ and $\vec{q}(t)$ can be expressed as

$$\vec{x}(t) = \frac{1}{\sqrt{n}}\begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & c & c^2 & \cdots & c^{(n-1)} \\ 1 & c^2 & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & \vdots \\ 1 & c^{(n-1)} & \cdots & \cdots & c^{(n-1)^2} \end{pmatrix}\begin{pmatrix} q_1(t) \\ q_2(t) \\ q_3(t) \\ \vdots \\ q_n(t) \end{pmatrix},$$

$$\vec{q}(t) = \frac{1}{\sqrt{n}}\begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & c^{-1} & c^{-2} & \cdots & c^{-(n-1)} \\ 1 & c^{-2} & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & \vdots \\ 1 & c^{-(n-1)} & \cdots & \cdots & c^{-(n-1)^2} \end{pmatrix}\begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ \vdots \\ x_n(t) \end{pmatrix}.$$

We show the explicit form of each element:

$$x_i(t) = \frac{1}{\sqrt{n}}\sum_{j=1}^{n} c^{(i-1)(j-1)}q_j(t), \quad i = 1, 2, \ldots, n, \tag{B16}$$

$$q_k(t) = \frac{1}{\sqrt{n}}\sum_{l=1}^{n} c^{-(k-1)(l-1)}x_l(t), \quad k = 1, 2, \ldots, n. \tag{B17}$$

We also transform $\frac{dq_i}{dt}(t_0)$ and $R_i^q(t)$ into the following:

$$\frac{dq_i}{dt}(t_0) = \frac{1}{\sqrt{n}}\sum_{j=1}^{n} c^{-(i-1)(j-1)}u_j(t_0), \tag{B18}$$

$$R_i^q(t) = \frac{1}{\sqrt{n}}\sum_{j=1}^{n} c^{-(i-1)(j-1)}[R_j(t) + f_j(t)], \tag{B19}$$

where $u_i(t) := \frac{dx_i}{dt}(t)$. Substituting Eqs. (B16), (B17), (B18), and (B19) into Eqs. (B14) and (B15), we obtain the following equations:

$$\begin{aligned} q_i(t) =& \frac{1}{\sqrt{n}}\sum_{j=1}^{n}\frac{c^{-(i-1)(j-1)}\big[u_j(t_0)-\lambda_{\rm B}^{(i)}x_j(t_0)\big]}{\big(\lambda_{\rm A}^{(i)}-\lambda_{\rm B}^{(i)}\big)}e^{\lambda_{\rm A}^{(i)}(t-t_0)} \\ &-\frac{1}{\sqrt{n}}\sum_{j=1}^{n}\frac{c^{-(i-1)(j-1)}\big[u_j(t_0)-\lambda_{\rm A}^{(i)}x_j(t_0)\big]}{\big(\lambda_{\rm A}^{(i)}-\lambda_{\rm B}^{(i)}\big)}e^{\lambda_{\rm B}^{(i)}(t-t_0)} \\ &+\frac{1}{\sqrt{n}}\sum_{j=1}^{n}c^{-(i-1)(j-1)}\int_{t_0}^{t}ds\frac{[R_j(s)+f_j(s)]e^{\lambda_{\rm A}^{(i)}(t-s)}}{m\big(\lambda_{\rm A}^{(i)}-\lambda_{\rm B}^{(i)}\big)} \\ &-\frac{1}{\sqrt{n}}\sum_{j=1}^{n}c^{-(i-1)(j-1)}\int_{t_0}^{t}ds\frac{[R_j(s)+f_j(s)]e^{\lambda_{\rm B}^{(i)}(t-s)}}{m\big(\lambda_{\rm A}^{(i)}-\lambda_{\rm B}^{(i)}\big)}, \end{aligned} \tag{B20}$$

$$\begin{aligned} \frac{dq_i}{dt}(t) \\ =& \frac{1}{\sqrt{n}}\sum_{j=1}^{n}\frac{c^{-(i-1)(j-1)}\big[u_j(t_0)-\lambda_{\rm B}^{(i)}x_j(t_0)\big]}{\big(\lambda_{\rm A}^{(i)}-\lambda_{\rm B}^{(i)}\big)}\lambda_{\rm A}^{(i)}e^{\lambda_{\rm A}^{(i)}(t-t_0)} \\ &-\frac{1}{\sqrt{n}}\sum_{j=1}^{n}\frac{c^{-(i-1)(j-1)}\big[u_j(t_0)-\lambda_{\rm A}^{(i)}x_j(t_0)\big]}{\big(\lambda_{\rm A}^{(i)}-\lambda_{\rm B}^{(i)}\big)}\lambda_{\rm B}^{(i)}e^{\lambda_{\rm B}^{(i)}(t-t_0)} \\ &+\frac{1}{\sqrt{n}}\sum_{j=1}^{n}c^{-(i-1)(j-1)}\lambda_{\rm A}^{(i)}\int_{t_0}^{t}ds\frac{[R_j(s)+f_j(s)]e^{\lambda_{\rm A}^{(i)}(t-s)}}{m\big(\lambda_{\rm A}^{(i)}-\lambda_{\rm B}^{(i)}\big)} \\ &-\frac{1}{\sqrt{n}}\sum_{j=1}^{n}c^{-(i-1)(j-1)}\lambda_{\rm B}^{(i)}\int_{t_0}^{t}ds\frac{[R_j(s)+f_j(s)]e^{\lambda_{\rm B}^{(i)}(t-s)}}{m\big(\lambda_{\rm A}^{(i)}-\lambda_{\rm B}^{(i)}\big)}. \end{aligned} \tag{B21}$$

Thus, we obtain the solution of Eq. (B1), which is expressed as

$$\begin{aligned} x_i(t) =& \frac{1}{n}\sum_{j=1}^{n}\sum_{k=1}^{n}c^{(j-1)(i-k)}\frac{\big[u_k(t_0)-\lambda_{\rm B}^{(j)}x_k(t_0)\big]}{\big(\lambda_{\rm A}^{(j)}-\lambda_{\rm B}^{(j)}\big)}e^{\lambda_{\rm A}^{(j)}(t-t_0)} \\ &-\frac{1}{n}\sum_{j=1}^{n}\sum_{k=1}^{n}c^{(j-1)(i-k)}\frac{\big[u_k(t_0)-\lambda_{\rm A}^{(j)}x_k(t_0)\big]}{\big(\lambda_{\rm A}^{(j)}-\lambda_{\rm B}^{(j)}\big)}e^{\lambda_{\rm B}^{(j)}(t-t_0)} \\ &+\frac{1}{n}\sum_{j=1}^{n}\sum_{k=1}^{n}c^{(j-1)(i-k)}\int_{t_0}^{t}ds\frac{\big[f_k(s)+R_k(s)\big]e^{\lambda_{\rm A}^{(j)}(t-s)}}{m\big(\lambda_{\rm A}^{(j)}-\lambda_{\rm B}^{(j)}\big)} \\ &-\frac{1}{n}\sum_{j=1}^{n}\sum_{k=1}^{n}c^{(j-1)(i-k)}\int_{t_0}^{t}ds\frac{\big[f_k(s)+R_k(s)\big]e^{\lambda_{\rm B}^{(j)}(t-s)}}{m\big(\lambda_{\rm A}^{(j)}-\lambda_{\rm B}^{(j)}\big)}. \end{aligned} \tag{B22}$$

The differentiation of the solution, $u_i(t)$, can be expressed as

$$\begin{aligned} u_i(t) \\ =& \frac{1}{n}\sum_{j=1}^{n}\sum_{k=1}^{n}c^{(j-1)(i-k)}\frac{\big[u_k(t_0)-\lambda_{\rm B}^{(j)}x_k(t_0)\big]}{\big(\lambda_{\rm A}^{(j)}-\lambda_{\rm B}^{(j)}\big)}\lambda_{\rm A}^{(j)}e^{\lambda_{\rm A}^{(j)}(t-t_0)} \\ &-\frac{1}{n}\sum_{j=1}^{n}\sum_{k=1}^{n}c^{(j-1)(i-k)}\frac{\big[u_k(t_0)-\lambda_{\rm A}^{(j)}x_k(t_0)\big]}{\big(\lambda_{\rm A}^{(j)}-\lambda_{\rm B}^{(j)}\big)}\lambda_{\rm B}^{(j)}e^{\lambda_{\rm B}^{(j)}(t-t_0)} \\ &+\frac{1}{n}\sum_{j=1}^{n}\sum_{k=1}^{n}c^{(j-1)(i-k)}\lambda_{\rm A}^{(j)}\int_{t_0}^{t}ds\frac{[f_k(s)+R_k(s)]e^{\lambda_{\rm A}^{(j)}(t-s)}}{m\big(\lambda_{\rm A}^{(j)}-\lambda_{\rm B}^{(j)}\big)} \\ &-\frac{1}{n}\sum_{j=1}^{n}\sum_{k=1}^{n}c^{(j-1)(i-k)}\lambda_{\rm B}^{(j)}\int_{t_0}^{t}ds\frac{[f_k(s)+R_k(s)]e^{\lambda_{\rm B}^{(j)}(t-s)}}{m\big(\lambda_{\rm A}^{(j)}-\lambda_{\rm B}^{(j)}\big)}. \end{aligned} \tag{B23}$$

APPENDIX C: LINEAR RESPONSE FUNCTION

We derive the response function of Eq. (B23). When we set the external forces to

$$\vec{R}(t) = \begin{pmatrix} R_1(t) \\ \vdots \\ f_k(t)+R_k(t) \\ \vdots \\ R_n(t) \end{pmatrix}, \qquad (k=1,2,\dots,n),$$

we can transform Eq. (B23) into the following:

$$\begin{aligned} \langle u_i(t)\rangle_{f_k} \\ =& \frac{1}{n}\sum_{p=1}^{n}\sum_{q=1}^{n}c^{(p-1)(i-q)}\frac{\big[u_q(t_0)-\lambda_{\rm B}^{(p)}x_q(t_0)\big]}{\big(\lambda_{\rm A}^{(p)}-\lambda_{\rm B}^{(p)}\big)}\lambda_{\rm A}^{(p)}e^{\lambda_{\rm A}^{(p)}(t-t_0)} \\ &-\frac{1}{n}\sum_{p=1}^{n}\sum_{q=1}^{n}c^{(p-1)(i-q)}\frac{\big[u_q(t_0)-\lambda_{\rm A}^{(p)}x_q(t_0)\big]}{\big(\lambda_{\rm A}^{(p)}-\lambda_{\rm B}^{(p)}\big)}\lambda_{\rm B}^{(p)}e^{\lambda_{\rm B}^{(p)}(t-t_0)} \\ &+\frac{1}{n}\sum_{p=1}^{n}\sum_{q=1}^{n}\delta_{qk}c^{(p-1)(i-q)}\lambda_{\rm A}^{(p)}\int_{t_0}^{t}ds\frac{f_q(s)e^{\lambda_{\rm A}^{(p)}(t-s)}}{m\big(\lambda_{\rm A}^{(p)}-\lambda_{\rm B}^{(p)}\big)} \\ &-\frac{1}{n}\sum_{p=1}^{n}\sum_{q=1}^{n}\delta_{qk}c^{(p-1)(i-q)}\lambda_{\rm B}^{(p)}\int_{t_0}^{t}ds\frac{f_q(s)e^{\lambda_{\rm B}^{(p)}(t-s)}}{m\big(\lambda_{\rm A}^{(p)}-\lambda_{\rm B}^{(p)}\big)}. \end{aligned}$$

Using $\Re(\lambda_{\rm A}^{(i)}) \leqslant 0$ and $\Re(\lambda_{\rm B}^{(i)}) \leqslant 0 (i=1,2,\dots,n)$, we rewrite the terms that involve $e^{\lambda_{\rm A}^{(j)}(t-t_0)}$ or $e^{\lambda_{\rm B}^{(j)}(t-t_0)}$ as $\langle u_i\rangle_0 \equiv \bar{u}_i$.

Thus, we obtain the following response function:

$$\begin{aligned} \langle u_i(t)\rangle_{f_k} - \langle u_i\rangle_0 \\ =& \frac{1}{n}\sum_{p=1}^{n}c^{(p-1)(i-k)}\lambda_{\rm A}^{(p)}\int_{t_0}^{t}ds\frac{f_k(s)e^{\lambda_{\rm A}^{(p)}(t-s)}}{m\big(\lambda_{\rm A}^{(p)}-\lambda_{\rm B}^{(p)}\big)} \\ &-\frac{1}{n}\sum_{p=1}^{n}c^{(p-1)(i-k)}\lambda_{\rm B}^{(p)}\int_{t_0}^{t}ds\frac{f_k(s)e^{\lambda_{\rm B}^{(p)}(t-s)}}{m\big(\lambda_{\rm A}^{(p)}-\lambda_{\rm B}^{(p)}\big)}. \\ =& \int_{t_0}^{t}ds f_k(s)\phi_{ik}(t-s), \end{aligned}$$

$$\phi_{ik}(t) := \frac{1}{n}\sum_{p=1}^{n}c^{(p-1)(i-k)}\left[\frac{\lambda_{\rm A}^{(p)}e^{\lambda_{\rm A}^{(p)}t}-\lambda_{\rm B}^{(p)}e^{\lambda_{\rm B}^{(p)}t}}{m\big(\lambda_{\rm A}^{(p)}-\lambda_{\rm B}^{(p)}\big)}\right],$$

where $\phi_{ik}(t)$ is the linear response function for velocity $u_i(t)$ with external force $f_k(t)$.

APPENDIX D: CORRELATION FUNCTION

We derive the correlation function of Eq. (B23). We set the external forces to independent white noise:

$$\vec{R}(t)=\begin{pmatrix}R_1(t)\\R_2(t)\\\vdots\\\vdots\\R_n(t)\end{pmatrix},\quad \begin{aligned}\langle R_i(t)\rangle_0&=0,\\ \langle R_i(t)R_j(s)\rangle_0&=2\gamma k_B T\delta_{ij}\delta(t-s)\\&=D\delta_{ij}\delta(t-s).\end{aligned}$$

In Eq. (B23), we set t_0 to $-\infty$ and obtain

$$\begin{aligned}u_i(t)&=\frac{1}{n}\sum_{p=1}^{n}\sum_{q=1}^{n}c^{(p-1)(i-q)}\lambda_A^{(p)}\int_{-\infty}^{t}dr\frac{R_q(r)e^{\lambda_A^{(p)}(t-r)}}{m(\lambda_A^{(p)}-\lambda_B^{(p)})}\\&\quad-\frac{1}{n}\sum_{p=1}^{n}\sum_{q=1}^{n}c^{(p-1)(i-q)}\lambda_B^{(p)}\int_{-\infty}^{t}dr\frac{R_q(r)e^{\lambda_B^{(p)}(t-r)}}{m(\lambda_A^{(p)}-\lambda_B^{(p)})}\\&=\frac{1}{n}\sum_{p=1}^{n}\sum_{q=1}^{n}c^{(p-1)(i-q)}\\&\quad\times\int_{-\infty}^{t}dr\frac{R_q(r)\big[\lambda_A^{(p)}e^{\lambda_A^{(p)}(t-r)}-\lambda_B^{(p)}e^{\lambda_B^{(p)}(t-r)}\big]}{m(\lambda_A^{(p)}-\lambda_B^{(p)})}.\end{aligned}\tag{D1}$$

Using Eq. (D1), we obtain the following correlation function:

$$\begin{aligned}C_{ij}(t-s)&=\langle(u_i(t)-\bar{u}_i)(u_j(s)-\bar{u}_j)\rangle_0\\&=\frac{D}{n^2}\sum_{k=1}^{n}\sum_{l=1}^{n}\sum_{p=1}^{n}\frac{c^{(k-1)(i-l)+(p-1)(j-l)}}{m^2(\lambda_A^{(k)}-\lambda_B^{(k)})(\lambda_A^{(p)}-\lambda_B^{(p)})}\\&\quad\times\bigg[-\frac{\lambda_A^{(k)}\lambda_A^{(p)}}{\lambda_A^{(k)}+\lambda_A^{(p)}}e^{\lambda_A^{(k)}t+\lambda_A^{(p)}s}e^{-(\lambda_A^{(k)}+\lambda_A^{(p)})\min(t,s)}\\&\quad+\frac{\lambda_A^{(k)}\lambda_B^{(p)}}{\lambda_A^{(k)}+\lambda_B^{(p)}}e^{\lambda_A^{(k)}t+\lambda_B^{(p)}s}e^{-(\lambda_A^{(k)}+\lambda_B^{(p)})\min(t,s)}\\&\quad+\frac{\lambda_B^{(k)}\lambda_A^{(p)}}{\lambda_B^{(k)}+\lambda_A^{(p)}}e^{\lambda_B^{(k)}t+\lambda_A^{(p)}s}e^{-(\lambda_B^{(k)}+\lambda_A^{(p)})\min(t,s)}\\&\quad-\frac{\lambda_B^{(k)}\lambda_B^{(p)}}{\lambda_B^{(k)}+\lambda_B^{(p)}}e^{\lambda_B^{(k)}t+\lambda_B^{(p)}s}e^{-(\lambda_B^{(k)}+\lambda_B^{(p)})\min(t,s)}\bigg].\end{aligned}\tag{D2}$$

In the case of $t\geqslant s$, we obtain

$$\begin{aligned}C_{ij}(t-s)&=\frac{D}{n^2}\sum_{k=1}^{n}\sum_{l=1}^{n}\sum_{p=1}^{n}\frac{c^{(k-1)(i-l)+(p-1)(j-l)}}{m^2(\lambda_A^{(k)}-\lambda_B^{(k)})(\lambda_A^{(p)}-\lambda_B^{(p)})}\\&\quad\times\bigg[\bigg(-\frac{\lambda_A^{(k)}\lambda_A^{(p)}}{\lambda_A^{(k)}+\lambda_A^{(p)}}+\frac{\lambda_A^{(k)}\lambda_B^{(p)}}{\lambda_A^{(k)}+\lambda_B^{(p)}}\bigg)e^{\lambda_A^{(k)}(t-s)}\\&\quad+\bigg(\frac{\lambda_B^{(k)}\lambda_A^{(p)}}{\lambda_B^{(k)}+\lambda_A^{(p)}}-\frac{\lambda_B^{(k)}\lambda_B^{(p)}}{\lambda_B^{(k)}+\lambda_B^{(p)}}\bigg)e^{\lambda_B^{(k)}(t-s)}\bigg].\end{aligned}\tag{D3}$$

Defining

$$\begin{aligned}\Gamma(k,p;t-s)&:=\frac{1}{(\lambda_A^{(k)}-\lambda_B^{(k)})}\bigg[\frac{-\lambda_A^{(k)}\lambda_A^{(k)}e^{\lambda_A^{(k)}(t-s)}}{(\lambda_A^{(k)}+\lambda_A^{(p)})(\lambda_A^{(k)}+\lambda_B^{(p)})}\\&\quad+\frac{\lambda_B^{(k)}\lambda_B^{(k)}e^{\lambda_B^{(k)}(t-s)}}{(\lambda_B^{(k)}+\lambda_A^{(p)})(\lambda_B^{(k)}+\lambda_B^{(p)})}\bigg],\end{aligned}$$

we can express correlation function Eq. (D3) as

$$\begin{aligned}C_{ij}(t-s)&=\frac{-De^{\lambda_B^{(1)}(t-s)}}{2nm^2\lambda_B^{(1)}}\\&\quad+\frac{D}{nm^2}\sum_{k=2}^{n}c^{i(k-1)-j(k-1)}\Gamma(k,n+2-k;t-s).\end{aligned}$$

In the case of $s\geqslant t$, we obtain

$$\begin{aligned}C_{ij}(t-s)&=\frac{D}{n^2}\sum_{k=1}^{n}\sum_{l=1}^{n}\sum_{p=1}^{n}\frac{c^{(k-1)(i-l)+(p-1)(j-l)}}{m^2(\lambda_A^{(k)}-\lambda_B^{(k)})(\lambda_A^{(p)}-\lambda_B^{(p)})}\\&\quad\times\bigg[\bigg(-\frac{\lambda_A^{(k)}\lambda_A^{(p)}}{\lambda_A^{(k)}+\lambda_A^{(p)}}+\frac{\lambda_B^{(k)}\lambda_A^{(p)}}{\lambda_B^{(k)}+\lambda_A^{(p)}}\bigg)e^{\lambda_A^{(p)}(s-t)}\\&\quad+\bigg(\frac{\lambda_A^{(k)}\lambda_B^{(p)}}{\lambda_A^{(k)}+\lambda_B^{(p)}}-\frac{\lambda_B^{(k)}\lambda_B^{(p)}}{\lambda_B^{(k)}+\lambda_B^{(p)}}\bigg)e^{\lambda_B^{(p)}(s-t)}\bigg].\end{aligned}\tag{D4}$$

Using $\Gamma(k,p;t)$, we can also rewrite this result as

$$\begin{aligned}C_{ij}(t-s)&=\frac{-De^{\lambda_B^{(1)}(s-t)}}{2nm^2\lambda_B^{(1)}}\\&\quad+\frac{D}{nm^2}\sum_{p=2}^{n}c^{-i(p-1)+j(p-1)}\Gamma(p,n+2-p;s-t).\end{aligned}$$

APPENDIX E: CASE OF THE SYMMETRIC INTERACTION

We show the FRR in the case of the symmetric interaction, $k_L=k_R=\kappa$.

1. Case of $t>s$

We rewrite the correlation function as

$$\begin{aligned}C_{ij}(t-s)&=\frac{D}{n^2}\frac{1}{m^2}\sum_{k=1}^{n}\sum_{p=1}^{n}\sum_{l=1}^{n}c^{(k-1)(i-l)(p-1)(j-l)}\\&\quad\times\Gamma(k,p;t-s),\end{aligned}$$

where

$$\begin{aligned}\Gamma(k,p;t-s)&:=\frac{1}{(\lambda_A^{(k)}-\lambda_B^{(k)})}\bigg[\frac{-\lambda_A^{(k)}\lambda_A^{(k)}e^{\lambda_A^{(k)}(t-s)}}{(\lambda_A^{(k)}+\lambda_A^{(p)})(\lambda_A^{(k)}+\lambda_B^{(p)})}\\&\quad+\frac{\lambda_B^{(k)}\lambda_B^{(k)}e^{\lambda_B^{(k)}(t-s)}}{(\lambda_B^{(k)}+\lambda_A^{(p)})(\lambda_B^{(k)}+\lambda_B^{(p)})}\bigg].\end{aligned}$$

We change the orders of summations and obtain

$$C_{ij}(t-s) = \frac{D}{n^2}\frac{1}{m^2}\sum_{k=1}^{n}\sum_{p=1}^{n} c^{i(k-1)+j(p-1)}\Gamma(k,p;t-s) \times \sum_{l=1}^{n} c^{-l(k+p-2)}. \quad \text{(E1)}$$

We note that c has the following property:

$$1 + c^y + c^{2y} + \cdots + c^{(n-1)y} = 0, \quad y \neq nx, x \in \mathbb{Z}. \quad \text{(E2)}$$

This property is shown by multiplying c^y and

$$1 + c^y + c^{2y} + \cdots + c^{(n-1)y} = z, \quad z \in \mathbb{C}.$$

Then, we obtain

$$1 + c^y + c^{2y} + \cdots + c^{(n-1)y} = c^y z.$$

We recall that $a^y \neq 1$ and obtain $z = 0$.

In addition, we note that

$$g(c^k) = 2[\cos(k\theta) - 1].$$

Then, part of $\lambda_A^{(j)}$ and $\lambda_B^{(j)}$, $\sqrt{\gamma^2 + 4mg(c^{j-1})}$, is transformed into the following:

$$\sqrt{\gamma^2 + 4mg(c^{j-1})} = \sqrt{\alpha^{(j)}},$$
$$\alpha^{(j)} = \gamma^2 + 8m\kappa\{\cos[(j-1)\theta] - 1\}.$$

$\lambda_A^{(j)}$ and $\lambda_B^{(j)}$ are rewritten as

$$\lambda_A^{(j)} = \frac{-\gamma + \sqrt{\alpha^{(j)}}}{2m},$$
$$\lambda_B^{(j)} = \frac{-\gamma - \sqrt{\alpha^{(j)}}}{2m}.$$

Since $\alpha^{(j)}$ can be rewritten as

$$\begin{aligned}\alpha^{(j)} &= \gamma^2 + 8m\kappa\{\cos[(j-1)\theta] - 1\} \\ &= \gamma^2 + 8m\kappa(\cos\{[(n-j+2)-1]\theta\} - 1) \\ &= \alpha^{(n-j+2)},\end{aligned}$$

$\lambda_A^{(j)}$ and $\lambda_B^{(j)}$ satisfy

$$\lambda_A^{(j)} = \lambda_A^{(n+2-j)}, \quad \lambda_B^{(j)} = \lambda_B^{(n+2-j)}. \quad \text{(E3)}$$

When the index is other than $k+p-2=0$ or $k+p-2=n$, from Eq. (E2), the summation $\sum_{l=1}^{n} c^{-l(k+p-2)}$ vanishes. In other words, Eq. (E1) is reduced to

$$C_{ij}(t-s) = \frac{D}{n^2}\frac{n}{m^2}\sum_{k=1}^{n}\sum_{p=1}^{n}(\delta_{k,1}\delta_{p,1} + \delta_{p,n+2-k}) \times c^{i(k-1)+j(p-1)}\Gamma(k,p;t-s).$$

We expand this expression to

$$C_{ij}(t-t') = \frac{D}{n^2}\frac{n}{m^2}\sum_{k=1}^{1}\Gamma(k,k;t-s) + \frac{D}{n^2}\frac{n}{m^2}\sum_{k=2}^{n} c^{i(k-1)-j(k-1)}\Gamma(k,n+2-k;t-s).$$

Using Eq. (E3), we reduce $\Gamma(k,n+2-k;t-s)$ to

$$\begin{aligned}\Gamma(k,n+2-k;t-s) &= \frac{m}{2\gamma}\frac{\left[\lambda_A^{(k)}e^{\lambda_A^{(k)}(t-s)} - \lambda_B^{(k)}e^{\lambda_B^{(k)}(t-s)}\right]}{\lambda_A^{(k)} - \lambda_B^{(k)}} \\ &= \Gamma(k,k;t-s),\end{aligned}$$

where we use the following: $(\lambda_A^{(k)} + \lambda_B^{(k)}) = -\frac{\gamma}{m}$. Thus, the correlation function is reduced to

$$C_{ij}(t-s) = k_B T \sum_{k=1}^{n}\frac{c^{(k-1)(i-j)}}{nm} \times \frac{\left[\lambda_A^{(k)}e^{\lambda_A^{(k)}(t-s)} - \lambda_B^{(k)}e^{\lambda_B^{(k)}(t-s)}\right]}{\lambda_A^{(k)} - \lambda_B^{(k)}}.$$

2. Case of $s > t$

Similar to the case of $t > s$, $C_{ij}(t-s)$ is reduced to

$$C_{ij}(t-s) = \frac{1}{2\gamma}\frac{D}{n}\frac{1}{m}\sum_{k=1}^{n} c^{(k-1)(i-j)} \times \frac{\left[\lambda_A^{(k)}e^{\lambda_A^{(k)}(s-t)} - \lambda_B^{(k)}e^{\lambda_B^{(k)}(s-t)}\right]}{\lambda_A^{(k)} - \lambda_B^{(k)}}.$$

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