Dynamic critical behavior of the two-dimensional Ising model with nonextensive statistics

V. N. Borodikhin*
Dostoevsky Omsk State University, pr. Mira 55a, Omsk 644077, Russia

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The dynamic critical behavior of the two-dimensional Ising model with nonextensive Tsallis statistics has been studied. The values of the dynamic critical index z as well as the values of the indices ν and β for different values of the deformation parameter q have been obtained. The emergence of a new type of critical behavior has been revealed.

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I. INTRODUCTION

A two-dimensional uniform Ising model within the nonextensive Tsallis statistics was considered in the works [1,2]. The investigation of this model is of theoretical and experimental interest, since the critical behavior of this model agrees qualitatively with that of real systems, in particular, of so-called manganites, e.g., $La_{0.60}Y_{0.07}Ca_{0.33}Mn_{O3}$ [3,4]. However, the possibility of describing these systems within the nonextensive statistics is still unclear.

It is of interest to study the system in the nonequilibrium case because in dynamics the effects of nonadditivity can manifest themselves more strongly than in the equilibrium case.

The nonextensive statistics is characterized by the deformation parameter q (entropy index), which corresponds to the deviation from the Gibbs statistics.

The dynamic critical behavior of the two-dimensional Ising model with nonextensive statistics has been studied. Both the homogeneous system and the system with a low concentration of impurities were considered. The dynamic critical index z has been determined as well as the value of critical indicators ν and β for different values of the deformation parameter q have been specified.

II. SYSTEMS WITH NONEXTENSIVE STATISTICS

The nonadditive Tsallis entropy is described by the following expression [5]:

$$S_q = k\frac{1-\sum_i p_i^q}{q-1}, \tag{1}$$

where p_i is the probability of finding the system in a state i, and q is the parameter of the deformation of the following statistics: $0 < p_i < 1$ $p_i^q > p_i$ for $q < 1$, and $p_i^q < p_i$ for $q > 1$. If $q = 1$ the statistics is not deformed. In the nonextensive statistical theory [5], the following definition is introduced:

$$\langle H\rangle_q = \sum_{i=1}^{\Omega} P_i\varepsilon_i = U_q, \tag{2}$$

where H is the Hamiltonian of the system, ε_i is one of the possible energy states, U_q is the internal energy, and P_i is the so-called escort distribution [6]

$$P_i = \frac{p_i^q}{\sum_{j=1}^{\Omega} p_j^q}, \tag{3}$$

$$p_i = \frac{e_q^{-\beta'\varepsilon_i}}{\sum_{j=1}^{\Omega} e_q^{-\beta'\varepsilon_j}}, \tag{4}$$

where e_q^{-x} is the deformed exponent that satisfies the expression

$$[1-(1-q)\beta'\varepsilon_i]^{1/(1-q)} = \begin{cases} e_q^{-x}, & 1-(1-q)x \geqslant 0; \\ 0, & 1-(1-q)x < 0, \end{cases} \tag{5}$$

$$\beta' = \frac{\beta}{\sum_{j=1}^{\Omega} p_j^q + (1-q)\beta U_q}, \tag{6}$$

with β a Lagrange multiplier.

Following [1] we chose $\beta' = (kT)^{-1}$ as the physical temperature.

III. SIMULATION OF SYSTEMS WITH NONEXTENSIVE STATISTICS

In the computer simulation, we employed the Metropolis algorithm with the modified probability of spin flips caused by the nonextensive statistics:

$$w_q = \frac{P_i^{(2)}}{P_i^{(1)}} = \frac{e_q^{-\varepsilon_i^{(2)}/T}}{e_q^{-\varepsilon_i^{(1)}/T}}, \tag{7}$$

where $P_i^{(2)}$ is the escortdistribution (3) after the spin flip, and $P_i^{(1)}$ is the escort distribution (3) before the spin flip.

The Metropolis algorithm is used in the following form [7]. Each Monte Carlo step can be resumed as the following: compute the interaction energy of a given spin i of the lattice with its neighbors $\varepsilon_i^{(1)} = \sum_{j=1}^{4}\varepsilon_{ij}$. After that, change the state of this spin and compute again its interaction energy, $\varepsilon_i^{(2)}$. If $\varepsilon_i^{(2)} < \varepsilon_i^{(1)}$, accept the change of state. If the energy is not lower, using (5), compute the probability w. Compare this quantity to a number that belongs to the interval [0, 1]

*borodikhin@inbox.ru

TABLE I. Relations of critical indices of the two-dimensional homogeneous Ising model with nonextensive statistics ($p = 1$).

q	T_c	$\beta/\nu z$	d/z	$1/\nu z$
0.9	2.057(7)	0.0562(5)	0.9297±0.0014	0.46967±0.0014
0.8	1.888(5)	0.0570(7)	0.929±0.001	0.4774±0.0012
0.7	1.778(6)	0.0563(9)	0.934±0.0023	0.49±0.0015
0.6	1.773(7)	0.057(1)	0.93±0.0012	0.4867±0.002

generated randomly. Being that this random number is smaller than w then accept the change of state, otherwise not.

As can be seen, this is the ordinary Metropolis algorithm in which the probability of state was changed from the Boltzmann weight to the Tsallis factor (5).

The energy state ε_i is determined by the Hamiltonian of the two-dimensional Ising model taking into account the interaction between the nearest-neighbor spins s_i as follows:

$$H = \frac{1}{2}\sum_{i,j} J_{ij} p_i p_j s_i s_j, \tag{8}$$

where J_{ij} is the exchange constant of the ferromagnetic interaction, and p_i and p_j are random variables characterized by the distribution function

$$P(p_i) = p\delta(p_i - 1) + (1-p)\delta(p_i), \tag{9}$$

which are introduced to describe quenched nonmagnetic impurity atom vacancies distributed over the lattice and characterized by the concentration $c_{\text{imp}} = 1 - p$, where p is spin concentration.

The system is represented by a square lattice with the side L and the periodic boundary conditions. Spin s_i, which can take values ± 1 are associated with the lattice sites there.

In the course of the simulation, the magnetization m per spin and the second-order cumulant g_2, were calculated as follows:

$$g_2 = \langle m^2\rangle - \langle m\rangle^2, \tag{10}$$

as well as the logarithmic derivative:

$$\partial \ln(m) = \frac{|m(T_c \pm \Delta T) - m(T_c)|}{2\Delta T m(T_c)}, \tag{11}$$

where T_c is the critical temperature.

The main feature of the short-term dynamics method is that information about the critical behavior of the system is obtained from a relatively small macroscopic period of time at an early stage of the system development at a critical point or its vicinity. For lattices with sufficiently large dimensions L, the dynamic scaling dependence for magnetization acquires the following form in the critical region [8]:

$$m(t, \tau) = t^{-\beta/\nu z} F(t^{1/\nu z}, \tau), \tag{12}$$

where t is the time, $\tau = \frac{T - T_c}{T_c}$ is the reduced temperature, β is the critical index of the order parameter (magnetization), ν is the critical index of the correlation length, z is the dynamic critical index, and F is the scaling function.

At the critical point ($\tau = 0$), the relaxation of magnetization is characterized by a power law

$$m(t) \sim t^{-\beta/\nu z}. \tag{13}$$

The indicator $1/\nu z$ can be determined from the scaling dependence of the logarithmic derivative:

$$\partial \ln[m(t, \tau)] \sim t^{1/\nu z}. \tag{14}$$

The dynamic critical indicator z can be determined from the scaling dependence of the cumulant g_2:

$$g_2 \sim t^{d/z}, \tag{15}$$

where d is the system dimension.

TABLE II. Relations of critical indices of the two-dimensional Ising model with nonextensive statistics ($p = 0.95$).

q	T_c	$\beta/\nu z$	d/z	$1/\nu z$
0.9	1.900(6)	0.0655(45)	0.8956±0.0013	0.4659±0.0012
0.8	1.7478(9)	0.0681(8)	0.91±0.0016	0.4701±0.0014
0.7	1.656(7)	0.0716(87)	0.9168±0.0015	0.4824±0.0017
0.6	1.665(8)	0.074(12)	0.9308±0.0014	0.497±0.0023

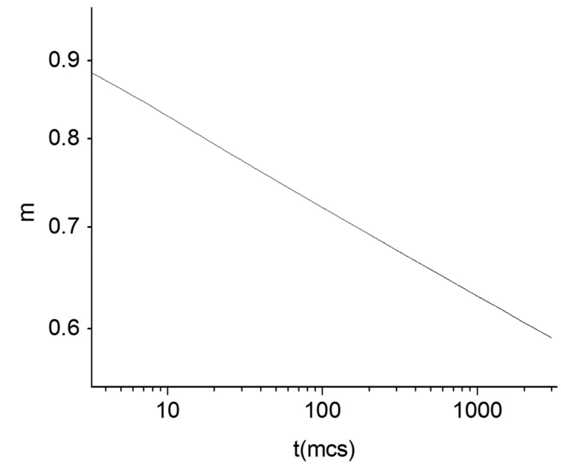

FIG. 1. Time dependence of magnetization m at $q = 0.8$, $p = 1$ in the double logarithmic scale.

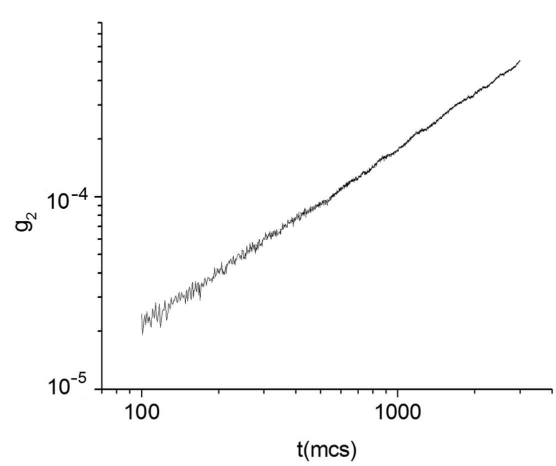

FIG. 2. Time dependence of cumulant g_2 at $q = 0.8$, $p = 1$ in the double logarithmic scale.

TABLE III. Critical indices of the two-dimensional homogeneous Ising model with nonextensive statistics ($p = 1$).

q	z	ν	β
1	2.24±0.007	1	0.125
0.9	2.151±0.0055	0.98975±0.0074	0.11987±0.00043
0.8	2.1528±0.005	0.9729±0.006	0.1194±0.0004
0.7	2.141±0.006	0.9589±0.0071	0.1156±0.0005
0.6	2.1505±0.007	0.955±0.008	0.1173±0.00048

IV. SIMULATION RESULTS

The simulation of the two-dimensional uniform ($p = 1$) and with spin concentrations $p = 0.95$ Ising models with nonextensive statistics was performed.

The size of the system was chosen to be $L = 1024$, and the deformation parameter $q = 0.9, 0.8, 0.7, 0.6$.

In the course of the simulation, we used 3200–6400 trials or impurity configuration and 3000 Monte Carlo steps from the initial ordered state.

The critical temperatures were determined for $p = 1$, $q = 0.9$ and 0.7, and for $p = 0.95$ in the entire range of the deformation parameters (Tables I and II).

Figure 1 shows the graph of magnetization at a critical point at $q = 0.8$, $p = 1$; Fig. 2 shows the graph of cumulant g_2 at a critical point at $q = 0.8$ in the double logarithmic scale.

In Tables I and III is given the found relations of critical indices; in Tables III and IV is given their resulting values ($p = 1$ and 0.95).

In this work the critical temperatures were determined by the behavior of magnetization and cumulant (Figs. 2 and 3). If the system is at a critical point, the graph resembles a straight line. If the system was above T_c, the magnetization curve would bend down, and at $T < T_c$ it would bend up.

The value of the dynamic critical index z in the homogeneous two-dimensional Ising model varies in a quite wide range. The table shows the value taken from [9]; other values were obtained in study [10].

At spin concentration $p = 0.95$ the critical exponent z decreases with decreasing of q. The dependence of the dynamic critical index z on the deformation parameter q is approximated by a quadratic dependence: $z(q) = a_0 + a_1 q + a_2 q^2$ (Fig. 3), where $a_0 = 1.724 \pm 0.06$, $a_1 = 1.05 \pm 0.16$, $a_2 = -0.57 \pm 0.107$.

With decreasing of deformation parameter q the values of the critical exponent ν decrease both for the spin concentration $p = 1$ and for $p = 0.95$. The dependence of the critical index ν on q is approximated by the straight line $\nu(q) = b_0 + b_1 q$ (Figs. 4 and 5), where the coefficients for $p = 1$ are $b_0 = 0.8786 \pm 0.001$, $b_1 = 0.1208 \pm 0.0123$, and for $p = 0.95$ are $b_0 = 0.8414 \pm 0.0078$, $b_1 = 0.156 \pm 0.01$.

TABLE IV. Critical indices of the two-dimensional Ising model with nonextensive statistics ($p = 0.95$).

q	z	ν	β
0.9	2.208±0.0067	0.9776±0.007	0.1414±0.0005
0.8	2.198±0.007	0.9678±0.0054	0.14486±0.0006
0.7	2.1814±0.0082	0.95±0.0068	0.1484±0.0007
0.6	2.1486±0.0095	0.9358±0.0077	0.1488±0.0007

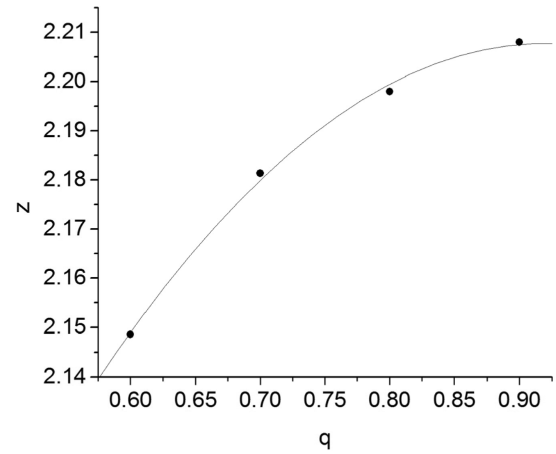

FIG. 3. The dependence of the dynamic critical index z on the deformation parameter q, $p = 0.95$.

V. CONCLUSIONS

Comparing the values of the critical indices of the two-dimensional homogeneous Ising model and the model with nonextensive statistics, we can conclude that a new type of critical behavior occurs in the model with nonextensive statistics.

In [1] it was found that when modeling the two-dimensional Ising model in the equilibrium case, the value of the critical index ν does not change when taking into account the deformation of statistics. However, calculations for the nonequilibrium model that were carried out in this paper showed that the critical index ν changes when taking into account the nonextensive statistics and, in addition, there is a slight change in this indicator for different values of the deformation parameter q.

In [1] it was shown that for $q \neq 1$ the critical index is $\alpha > 0$. According to the scaling relation $\nu d = 2 - \alpha$, where d is

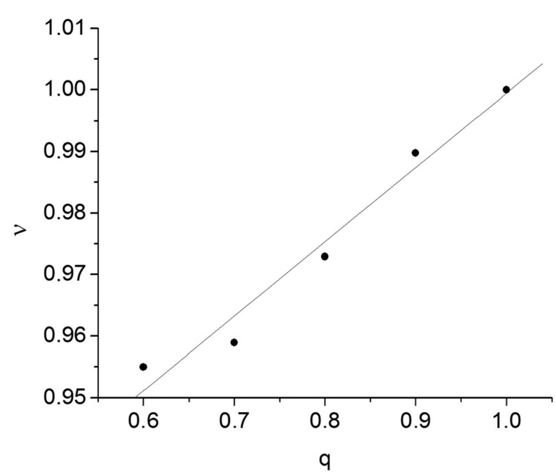

FIG. 4. The dependence of the dynamic critical index ν on the deformation parameter q, $p = 1$.

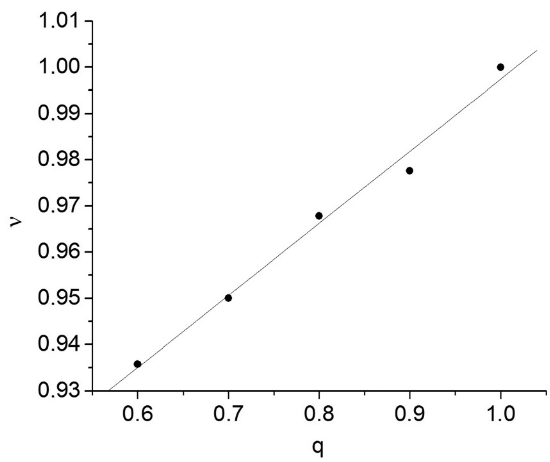

FIG. 5. The dependence of the dynamic critical index ν on the deformation parameter q, $p = 0.95$.

the dimension of the system, for $d = 2$ if $\alpha > 0$, the index ν should decrease. In this paper it was shown that in the case of not taking into account the influence of impurities as well as in the case of taking it into account, the index ν changes depending on the deformation parameter q by a linear law.

The values of the critical indexes z and β for the nonextensive model also differ from the values of the Ising model without taking into account the deformation statistics, but practically do not depend on the value of the deformation parameter q (within the margin of error).

In the two-dimensional Ising model impurities have a little effect on critical behavior. Comparing to the models with deformed statistics, the impurity case and the homogeneous one, it can be noted that the value of the dynamic critical index z has slightly changed, and the index β has also slightly increased. The index ν at a low concentration of impurities varies depending on the deformation parameter by a linear law, as in the homogeneous case of the nonextensive Ising model. Thus one can also speak of a new type of critical behavior for models with a low impurity concentration with deformed statistics. At $p = 0.95$ one can note decreasing of the critical exponent z with decreasing of q by a nonlinear law.

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