Influence of dust particles on ionization equilibrium in partially ionized plasmas

A. E. Davletov, F. Kurbanov, and Ye. S. Mukhametkarimov
Department of Physics and Technology, Al-Farabi Kazakh National University, 71 Al-Farabi av., 050040 Almaty, Kazakhstan

(Received 5 October 2019; revised manuscript received 18 April 2020; accepted 18 May 2020; published 5 June 2020)

A chemical model is proposed for a dusty plasma consisting of electrons, ions, neutrals, and positively charged dust particles all being at thermodynamic equilibrium. An expression is derived for the Helmholtz free energy, which comprises the ideal part, taking into account the charge of dust particles, and the excess part, evaluated in the framework of the self-consistent chemical model [Phys. Rev. E **83**, 016405 (2011)]. The ionization potential depression for a dust-free partially ionized hydrogen is analytically evaluated for weakly and strongly ionized states to consistently account for the presence of charged and neutral components. An *ad hoc* interpolation of the ionization potential depression, valid across the whole ionization region, is put forward and subsequent solution of the generalized Saha equation is found to be in a perfect agreement with exact calculations. Minimization of the Helmholtz free energy of dusty plasmas provides the number densities of free electrons, free ions, neutrals, and the dust electric charge as well. Based on consideration of weakly and strongly ionized states, a straightforward comparison is made of the ionization equilibrium in a partially ionized plasma with and without dust particles to demonstrate that at thermal equilibrium positively charged dusts are responsible for an increase in the number density of free electrons and a decrease in the number density of free ions. It is analytically proved that nonideality effects result in a growth of the number densities of free electrons and ions by obtaining the so-called electron and proton ionization potential depressions. Electric charge of dust particles is systematically studied as a full plasma component rather than considering a detailed balance of the electron and ion fluxes on the surface of a solitary dust grain.

DOI: 10.1103/PhysRevE.101.063203

I. INTRODUCTION

At present a dusty plasma still poses a great variety of challenges in both experimental and theoretical investigations [1], which is strictly prescribed to its rather complex nature. Indeed, a typical dusty plasma consists of electrons, ions, neutrals and dust particles, called grains, and the whole diversity of phenomena, observed in various external conditions, is just an interplay between the dusty plasma constituents having drastically different characteristic space and time evolution scales [2]. It has to be especially emphasized that an important role is bound to played by the ionization source, which can be either external electric fields and radiation in various gas discharges or the thermal motion as appropriate, for instance, for magnetically confined fusion devices [3].

Another origin of dusty plasma complexity is due to quite a broad range of its physical parameters spanning orders of magnitude in temperature and number density of each plasma component [4]. A great number of effects are thus invoked to compete with one another preventing further construction of a unified theoretical approach needed to understand all subtleties of events occurring both in technological installations [5] and in nature [6,7]. Nevertheless, dusty plasmas are well distinguished among all other objects of general scientific interest by the temporal behavior of macroscopic dust particles, which is promptly taped via high-speed and high-resolution videography [8,9], thereby serving as an extraordinary test tube for verification of distinct theoretical models already worked out for many-body systems [10].

It is physically evident that dust particles, being embedded in a partially ionized medium, can vitally alter the local plasma characteristics. First of all it is especially true for the ionization balance since dust grains are capable of attaining quite high, mostly negative, electric charge [11,12], thereby affecting the quasineutrality condition. Hence increasing the dust number density can easily cause a considerable depletion in the electron number density as compared to the case of a dust-free plasma [13,14]. Another expected implication of the dust injection into an ionized medium is the growth of the electron temperature, which, in accordance with the decline of the electron number density, seems to be an unavoidable response from the plasma to maintain the rate of electron impact ionization [15,16]. Furthermore, some specific electric properties of gas discharges, such as impedance and phase shift between the current and voltage, were found to be highly sensitive to the presence of even a small amount of dust [17].

Note that the situation with the ionization balance remains rather ambiguous for all plasma regions, which is especially intrinsic to the central void of a dust cloud. In particular, the optical emission spectroscopy provided a clear evidence that the electron temperature grows [18], while the converse was proved using direct measurements with the aid of electric probes [19]. Curiously enough, numerical simulations precisely demonstrated [20] that both an increased electron temperature and number density should appear in a well-developed void of a dusty plasma. Recently, beam microwave interferometry, which is both noninvasive and model-free, finally showed [21] that the whole effect might depend on the size of dust particles such that an increased electron number density was detected for rather large grains and vice versa.

It was Langmuir [22] who first noted that in the vast majority of practical situations dust particles, immersed in a ordinary plasma, promptly acquire negative electric charge, which is explained by the absorption of oppositely charged electrons and ions with drastically different mobilities. This is, however, not obligatory since positively charged dust grains are frequently encountered in some context of planetary sciences [23,24] as well as eagerly generated in plasma-wall interactions in the scrape-off layer in tokamaks [25,26]. In the last two examples the thermionic and secondary electron emissions are dominating phenomena that determine the physical properties of the dust component. Hereinafter we are in a position to study some general aspects of the ionization equilibrium in a partially ionized plasma with positively charged dusts.

Obviously, the ionization balance and the charge of dust grains in a plasma are mutually interrelated with the former being almost unchanged when the dust charge density is much smaller in magnitude than the charge density of plasma particles. At the moment there exist only few theoretical approaches that are capable of estimating positive electric charge of dust particles. The paramount one is certainly the orbital motion limited (OML) approximation [27,28], which particularly relies on several simplifying assumptions: (i) dust particles are much smaller in size than the Debye screening length; (ii) interaction potentials between plasma and dust particles remain monotonic at all separations; (iii) plasma particles' trajectories are exclusively ballistic such that interparticle collisions can be totally ignored. It is then no surprise that the pure OML method is only substantiated for rather dilute collisionless plasmas and further amendments are provided within the OML^+ [29,30] and the modified OML (MOML) [31] approximations. It is inevitable in all OML-like formulations that an average should be taken over the plasma particles distribution functions, which is assumed to be unperturbed at rather large separations from a grain and thus must be *a priori* known. In this regard a more plausible concept is advocated within the orbital motion (OM) theory [32] that consistently incorporates a set of the Poisson and Vlasov kinetic equations for the case of a collisionless medium. However, with growing plasma density pairwise interparticle collisions become of importance, which is traditionally handled by introducing collision integrals into the Vlasov kinetic equation [33,34]. As for a fully collisional regime even hydrodynamics is conventionally employed to describe the charging of dust grains [35] to attain a fairly good agreement with experimental data. Note that if the magnetic field comes to play a crucial part, like it is the case for magnetic fusion devices, the thin-sheath ion model [36] proved to be very fruitful.

Thus it is readily inferred that theory usually treats the charging of a solitary dust particle, whereas the influence of a dust cloud on the ionization equilibrium is accounted for via the quasineutrality condition. On the other hand, a whole series of experiments were performed with a purpose of establishing the effect of a dust component on the plasma ionization degree, particularly the bulk electron number density. The present consideration undertakes a systematic theoretical attempt to consistently treat dust particles as a full plasma component with a strong emphasis on their electric charge and the ionization equilibrium in the medium. Specifically, we develop a meaningful extension of the self-consistent chemical model of a partially ionized medium [37] thoroughly modified for a thermal dusty plasma by incorporating dust particles into the Helmholtz free energy of the system. In order to do so, one has to keep in mind that, to pull an electron out of a dust particle, some work function, say W, should be done and this creates a free electron and a positively charged dust grain. It is therefore clear that a neutral dust particle can be viewed as a "unionized" entity, whereby a strict analogy with a neutral atom in a partially ionized plasma is convincingly drawn. Thus the resulting Helmholtz free energy turns out to be a function of free electron and free proton number densities as well as the dust charge and its further multiparametric minimization under quasineutrality condition provides the corresponding equilibrium quantities.

The rest is organized as follows. In Sec. II dimensionless plasma parameters, relevant to the description of the plasma state, are introduced. A chemical model of a dusty plasma is accurately formulated in quite clear physics terms in Sec. III. Section IV is completely devoted to the ionization equilibrium in a partially ionized plasma needed to make a further comparison with the case when dust particles are present. In Sec. V an equilibrium state of dusty plasmas is comprehensively studied with a strong emphasis on the ionization balance and related Sec. VI is aimed at evaluating the electric charge of the dust component from the imposed quasineutrality condition. At the end conclusions are broadly drawn in Sec. VII and important provisions for future work are briefly stated.

II. PLASMA PARAMETERS

A typical dusty plasma is known to constitute a multicomponent aggregation in which solid dust particles are trapped in a partially ionized plasma that itself contains at least three sorts of particles, i.e., electrons, ions, and neutrals. Despite this evident experimental fact, many theoretical approaches are solely concentrated on studying the behavior of dust particles, thus regarding a dusty plasma as a one-component system in which the role of electrons and ions is basically reduced to shielding of interdust electrostatic interactions. This widely accepted model, called the Yukawa one-component plasma, proved to be especially beneficial in accurate description of experiments with the strongly coupled dusty plasma [38,39], which is plainly rationalized as follows. First of all, in the strongly coupled regime the electrostatic interaction between dust grains largely predominates over all other types of forces, such as the ion drag, thermoforetic force, etc. Secondly, in many practical situations the dust component hardly modifies the ionization balance in the surrounding plasma [40,41]; otherwise, the electron and ion dynamics must be taken into account to comprise the dust charge variation [42,43]. In contrast to the aforesaid the present consideration evenly treats all dusty plasma components since the ionization degree and electric charge of dust particles are both placed in spotlight and jointly handled within the thermodynamical approach. Even the presence of neutrals is thoroughly included into the whole analysis since electrons and ions permanently neutralize at grains surfaces, which, according to the ergodic hypothesis, surely requires the existence of at least a small portion of atoms.

Just for the reason of definiteness in the succeeding analytical approximations and numerical calculations we assume that the dusty plasma contains four sorts of particles: (i) free electrons with the number density n_e, mass m_e, and electric charge $-e$; (ii) free protons with the number density n_p, mass m_p, and electric charge e; (iii) neutrals, i.e., electrons and protons bounded in hydrogen atoms, with the number density n_n and mass m_n; (iv) dust particles with the number density n_d, mass m_d, and positive electric charge Ze. However, the exact plasma composition that incorporates the number densities of free electrons, free protons, and neutrals, as well as the electric charge of dust particles, remains initially unknown and is to be determined in the subsequent consideration. Note that the medium temperature is implied to be rather high such that the formation of hydrogen molecules is effectively prevented.

In order to theoretically describe the state of the dusty plasma, we keep the total number density of protons in the system, $n = n_p + n_n$, fixed such that the dimensionless density parameter is conventionally defined as

$$r_s = \frac{a}{a_B}, \tag{1}$$

where $a = (3/4\pi n)^{1/3}$ stands for the Wigner-Seitz radius, $a_B = \hbar^2/m_e e^2$ denotes the first Bohr radius, and $\hbar$ signifies the reduced Planck constant. The density parameter (1) can vary in quite a broad range but should always obey the inequality $r_s > 1.5$ to avoid consideration of the pressure ionization [44].

Another magnitude, appropriate for the description of the dusty plasma state, is the coupling parameter that measures the degree of plasma nonideality by representing the ratio of the average energy of Coulomb interaction to the thermal kinetic energy, viz.

$$\Gamma = \frac{e^2}{ak_BT}, \tag{2}$$

with k_B symbolizing the Boltzmann constant and T being the system temperature. Due to the method used below and since the thermionic emission is under scrutiny, the temperature must be high enough, effectively restricting our consideration to the case of $\Gamma < 1$.

As for the dust component its number density is quantified by the grain density parameter

$$\gamma = n_d/n, \tag{3}$$

which is usually very small, $\gamma \ll 1$.

To simplify matters, each of the dust particles is assumed to be a hard ball of radius R with the dust material being characterized by some work function W for the electrons. It is crucial for the following that the dust particles have finite dimensions specified by the packing fraction as

$$\eta = \tfrac{4}{3}\pi n_d R^3, \tag{4}$$

which cannot exceed its upper bound theoretical value of $\pi/\sqrt{18}$.

It is, of course, obligatory that the dusty plasma retains its local quasineutrality by imposing the following condition:

$$n_e = n_p + Zn_d. \tag{5}$$

Note that the parameters above imply that the whole system is in its thermal equilibrium such that the temperatures of all plasma components are essentially the same, which is referred to in the literature as a thermal dusty plasma [45]. Such objects are not rare in nature and laboratory, see [46,47] and references therein, as exemplified by stellar atmospheres and meteors, active media in flames, magnetohydrodynamic generators, rocket exhausts, radiatively heated dust clouds, and specifically designed plasma generators, in all of which the thermionic emission is presumed to play an essential part in determining their rather interesting physical characteristics [48].

III. CHEMICAL MODEL

It is well known that a standard chemical model of partially ionized plasmas is constructed in the following way. An expression for the ideal part of the Helmholtz free energy is literally postulated to consist of independent contributions from all plasma constituents [49,50], thereby discounting their reciprocal interrelations. This is then called the linear mixing rule that only validates for a system of noninteracting particles [51] but undoubtedly breaks down when the role of interparticle correlations significantly grows. Chemical models are closely articulated by the Helmholtz free energy minimization procedure [52,53] and known to suffer from thermodynamic inconsistencies [54] approving their verification against results of the physical picture [55,56]. Quite recently the self-consistent version of the chemical model was put forward to correctly treat microscopic interinfluence of the charged and neutral components of partially ionized plasmas [37], which fundamentally relies on the generalized Poisson-Boltzmann equation, first derived in [57] from the Bogolyubov chain of equations for equilibrium distribution functions in the pair correlation approximation. It is a foremost gist of this section to extend the self-consistent chemical model to account for the presence of positively charged dust particles.

Strictly speaking, both the ionization equilibrium and the dust charge are governed by competing kinetic processes, which demands accurate knowledge of the corresponding cross sections. In the case of the ionization balance those competing processes are the thermal ionization and recombination, whereas for the electric charge of dust particles they are charging and discharging currents of oppositely charged plasma particles. Herein it is demonstrated that at thermal equilibrium the ionization degree as well as the dust charge are independent of the details of those kinetic processes and can predominantly be extracted in the framework of classical thermodynamics by minimizing an appropriately defined Helmholtz free energy. It has to be admitted, however, that such a chemical model pays off for its success in that it is totally incapable of predicting the electric charge of a solitary dust particle—only electric charge of dust particles in a dust cloud with a certain number density n_d can be determined, which makes the principal distinction from the OML-like approaches mentioned in the introductory Sec. I.

The chemical model conventionally starts from the well established assumption that the Helmholtz free energy of any system, F_{tot}, is ultimately found as a sum of the ideal, F_{id}, and excess, $F_{\rm exc}$, parts

$$F_{\rm tot} = F_{id} + F_{\rm exc}. \tag{6}$$

As it has been remarked above the ideal part of the Helmholtz free energy is additive due to the linear mixing rule and is to be defined in Sec. III A, while the same does not hold for the excess part, which is to be further qualified in the spirit of the self-consistent chemical model of partially ionized plasmas in Sec. III B.

A. Ideal part

The keystone of the whole present consideration is how to incorporate dust particles into the chemical model of a plasma. It is believed in the sequel that each dust particle is capable of emitting electrons, but cannot absorb protons. Despite the fact that electrons and protons are constantly neutralized on dust surfaces, this process is accounted for in the following by the presence of neutrals, as assured by the ergodic hypothesis. The main idea is to consider the emission of electrons by a dust grain as a certain effective ionization process, i.e., an uncharged dust particle is viewed as a neutral entity, whereas a charged dust particle is thought of as its ionized counterpart. This virtually completes a direct analogy with an atom, but the big difference is that all dust particles are thus charged or ionized in the above sense. On the contrary, from a purely theoretical point of view neutral atoms are inevitably present in a partially ionized plasma, as absolutely guaranteed by the detailed balance between the processes of ionization and recombination.

To continue with our approach, let us determine the minimal work that needs to be done in order to pull Z electrons out of a single neutral dust particle, whose material is characterized by the work function W, and bring them all to infinite separation one after another, thereby leaving the grain with the positive charge Ze. This work is easily derived within the framework of classical electrodynamics to be equal to

$$A = ZW + \frac{Z(Z+1)e^2}{2R}. \tag{7}$$

Note that the dust particles are treated as if they would have a fixed charge, but in reality the dust charge fluctuates around its mean value Z.

It is now uncomplicated to obtain an expression for the Helmholtz free energy of charged dust particles. Namely, we start from the ideal part of the Helmholtz free energy of N_d neutral dust particles which reads as [58]

$$F_0 = N_d k_B T\left[\ln\left(n_d\lambda_d^3\right) - 1\right], \tag{8}$$

with the dust thermal de Broglie wavelength λ_d defined below.

Then, at fixed temperature, the work (7) is done to charge each of N_d dust particles, which, in accordance with the general laws of thermodynamics, immediately provides their Helmholtz free energy in the following form:

$$F = F_0 + N_d A. \tag{9}$$

After some trivial transformation, the expression for the ideal part, F_{id}, of the Helmholtz free energy of the whole system can be cast as

$$\frac{F_{id}}{Vk_BT} = n_e\left[\ln\left(n_e\lambda_e^3/2\right) - 1\right] + n_p\left[\ln\left(n_p\lambda_p^3\right) - 1\right] + n_n\left[\ln\left(n_n\lambda_n^3/\sigma\right) - 1\right] + n_d\left[\ln\left(n_d\lambda_d^3/\Sigma\right) - 1\right], \tag{10}$$

where V denotes the system volume, $\lambda_a = (2\pi\hbar^2/m_a k_B T)^{1/2}$ stands for the de Broglie wavelength of particles of species a, and the atomic partition function, σ, of hydrogen atoms is chosen in the following form, proposed by Planck and Larkin [59]:

$$\sigma = \sum_{n=1}^{\infty} 2n^2\left[\exp\left(\frac{I_n}{k_BT}\right) - 1 - \frac{I_n}{k_BT}\right], \tag{11}$$

with $I_n = -I/n^2$ referring to the energetic spectrum of the hydrogen atom and $I = -m_e m_p e^4/2(m_e+m_p)\hbar^2$ being the ground state energy, and the dust partition function is defined as

$$\Sigma = \exp\left(-\frac{A}{k_BT}\right). \tag{12}$$

Expression (10) is rather remarkable for it allows one to evaluate the thermodynamic properties of the entire system as a whole, not just those of the dust component. Particularly, the first three terms are very well known and widely used in chemical models of partially ionized plasmas [60], whereas the fourth term stands for the dust component and is derived by combining Eqs. (7)–(9). Note that the third term on the right hand side of the relation (10) strictly corresponds to the neutral atoms, while the fourth precisely pertains to the charged dust particles. It should necessarily be emphasized that in the limiting case of the absence of protons and atoms, as well as of the contribution from the neutral dust particles, expression (10) promptly turns into the Helmholtz free energy of an electron-dust plasma [61], which enables one to establish exact correspondence with the results of the respective OML-like approximation.

B. Excess part

In a great variety of experimental and natural settings interparticle interactions play a noticeable role in dusty plasmas, which is especially true for dusts capable of acquiring a significant electric charge. One of the simplest options for taking into account interparticle interactions is a self-consistent chemical model [62], which is entirely based on the renormalization procedure of particle interactions [63] and leads to the following generalized Poisson-Boltzmann equation for the macroscopic potential $\Phi_{ab}(\mathbf{r}_i^a, \mathbf{r}_j^b)$ of interaction of particle species a and b, taking into account collective events in the medium

$$\Delta_i\Phi_{ab}\left(\mathbf{r}_i^a, \mathbf{r}_j^b\right) = \Delta_i\varphi_{ab}\left(\mathbf{r}_i^a, \mathbf{r}_j^b\right) - \frac{n_c}{k_BT}\int \Delta_i\varphi_{ac}\left(\mathbf{r}_i^a, \mathbf{r}_k^c\right)\Phi_{cb}\left(\mathbf{r}_j^b, \mathbf{r}_k^c\right)d\mathbf{r}_k^c. \tag{13}$$

Here $\varphi_{ab}(\mathbf{r}_i^a, \mathbf{r}_j^b)$ denotes the genuine microscopic interaction potential, $\mathbf{r}_i^a$ stands for the radius vector of the ith particle with

Δ_i being the corresponding Laplace operator, and n_c signifies the number density of particles of sort c. Note that above and everywhere below the summation is implied over the repeated subscripts of particle species.

In Fourier space the set of equations (13) for particle species converts into a set of linear algebraic equations, whose solution for the Fourier transform of the macropotential $\tilde{\Phi}_{ab}(k)$ is neatly expressed in terms of the Fourier transform of the microscopic potential $\tilde{\varphi}_{ab}(k)$ as follows [64]:

$$\begin{aligned}\tilde{\Phi}_{ab}(k) &= \frac{1}{\Delta}\bigg(\tilde{\varphi}_{ab}(k) + A_c[\tilde{\varphi}_{cc}(k)\tilde{\varphi}_{ab}(k) - \tilde{\varphi}_{ac}(k)\tilde{\varphi}_{bc}(k)] \\ &\quad + \delta_{ab}A_cA_d\bigg[\tilde{\varphi}_{ac}(k)\tilde{\varphi}_{ad}(k)\tilde{\varphi}_{cd}(k) + \frac{\tilde{\varphi}_{aa}(k)\tilde{\varphi}_{cc}(k)\tilde{\varphi}_{dd}(k)}{2} \\ &\quad - \frac{\tilde{\varphi}_{aa}(k)\tilde{\varphi}_{cd}(k)^2 + \tilde{\varphi}_{cc}(k)\tilde{\varphi}_{ad}(k)^2 + \tilde{\varphi}_{dd}(k)\tilde{\varphi}_{ac}(k)^2}{2}\bigg]\bigg),\end{aligned} \tag{14}$$

where

$$\begin{aligned}\Delta &= 1 + A_a\tilde{\varphi}_{aa}(k) + A_aA_b\big[\tilde{\varphi}_{aa}(k)\tilde{\varphi}_{bb}(k) - \tilde{\varphi}_{ab}(k)^2\big] + A_aA_bA_c \\ &\quad \times \bigg[\frac{\tilde{\varphi}_{ab}(k)\tilde{\varphi}_{bc}(k)\tilde{\varphi}_{ac}(k)}{3} + \frac{\tilde{\varphi}_{aa}(k)\tilde{\varphi}_{bb}(k)\tilde{\varphi}_{cc}(k)}{6} \\ &\quad - \frac{\tilde{\varphi}_{aa}(k)\tilde{\varphi}_{bc}(k)^2 + \tilde{\varphi}_{bb}(k)\tilde{\varphi}_{ac}(k)^2 + \tilde{\varphi}_{cc}(k)\tilde{\varphi}_{ab}(k)^2}{6}\bigg].\end{aligned} \tag{15}$$

Here $A_c = n_c/k_BT$ and δ_{ab} denotes the Kronecker delta.

In order to initiate application of the chemical model, the microscopic potentials must be appropriately chosen. The interactions between the electrons and protons of the plasma medium are taken to be pure Coulombic so that the Fourier transforms of the corresponding micropotentials are written as

$$\tilde{\varphi}_{ee}(k) = \tilde{\varphi}_{pp}(k) = -\tilde{\varphi}_{ep}(k) = \frac{4\pi e^2}{k^2}. \tag{16}$$

The microscopic potentials, involving electrons, protons, and neutrals, are picked out for the hydrogen plasma in the simplest available static form with the following Fourier transforms [65]:

$$\begin{aligned}\tilde{\varphi}_{pn}(k) &= -\tilde{\varphi}_{en}(k) = \frac{4\pi e^2\big(k^2 + 8/a_B^2\big)}{\big(k^2 + 4/a_B^2\big)^2}, \\ \tilde{\varphi}_{nn}(k) &= \frac{4\pi e^2}{\big(k^2 + 2/a_B^2\big)}.\end{aligned} \tag{17}$$

As for the microscopic potentials of interaction between dust particles and dust particles with electrons and protons, the following Fourier transforms are engaged [66]:

$$\begin{aligned}\tilde{\varphi}_{dd}(k) &= \frac{4\pi Z^2e^2}{k^2} - \frac{8\pi Z^2e^2R}{k}\bigg[\mathrm{Ci}(2kR)\sin(2kR) \\ &\quad + \frac{1}{2}\cos(2kR)[\pi - 2\mathrm{Si}(2kR)]\bigg],\end{aligned} \tag{18}$$

$$\begin{aligned}\tilde{\varphi}_{ed}(k) &= -\tilde{\varphi}_{pd}(k) = \frac{4\pi Ze^2}{k^2} - \frac{4\pi Ze^2R}{k}\bigg[\mathrm{Ci}(kR)\sin(kR) \\ &\quad + \frac{1}{2}\cos(kR)[\pi - 2\mathrm{Si}(kR)]\bigg],\end{aligned} \tag{19}$$

with $\mathrm{Ci}(x) = -\int_x^\infty \frac{\cos t}{t}dt$ and $\mathrm{Si}(x) = \int_0^x \frac{\sin t}{t}dt$ being the cosine and sine integral functions, respectively. It seems reasonable to assume that the interaction between hydrogen atoms and dust particles is really negligible because of their neutrality as compared to other kinds of forces described above and it is therefore dropped out in the subsequent analysis.

The excess part, F_{exc}, of the Helmholtz free energy is finally deduced in the framework of the self-consistent chemical model as [37,62]

$$\begin{aligned}\frac{F_{exc}}{Vk_BT} &= \frac{1}{2}n_an_b\frac{\tilde{\varphi}_{ab}(0)}{k_BT} \\ &\quad + \frac{1}{16\pi^3k_B^2}n_an_b\int d\mathbf{k}\,\tilde{\varphi}_{ab}(k)\int dT\,\frac{\tilde{\Phi}_{ab}(k)}{T^3}.\end{aligned} \tag{20}$$

Expression (20) for the excess part of the Helmholtz free energy is quite a broad generalization of the Debye-Hückel approximation, which is only valid in the pair correlation approximation for the Bogolyubov chain of equations for equilibrium distribution functions to soundly account for mutual interactions between all plasma components..

IV. IONIZATION EQUILIBRIUM IN PARTIALLY IONIZED PLASMAS

The chief objective of the present consideration, as its title states, is to establish the effect of positively charged dust particles on the ionization equilibrium in a plasma. As a first step and for the sake of comparison, we begin with studying the ionization balance in a partially ionized plasma, completely devoid of dust grains. To achieve this, it suffices to mathematically take the limit $n_d \to 0$ in the corresponding expressions for the ideal and excess parts of the Helmholtz free energy. In the framework of the self-consistent chemical model, numerical results were previously obtained elsewhere [37]; therefore, the rest of this section concentrates on derivation of pure analytical results. An important consequence is a generalization of the Saha equation, which is primarily written as [59,67]

$$\frac{n_en_p}{n_n} = \frac{2\lambda_n^3}{\lambda_e^3\lambda_p^3\sigma}. \tag{21}$$

As is straightly inferred from the laws of thermodynamics, the Helmholtz free energy of the system should take its minimum value at fixed magnitudes of temperature and total number of particles. It is convenient to introduce the ionization degree as $\alpha = n_e/n$, and it then follows from the quasineutrality condition (5) at $n_d = 0$ that $n_p = \alpha n$, as well as $n_n = (1-\alpha)n$, comes from the conservation of the total number of protons. Hence, at fixed magnitudes of parameters (1) and (2), the Helmholtz free energy of a partially ionized plasma becomes a function of the single parameter α, which is then minimized to obtain the equilibrium value of α. This main procedure in the framework of the chemical picture is consistently implemented below in order to obtain new analytical results on the so-called ionization potential depression [68–70], which is most correctly described within *ab initio* simulations at high densities [71] and quantum statistical theory at low densities [72].

A. Ideal plasma

Let a plasma be in an ideal state of matter when interparticle interactions can be totally omitted. Then, minimizing the ideal part of the Helmholtz free energy results in the following form of the Saha equation (21):

$$\frac{\alpha^2}{1-\alpha} = \frac{2\lambda_n^3}{n\lambda_e^3\lambda_p^3\sigma}, \tag{22}$$

which is quadratic with the only positive root

$$\alpha = \frac{\lambda_n^3}{n\lambda_e^3\lambda_p^3\sigma}\left(\sqrt{1+\frac{2n\lambda_e^3\lambda_p^3\sigma}{\lambda_n^3}}-1\right). \tag{23}$$

Hence, as it is well known, the Saha equation strictly corresponds to the ideal gas approximation.

Let us consider two limiting cases which are particularly needed for further treatment of analytical results obtained below.

Case 1 of weakly ionized state. If $2\lambda_n^3/n\lambda_e^3\lambda_p^3\sigma \ll 1$, then the ionization degree is very low and is approximated from the exact solution (23) as

$$\alpha_{id}^0 = \sqrt{\frac{2\lambda_n^3}{n\lambda_e^3\lambda_p^3\sigma}}. \tag{24}$$

Case 2 of strongly ionized state. If $2\lambda_n^3/n\lambda_e^3\lambda_p^3\sigma \gg 1$, then the ionization degree is very high and is approximated from the exact solution (23) as

$$\alpha_{id}^1 = 1 - \frac{n\lambda_e^3\lambda_p^3\sigma}{2\lambda_n^3}. \tag{25}$$

It should be mentioned that formulas (24) and (25), albeit elementary in derivation, are crucial for understanding the results of the following subsection.

B. Nonideal plasma

If a plasma is found in a nonideal state such that interparticle correlations can no longer be ignored, the minimization of the Helmholtz free energy can only be fulfilled numerically because the contribution of the excess part turns rather essential. Nevertheless, it is still possible to analytically study the same limiting cases as in the previous subsection. After all it seems much more practical to search for an approximate solution to the following equation:

$$\frac{dF_{\text{tot}}}{d\alpha} = 0, \tag{26}$$

strictly corresponding to the minimum of the Helmholtz free energy.

Case 1 of weakly ionized state. When the ionization is very low, $\alpha \ll 1$, the excess part of the Helmholtz free energy can be expanded in series in the vicinity of $\alpha = 0$ and the following formula holds for the ionization degree:

$$\alpha_{\text{tot}}^0 = \sqrt{\frac{2\lambda_n^3}{n\lambda_e^3\lambda_p^3\sigma\,\exp(-\Delta I_0/k_BT)}}, \tag{27}$$

where

$$\Delta I_0 = k_BTx + \frac{e^2}{\sqrt{2}a_B}f_0(x) \tag{28}$$

and

$$f_0(x) = 1 + \frac{x\sqrt{2(1+x)}(13-259x+127x^2-9x^3) - 16(1-3x-16x^2-4x^3+7x^4-x^5)}{16(1-x)^4\sqrt{1+x}}, \tag{29}$$

with $x = 2\pi ne^2a_B^2/k_BT$.

Comparison of formula (27) with expression (24) reveals that ΔI_0 can immediately be interpreted as the ionization potential depression, which is essentially caused by interatomic interactions in a plasma. This effect is well known in the literature and can be numerically taken into account [73,74] but we present here the analytical result (28).

Case 2 of strongly ionized state. When the ionization is rather high, $\alpha \approx 1$, the excess part of the Helmholtz free energy can be expanded in series in the vicinity of $\alpha = 1$ yielding the following expression for the ionization degree:

$$\alpha_{\text{tot}}^1 = 1 - \frac{n\lambda_e^3\lambda_p^3\sigma\,\exp(-\Delta I_1/k_BT)}{2\lambda_n^3}, \tag{30}$$

where

$$\Delta I_1 = \frac{e^2}{\sqrt{2}a_B}f_1(x), \tag{31}$$

with

$$f_1(x) = \frac{\sqrt{2x}(32+115\sqrt{x}+140x+75x^{3/2}+16x^2)}{16(1+\sqrt{x})^4} \tag{32}$$

and the same notation for x as above.

It is again concluded by contrasting formulas (25) and (30) that ΔI_1 can be treated as the ionization potential depression, which is now connected to the dominating contribution of the charged component of the plasma. It is rather simple to demonstrate that, at small values of $x \ll 1$, formula (31) exactly reproduces the classical ionization potential depression in the Debye approximation [75,76].

C. General case

The problem of determining the ionization potential depression in the medium or the so-called continuum lowering has a long history and dates back to the 1960s of the past century, when two competing approaches were proposed. The first method of Stewart and Pyatt [77] was based on the approximation of an electron-nucleus electrostatic interaction by the Thomas-Fermi potential at a finite temperature. The other approach was formulated by Ecker and Kröll [78] and stemmed from the generalized Saha equation incorporating the chemical potential of the surrounding plasma. The experimental capabilities of that time refrained from making a univocal choice in favor of one of the two concepts, but the situation has recently changed dramatically. Rather fresh experiments with matter at high-energy densities [79–81] have revived the subject by demonstrating that the problem of the ionization potential depression still requires further elaboration.

It is beyond our intent to go deep into the topic of the ionization potential depression; nevertheless, in the previous subsection we have been able to analytically evaluate the continuum lowering of hydrogen medium both in an almost completely ionized state and in the regime of weak ionization. A natural question arises whether it is possible to determine the continuum lowering for an arbitrary ionization state of a partially ionized medium. The answer is positive, but the ionization potential depression ΔI should thus become a function of the ionization degree α, i.e., $\Delta I(\alpha)$, such that it is necessary to solve the following generalized Saha equation [82,83]:

$$\frac{\alpha^2}{1-\alpha} = \frac{2\lambda_n^3}{n\lambda_e^3\lambda_p^3\sigma \exp[-\Delta I(\alpha)/k_BT]}. \quad (33)$$

The ionization potential depression can be explicitly found in the framework of the self-consistent chemical model of a partially ionized plasma, but in the most general case rather cumbersome calculations are involved. However, the formulas obtained in the expansions above enable the following prediction for the ionization potential lowering:

$$\Delta I(\alpha) = (1-\alpha)^{1/2}\Delta I_0 + \alpha^{1/2}\Delta I_1, \quad (34)$$

such that it gives rise to ΔI_0 in (27) at $\alpha \approx 0$, and turns into ΔI_1 in (30) when $\alpha \approx 1$.

Formula (34) together with the generalized Saha equation (33) provides a vary fast converging scheme for evaluation of the ionization degree of partially ionized hydrogen, which has been verified in quite a broad domain of plasma parameters with the representative results shown in Fig. 1 and its inset. It is seen that perfect agreement is observed with the self-consistent chemical model and with the results of (27) and (30) in the corresponding ranges of the ionization degree. Note that the *ad hoc* dependence (34) is hardly variable since the ionization potential depression appears in the exponent of the generalized Saha equation (33), which is, thus, very sensitive to even slight change in its form.

V. IONIZATION EQUILIBRIUM IN DUSTY PLASMAS

That the electron bulk density can exceed its magnitude prescribed by the Saha equation (21) was first observed a long time ago in experiments with rich hydrocarbon flames [84,85]. As was already mentioned in the introductory Sec. I, the recent experiment with the beam microwave interferometry clearly demonstrated [21] that an increased electron number density is a real effect for rather large dust particles even for gas discharge plasmas.

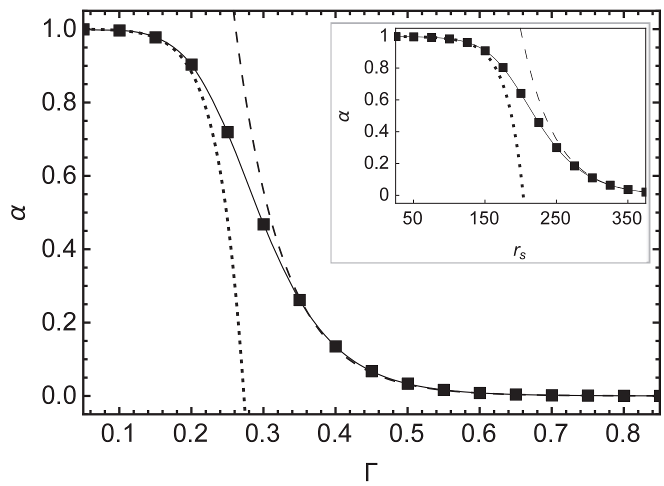

FIG. 1. Ionization degree α of a partially ionized hydrogen plasma as a function of Γ at $r_s = 40$ and as a function of r_s at $\Gamma = 0.2$ for the inset. Squares: [37]; solid lines: formulas (33) and (34); dashed lines: formulas (27) and (28); dotted lines: formulas (30) and (31).

In the past the influence of dust particles on the electron number density in a plasma was handled within the application of the Saha equation to charging or discharging of dusts in a plasma; see for details [47] and references therein. To do so, the ionization potential was formally replaced by $W + Ze^2/R$ in (21) written for the balance of electrons and dust grains, thereby implying that an uncharged dust particle is considered an analog of neutral atom in a partially ionized medium. Using this heuristic idea charge distribution in dust clouds was neatly determined for thermal equilibrium [86] and another phenomena taken into account in the method were the electric field emission process [87,88] and quantum tunneling [89]. However, it is worthwhile mentioning that such an employment of the Saha equation suffers from few drawbacks. First of all, the presence of positively charged ions is totally ignored, which vastly restricts the applicability of the results obtained. Secondly, interactions between plasma components are discarded as well, although the interdust correlations are usually quite strong. And finally, for a dust particle to attain a certain charge Z, the presence of all intermediate charge states of dusts, including neutral ones, is necessary.

Recently we have put forward an alternative approach [61], which is essentially deprived of the shortcomings mentioned in the previous paragraph. It is based on exact derivation of the Helmholtz free energy of electron-dust plasmas and has proven to explicitly reproduce the OML-like results for the charge of emitting dust particles. Herein this approach is extended to a principally different situation when positive ions and neutrals are present, thereby opening new grounds for simultaneously studying the ionization equilibrium and charging of dust grains.

In particular, in Sec. III the expression for the Helmholtz free energy of the four-component dusty plasma has been deduced to satisfy two limiting cases: (i) the electron-dust plasma of [61] and (ii) the partially ionized hydrogen plasma of [37]. At fixed values of the dimensionless parameters (1)–(4) as well as the work function, W, the total free energy $F_{\rm tot}$ of the dusty plasma depends on the number density of free electrons, n_e, the number density of free protons, n_p, and the charge number Z of dust grains, which are locally related via the quasineutrality condition (5). Introducing the reduced quantities $\alpha_e = n_e/n$ and $\alpha_p = n_p/n$ and excluding Z with the aid of the quasineutrality relation (5), the Helmholtz free energy ultimately turns out to be a function of the two variables, say α_e and α_p, and its further two-parametric minimization provides corresponding equilibrium magnitudes. Note that in general $\alpha_e \neq \alpha_p$ and, hence, there are two ionization degrees in dusty plasmas: α_e for free electrons and α_p for free protons. Instead of minimizing the system free energy it is simpler from the practical point of view to simultaneously solve the following set of equations:

$$\frac{\partial F_{\rm tot}}{\partial \alpha_e} = 0, \quad \frac{\partial F_{\rm tot}}{\partial \alpha_p} = 0, \tag{35}$$

which are both integral and transcendental with respect to α_e and α_p.

A. Ideal plasma

First of all, we study the ideal gas regime when the excess part can be neglected in expression (35). It is curious to mention that substituting the ideal part of the Helmholtz free energy into Eqs. (35) and taking their sum provide the following relation:

$$\frac{\alpha_e \alpha_p}{1-\alpha_p} = \frac{2\lambda_n^3}{n\lambda_e^3\lambda_p^3\sigma}, \tag{36}$$

which is a straightforward consequence of the Saha equation (21) with $\alpha_e \neq \alpha_p$. It is therefore inferred that the electron and proton ionization degrees are still related via the Saha equation, which is physically conceivable since the Saha equation is valid for the ideal gas approximation and is a result of the competition between the ionization and recombination processes even in a part of the system volume deprived of dust particles. Keep in mind, however, that in order to independently determine α_e and α_p, it is necessary to solve both equations (35).

Case 1 of weakly ionized state. When the ionization is very low, such that both $\alpha_e, \alpha_p \ll 1$, the set of equations (35) can be expanded in the vicinity of $\alpha_e = 0$, $\alpha_p = 0$ till the zeroth order resulting in the following formulas for the electron and proton ionization degrees:

$$\alpha^0_{id,e} = \frac{2}{n\lambda_e^3}\exp\left(-\frac{W}{k_BT} - \frac{e^2}{2Rk_BT}\right), \tag{37}$$

$$\alpha^0_{id,p} = \frac{\lambda_n^3}{\lambda_p^3\sigma}\exp\left(\frac{W}{k_BT} + \frac{e^2}{2Rk_BT}\right). \tag{38}$$

It is interesting to note in this case that the electron and proton ionization degrees do not actually depend on the number density of dust particles, but do depend on their size and the work function of the dust material. In virtue of the quasineutrality condition (5) this means that the charge density of the dust component maintains its invariance throughout a nonuniform dust cloud and, thus, the charge of dust particles is inversely proportional to their number density.

It is to be shown below that in the case of low ionization the inequalities $\alpha_p \ll \alpha_e \ll 1$ virtually hold; thus the set of equations (35) can be expanded in the vicinity of $\alpha_e = 0$ till the first order and in the vicinity of $\alpha_p = 0$ till the zeroth order to provide more accurate expressions for the electron and proton ionization degrees

$$\alpha^0_{id,e} = \frac{Rn_dk_BT}{ne^2}{\rm Lm}\left(\frac{2e^2}{Rn_dk_BT\lambda_e^3}\exp\left[-\frac{W}{k_BT} - \frac{e^2}{2Rk_BT}\right]\right), \tag{39}$$

$$\alpha^0_{id,p} = \frac{2e^2\lambda_n^3}{Rn_dk_BT\lambda_e^3\lambda_p^3\sigma} \times \left\{{\rm Lm}\left(\frac{2e^2}{Rn_dk_BT\lambda_e^3}\exp\left[-\frac{W}{k_BT} - \frac{e^2}{2Rk_BT}\right]\right)\right\}^{-1}, \tag{40}$$

where ${\rm Lm}(x)$ stands for the product logarithm or Lambert function.

When $x \ll 1$, expressions (39) and (40) reduce to formulas (37) and (38), respectively. However, for both sets of formulas the following relation is obviously secured:

$$\alpha^0_{id,e}\alpha^0_{id,p} = \frac{2\lambda_n^3}{n\lambda_e^3\lambda_p^3\sigma} = (\alpha^0_{id})^2, \tag{41}$$

with α^0_{id} being taken from expression (24). The striking regularity (41) is remarkable in many respects and bears simple physical meaning, stating that the product of the electron and proton ionization degrees in a dusty plasma is equal to the square of the ionization degree in a dust-free partially ionized plasma at the same values of the system temperature and the total number density of protons. It can be numerically demonstrated that at low ionization degrees $\alpha^0_{id,e} \gg \alpha^0_{id}$ and, therefore, $\alpha^0_{id,p} \ll \alpha^0_{id}$, which means that at the same external conditions injection of dust particles into a partially ionized plasma leads to a significant growth of the electron number density and, at the same time, to a depletion of the proton number density, as it was experimentally evidenced in [21].

Case 2 of strongly ionized state. When the plasma temperature grows, the proton ionization degree approaches unity, i.e., $\alpha_p \to 1$, whereas the electron ionization degree continues to rise further due to thermionic emission such that $\alpha_e \gg 1$. Expanding the set of equations (35) in the vicinity of $\alpha_e = \infty$, $\alpha_p = 1$ and solving it yield the following result:

$$\alpha^1_{id,e} = \frac{Rn_dk_BT}{ne^2}{\rm Lm} \times \left(\frac{2e^2}{Rn_dk_BT\lambda_e^3}\exp\left[-\frac{W}{k_BT} - \frac{e^2}{2Rk_BT} + \frac{e^2n}{Rn_dk_BT}\right]\right), \tag{42}$$

$$\alpha^1_{id,p} = 1 - \frac{Rn_dk_BT\lambda_e^3\lambda_p^3\sigma}{2\lambda_n^3e^2}{\rm Lm}\left(\frac{2e^2}{Rn_dk_BT\lambda_e^3} \times \exp\left[-\frac{W}{k_BT} - \frac{e^2}{2Rk_BT} + \frac{e^2n}{Rn_dk_BT}\right]\right). \tag{43}$$

In contrast to formulas (37) and (38), relations (42) and (43) contain direct dependence on the number density of dust particles. It is to be highlighted in Sec. VI that these relations have a lot to do with the results of the OML-like approach for an electron-dust plasma [61].

B. Nonideal plasma

When interactions between plasma particles are carefully taken into account, the Saha equation in the form of (36) definitely breaks down and the set of equations (35) has to be numerically solved, which, of course, cannot be performed analytically. Nevertheless, some approximations may still be adapted to attain very informative relations for the electron and proton ionization degrees.

Case 1 of weakly ionized state. If the electron and proton ionization degrees are very low, $\alpha_e, \alpha_p \ll 1$, the set of equations (35) can be expanded in the vicinity of $\alpha_e, \alpha_p = 0$ and the following formula is derived for the electron ionization degree:

$$\alpha^0_{\mathrm{tot},e} = \frac{2}{n\lambda_e^3}\exp\left(-\frac{W}{k_BT} - \frac{e^2}{2Rk_BT} + \frac{\Delta I_e^0}{k_BT}\right), \tag{44}$$

where the electron ionization potential depression is defined as

$$\Delta I_e^0 = k_BTx + \frac{e^2}{\sqrt{2}a_B}f_e^0(x), \tag{45}$$

with

$$f_e^0(x) = \frac{x[32x(1+x)(3-x)^2 + \sqrt{2(1+x)}(13-259x+127x^2-9x^3)]}{32(1-x)^4\sqrt{1+x}} \tag{46}$$

and the same definition for x as in Sec. IV.

Similar procedure for the proton ionization degree yields the following outcome:

$$\alpha^0_{\mathrm{tot},p} = \frac{\lambda_n^3}{\lambda_p^3\sigma}\exp\left(\frac{W}{k_BT} + \frac{e^2}{2Rk_BT} + \frac{\Delta I_p^0}{k_BT}\right), \tag{47}$$

with the proton ionization potential depression

$$I_p^0 = \frac{e^2}{\sqrt{2}a_B}f_p^0(x) \tag{48}$$

and

$$f_p^0(x) = 1 + \frac{\sqrt{2(1+x)}x(13-259x+127x^2-9x^3) - 32(1+x)^2(2x^2-5x+1)}{32(1-x)^4\sqrt{1+x}}. \tag{49}$$

It can immediately be stressed that both the electron (44) and proton (47) ionization degrees in nonideal plasmas exceed the corresponding values (37) and (38) in the ideal gas regime. Note that in dusty plasmas we have two ionization degrees, separately defined for electrons and protons, and, thus, two types of the ionization potential depression appear in the above analytical estimations.

The range of the applicability of formulas (44) and (47) is very well restricted and more practical expressions are delivered with the aid of expressions (39) and (40) by inserting the appropriate ionization potential depressions, which provide the electron and proton ionization degrees in the following forms:

$$\alpha^0_{\mathrm{tot},e} = \frac{Rn_dk_BT}{ne^2}\mathrm{Lm}\left(\frac{2e^2}{Rn_dk_BT\lambda_e^3}\exp\left[-\frac{W}{k_BT} - \frac{e^2}{2Rk_BT} + \frac{\Delta I_e^0}{k_BT}\right]\right), \tag{50}$$

$$\alpha^0_{\mathrm{tot},p} = \frac{2e^2\lambda_n^3}{Rn_dk_BT\lambda_e^3\lambda_p^3\sigma}\left\{\mathrm{Lm}\left(\frac{2e^2}{Rn_dk_BT\lambda_e^3}\exp\left[-\frac{W}{k_BT} - \frac{e^2}{2Rk_BT} - \frac{\Delta I_p^0}{k_BT}\right]\right)\right\}^{-1}. \tag{51}$$

Case 2 of strongly ionized state. It has already been mentioned above that if the system temperature rises, the proton ionization degree almost reaches its border value of unity, $\alpha_p \to 1$, while the electron ionization degree continues to grow much larger, $\alpha_e \gg 1$. Unlike the ideal plasma case, no rational approximations are possible for a nonideal plasma at high ionization degrees because the dominating contribution to the excess part of the Helmholtz free energy comes from the interdust interactions and the corresponding expansions and integrations cannot be altogether performed. As yet some simple estimations can be made to achieve the following formulas for the electron and proton ionization degrees:

$$\alpha^1_{\mathrm{tot},e} = \frac{\pi Rn_dk_BT}{(\pi-1)ne^2}\mathrm{Lm}\left(\frac{2(\pi-1)e^2}{\pi Rn_dk_BT\lambda_e^3}\exp\left[-\frac{W}{k_BT} - \frac{e^2}{2Rk_BT} + \frac{(\pi-1)e^2n}{\pi Rn_dk_BT}\right]\right), \tag{52}$$

$$\alpha^1_{\mathrm{tot},p} = 1 - \frac{\lambda_p^3\sigma}{2\lambda_n^3}\exp\left[-\frac{W}{k_BT} - \frac{e^2}{2Rk_BT} + \frac{(\pi-1)e^2n(1-\alpha^1_{\mathrm{tot},e})}{\pi Rn_dk_BT} + x(1-\alpha^1_{\mathrm{tot},e})\right]. \tag{53}$$

Accurate analysis clearly demonstrates that formulas (52) and (53) give rise to higher degrees of electron and proton ionizations than corresponding expressions (42) and (43) for an ideal system.

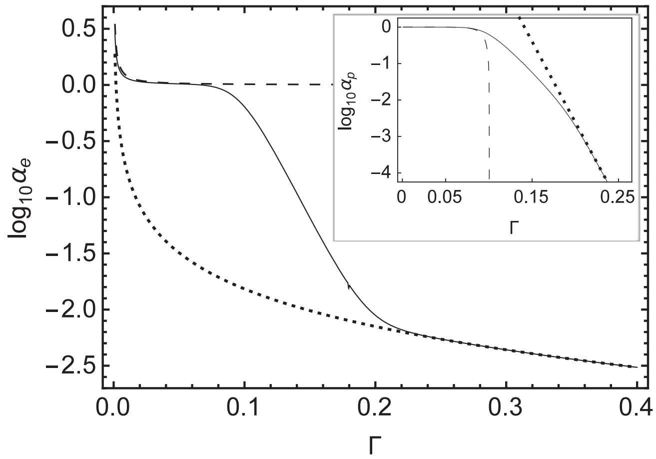

FIG. 2. Base-10 logarithms of the electron ionization degree α_e and of the proton ionization degree α_p for the inset as functions of Γ at $r_s = 200$, $\gamma = 0.0001$, $\eta = 1.25 \times 10^{-4}$, and $W = 1$ eV. Solid lines: present exact result; dashed lines: approximations (52) or (53); dotted lines: approximations (50) or (51).

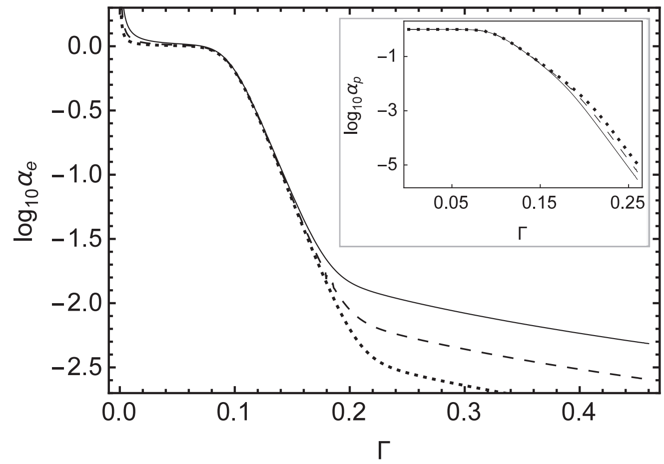

FIG. 4. Base-10 logarithms of the electron ionization degree α_e and of the proton ionization degree α_p for the inset as functions of Γ at $r_s = 200$, $\gamma = 10^{-4}$, and $W = 1$ eV. Solid lines: $\eta = 10^{-3}$; dashed lines: $\eta = 1.25 \times 10^{-4}$; dotted lines: $\eta = 1.5625 \times 10^{-5}$.

C. General case

In contrast to a partially ionized plasma the general case of arbitrary ionization in dusty plasmas can only be handled numerically and it is thus a chief goal of the following to inspect how deeply the ionization equilibrium is modified by the presence of dust particles in the plasma medium. First of all we check out the analytical approximations developed above against the results of exact numerical calculations, which is done in Fig. 2 and its inset for both the electron and the proton ionization degrees. It is clearly confirmed that the analytical approximations work excellently in the corresponding regions of ionization and, therefore, can be vastly used for practical purposes.

Figures 3 and 4 depict the receding electron and proton ionization degrees in dusty plasmas as a function of the coupling parameter Γ. Provided that the dust number density is fixed, as it is the case in Fig. 3 and its inset, both the electron and proton ionization degrees turn out to be independent of the density of the surrounding plasma medium at low magnitudes of the coupling parameter Γ, which is a characteristic feature of an electron-dust plasma in which the presence of free positive ions can be completely omitted because they are severely outnumbered by free electrons. However, this is not true for rather large values of the coupling when an increase in the density parameter r_s results in a drop of both the electron and proton ionization degrees. It has to be mentioned as well that further growth of the coupling parameter Γ should reveal the opposite behavior when the electron ionization degree diminishes with the growth of the density parameter r_s, which is due to an increasing contribution from the system nonideality. Note that with a decrease in the nonideality parameter Γ both the electron and proton degrees of ionization α_e and α_p increase with α_p approaching unity, and α_e persisting in rise due to the thermionic emission.

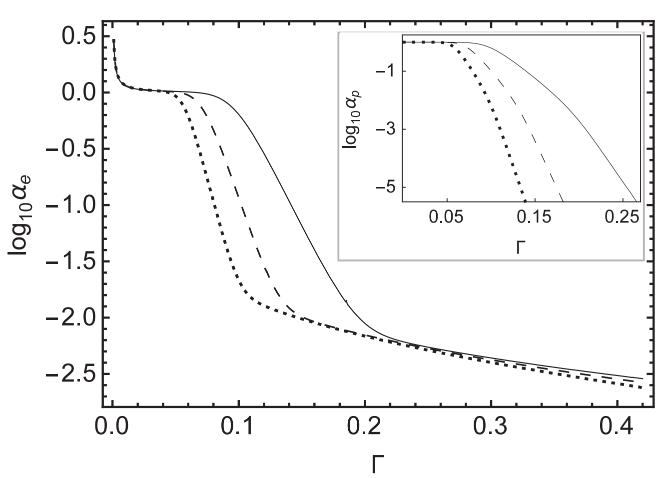

FIG. 3. Base-10 logarithms of the electron ionization degree α_e and of the proton ionization degree α_p for the inset as functions of Γ at $\gamma = 10^{-4}$, $\eta = 1.25 \times 10^{-4}$, and $W = 1$ eV. Solid lines: $r_s = 200$; dashed lines: $r_s = 300$; dotted lines: $r_s = 400$.

It is worthwhile identifying that the dependence of the ionization degrees on the work function W is obvious, i.e., the electron ionization degree should grow with regard to increasing W, and vice versa for the proton ionization degree. The similar behavior is observed in Fig. 4 and its inset for various values of the packing fraction η, which is straightly proportional to the cube of the dust particle radius. Note that there is always an intermediate range in the coupling parameter where the size of dust particles does not matter, whilst the same cannot be said about very high and very low temperatures. At rather large values of the coupling parameter when the dusty plasma is in a weakly ionized state, the electron ionization degree diminishes when the packing fraction η goes to zero, while the opposite tendency is discovered for the proton ionization degree.

It is rather self-evident that an increase in the density of dust particles n_d, which is proportional to γ, is responsible for a growth of the electron ionization degree α_e, and, at the same time, for a drop of the proton ionization degree α_p. This kind

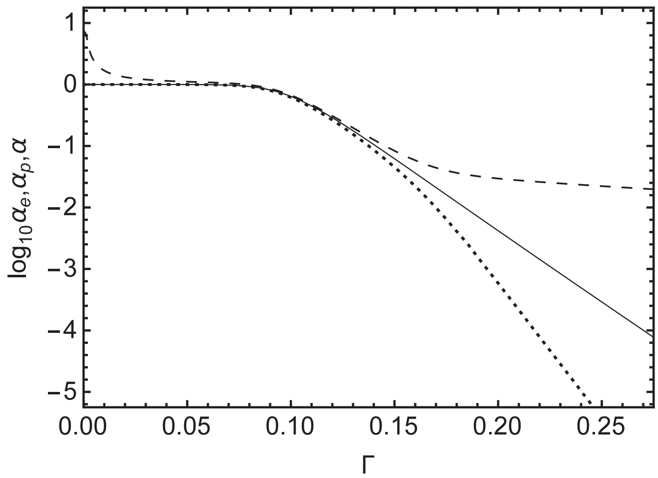

FIG. 5. Base-10 logarithms of the electron and proton ionization degrees α_e, α_p of dusty plasmas as functions of Γ at $r_s = 200$, $\gamma = 10^{-3}$, $\eta = 1.25 \times 10^{-4}$, and $W = 1$ eV. Dashed line: $\log_{10} \alpha_e$; dotted line: $\log_{10} \alpha_p$; solid line: $\log_{10} \alpha$ from formulas (33) and (34).

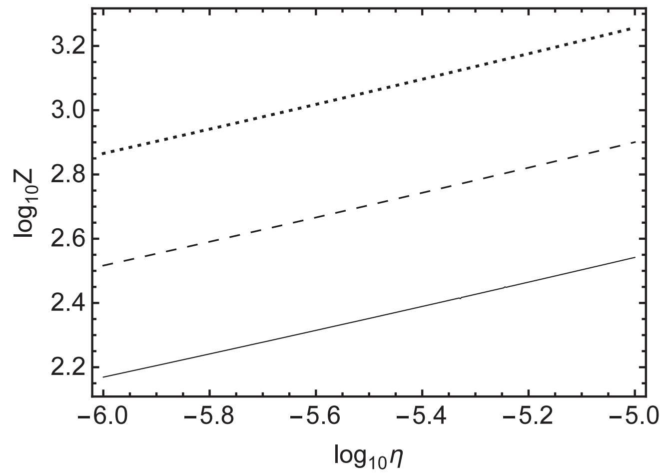

FIG. 6. Base-10 logarithm of the electric charge of the dust particles Z as a function of η at $\Gamma = 0.1$, $r_s = 200$, and $W = 1$ eV. Solid line: $\gamma = 10^{-3}$; dashed line: $\gamma = 10^{-4}$; dotted line: $\gamma = 10^{-5}$.

of pattern is firmly associated with the role of the thermionic emission, whose contribution is expected to grow at very low values of the coupling parameter Γ corresponding to the high temperature regime.

An important result is plotted in Fig. 5 in which the electron and proton ionization degrees are drawn as functions of the coupling parameter and compared with the ionization degree in a dust-free partially ionized plasma at fixed values of all other parameters. It is inevitably concluded that the self-consistent chemical model of partially ionized plasmas remains applicable to the dusty plasma at moderate degrees of ionization. However, for very low ionization degrees, relation (41) is numerically corroborated with $\alpha_e \gg \alpha$ and $\alpha_p \ll \alpha$ but $\alpha_e\alpha_p \approx \alpha^2$ or $\log_{10} \alpha_e + \log_{10} \alpha_p \approx 2 \log_{10} \alpha$, i.e., dust particles, brought into the partially ionized plasma, are responsible for an increase of the electron bulk density as it was recently reported in [21]. However, for high ionization degrees $\alpha_e \gg 1 \approx \alpha_{id}^0$ still holds due to the thermionic emission, whereas $\alpha_p \approx 1 \approx \alpha_{id}^0$.

It is required to mention that, in practice, the ionization degree in a partially ionized plasma can significantly vary, whereas in a dusty plasma it is almost always very low. Therefore, for a real laboratory dusty plasma formulas (50) and (51) seem only helpful but, in order to preserve the symmetry between the partially ionized plasma and the dusty plasma cases, the plasma parameters in Figs. 2–5 and below are chosen such that both the electron and proton ionization degrees could reach rather significant values.

VI. ELECTRIC CHARGE OF DUST PARTICLES

As noted above, the electric charge of dust particles in the framework of the proposed approach is not an independent quantity, being determined from the quasineutrality condition (5) as

$$Z = \frac{n_e - n_p}{n_d} = \frac{\alpha_e - \alpha_p}{\gamma}. \quad (54)$$

Since both the electron α_e and proton α_p degrees of ionization are determined by two-parametric minimization of the Helmholtz free energy, formula (54) gives an immediate opportunity to study the behavior of the electric charge of dust particles under various external conditions. In particular, the formulas of Sec. V can be directly used to obtain analytical expressions for the charge of dust particles in weakly and strongly ionized states, either taking into account the nonideality or not. These formulas have been sacrificed here because of their simple derivation in order to directly examine the general case.

In determining the charge of dust particles, the urgent question is its dependence on the grain size. For this purpose Fig. 6 is drawn to indicate the logarithm of the dust charge versus the logarithm of the packing fraction. The obtained dependence is almost linear, so that the charge of dust particles Z varies with their radius R according to the law $Z \sim R^n$ with the power exponent $n \approx 1.2$, which, nevertheless, slightly decreases with a growth of the number density of dust particles. It is thus concluded that the floating potential of dust particles in a dust cloud is no longer a constant but depends on the grain radius. Note that the numerical divergence in the charge of dust particles in Fig. 6 at fixed values of the number density of dust particles is due to their effect on the ionization equilibrium in the partially ionized hydrogen plasma.

Finally, the most important point is the influence of the dust number density on the dust grain charge, shown in Fig. 7. The use of the logarithmic scale again suggests a power-law dependence $Z \sim n_d^m$ with the power exponent $m \approx -0.36$ that has an insignificant trend to grow with decreasing the total number density of protons in the system or increasing the density parameter r_s. It is worthwhile emphasizing that a decrease in the charge of dust particles with an increase in their number density is caused by their effect on the ionization equilibrium in a partially ionized medium rather than by their absorption of plasma charge.

Of particular interest is a fundamental difference between the proposed approach to determining the charge of dust particles and the hitherto widely used OML-like

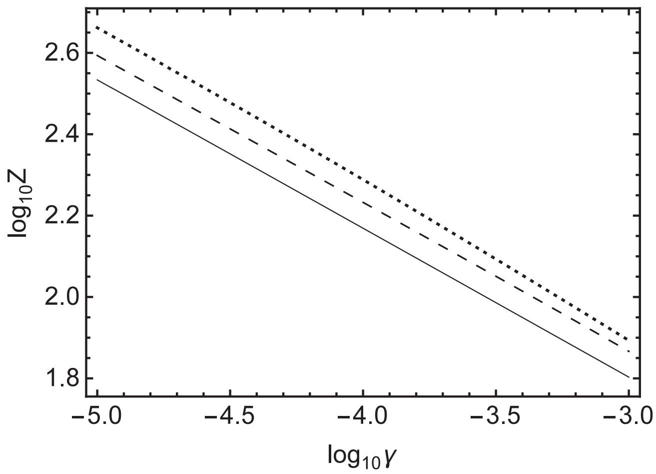

FIG. 7. Base-10 logarithm of the electric charge of the dust particles Z as a function of γ at $\Gamma = 0.1$, $\eta = 1.25 \times 10^{-4}$, and $W = 1$ eV. Solid line: $r_s = 200$; dashed line: $r_s = 300$; dotted line: $r_s = 400$.

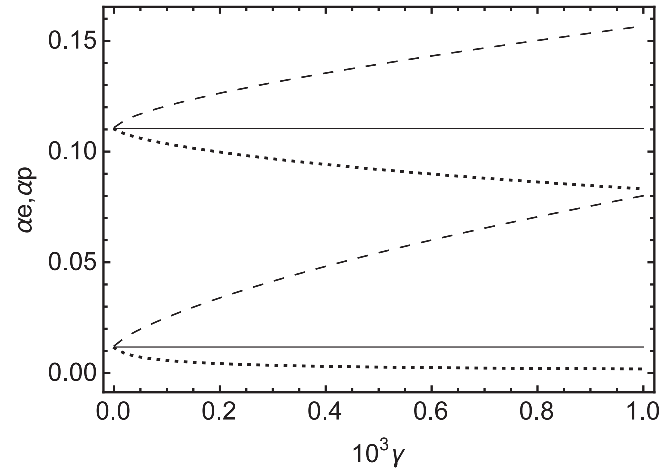

FIG. 8. Base-10 logarithm of the electron and proton ionization degrees α_e, α_p of dusty plasmas as functions of γ at $\Gamma = 0.1$, $\eta = 1.25 \times 10^{-4}$, and $W = 1$ eV. Dashed lines: α_e; dotted lines: α_p; solid lines: α from formulas (33) and (34). Upper lines for $r_s = 300$ and lower lines for $r_s = 400$.

approximations. The latter habitually considers the fluxes of electrons and ions on the surface of a solitary dust particle, whereas the method developed above works exclusively for the entire dust component with a certain number density n_d. There is, however, one exception when both concepts utterly agree with each other. Namely, in the case of an ideal electron-dust plasma the OML-like approach yields the following equation for the electric charge of dust particles [61]:

$$\frac{n_e}{2}\left(\frac{2\pi\hbar^2}{m_e k_B T}\right)^{3/2} = \exp\left(-\frac{W}{k_B T} - \frac{Ze^2}{Rk_B T}\right), \qquad (55)$$

whose solution is obtained under the quasineutrality condition $n_e = Zn_d$ as

$$Z = \frac{Rk_B T}{e^2}\mathrm{Lm}\left(\frac{2e^2}{Rn_d k_B T \lambda_e^3}\exp\left[-\frac{W}{k_B T}\right]\right). \qquad (56)$$

As remarked at the end of subsection V A, formula (56) exactly corresponds to the strong ionization case $\alpha_e \gg \alpha_p$ of an ideal dusty plasma (42) and (43) when the presence of protons can be entirely disregarded. It should be stressed that the nature of the agreement between the OML-like approach (55) and the present chemical model lies in imposing the quasineutrality condition $n_e = Zn_d$, which inescapably requires the dust grains be treated as a plasma component even within the OML-like approximation.

It is rather interesting to admit that the whole charge actually resides on a solitary dust particle when the limit $n_d \to 0$ is numerically taken, which Fig. 7 precisely exposes. In this case, the electron and proton degrees of ionization behave adequately, striving for the same limit of the ionization degree of a dust-free partially ionized plasma at $n_d \to 0$, which is decidedly demonstrated in Fig. 8 for two different sets of parameters.

Recognizing the fundamental difference between the proposed chemical model and the OML-like approximations, the question arises on under which conditions each of them validates. That the charge of dust particles in a dust cloud is reduced in comparison with the charge of a solitary dust grain was noted long ago by Havnes *et al.* [90], which was attributed to the plasma depletion on dusts. In reality, with an injection of dust particles, the ionization equilibrium in the plasma is itself affected, which must also be taken into account. Strictly speaking, the OML-like approximations work particularly well for a solitary dust particle, and their applicability to the dust component as a whole requires that the following inequality $a_d \gg \lambda_D$ be satisfied, where a_d is still the average interdust spacing and λ_D refers to the Debye screening length in the plasma. Indeed, in this case, the plasma sheaths of neighboring dust particles do not actually overlap, which allows us to consider them as solitary. However, such dust particles practically do not interact via their electric fields, remaining virtually isolated. Under such conditions, the method proposed herein does not work; nevertheless, it can yet be modified for the case of a solitary dust, which is to be done in the forthcoming investigations. Our chemical model is definitely valid in the opposite and practically more important case of $a_d \leqslant \lambda_D$, when the electrical interaction firmly ties all dust grains into a single plasma component.

VII. CONCLUSIONS

This article has advocated the thermodynamic point of view on thermal dusty plasmas by exclusively focusing on the effect of positively charged dust particles on the ionization equilibrium in a partially ionized plasma. To do so, an expression has been derived for the Helmholtz free energy of a four-component hydrogen plasma at thermal equilibrium containing free electrons, protons, neutral atoms, and dust particles. In the ideal part of the Helmholtz free energy, the work that needs to be done to positively charge all dust particles has been taken into account, while the excess part has been handled in a way similar to the self-consistent chemical model previously developed for a partially ionized plasma.

In the beginning, the ionization equilibrium in a partially ionized hydrogen has been considered. In the limit of high ionization degrees, an expression has been obtained for the ionization potential depression, which is somewhat identical to the classical expression of the Debye theory. Similarly, an expression has been predicted for the ionization potential depression at low ionization degrees, which is completely originated by the plasma neutral component. In the case of an arbitrary ionization degree, a simple interpolating expression has been proposed for the ionization potential depression, depending on the ionization degree itself, and the solution of the corresponding generalized Saha equation has been demonstrated to be in a perfect agreement with the results of calculations within the self-consistent chemical model.

As for a dusty plasma with positively charged particles, there exists two ionization degrees, the electron and proton ones, that correspond to the abundance of free electrons and protons in the system. As long as the Helmholtz free energy has been derived, its two-parametric minimization has allowed us to determine the equilibrium value of the proton ionization degree, which is always less than unity, as well as the equilibrium value of the electron ionization degree, which can exceed unity due to thermionic emission. Note that the electric charge of dust particles is not an independent quantity of the method, since it is excluded via the quasineutrality condition (5).

It has been shown that when a dusty plasma is treated as an ideal gas of particles, the electron and proton ionization degrees continue to be related via the Saha equation (36), which is not enough for their independent determination. It has been proven that in the case of very small electron and proton ionization degrees, they are practically independent of the number density of dust particles. Moreover, it then follows from the quasineutrality condition (5) that the dust charge density should remain constant throughout a dust cloud, although the number density of dust grains can vary from one point in space to another. More practical expressions for the electron and proton ionization degrees have also been obtained, such that their product is equal to the square of the plasma ionization degree in the absence of dust particles but under the same external conditions. In the high temperature limit, formulas have been found for the proton ionization degree, tending to unity when the coupling parameter vanishes, and for the electron ionization degree, growing unlimitedly due to the phenomenon of thermionic emission. In the limit of small degrees of ionization, it has been shown that the nonideality of dusty plasmas leads to a simultaneous increase in the number of free electrons and protons as compared with the ideal gas approximation. The corresponding values of the ionization potential depressions have been derived for electrons and protons, which has made it possible to obtain analytical expressions for the electron and proton ionization degrees valid in quite a broad domain of plasma parameters. In the case of high ionization degrees, plasma nonideality results in an increase in both the electron and proton ionization degrees as compared to the case of an ideal system. Note, however, that to embrace the whole range of ionization degrees the plasma parameters have been deliberately chosen such that the numerical examples are not exactly related to realistic laboratory dusty plasmas in which the ionization degree always remains very low.

The method developed has made it possible to determine the charge of dust particles without considering the fluxes of plasma particles on their surfaces, which is typical for the OML-like approximations, valid for a solitary dust grain. On the contrary, within the framework of the present approach, the charge of dust particles with the number density n_d in the dust cloud has been evaluated when the plasma sheaths of neighboring dust grains overlap to a large extent. It is demonstrated that the dust charge can be approximated as $Z \sim n_d^m R^n$ with the power exponents $m \approx -0.36$ and $n \approx 1.2$ with R being the dust particle radius.

The results presented herein can be expanded in a whole plethora of interesting directions, which thus form provisions for future work. First of all, we have obtained an expression for the Helmholtz free energy of dusty plasmas in the state of thermodynamic equilibrium, which opens up the possibility of determining all thermodynamic functions and tracing their influence on the spectra of collective modes of the dust component. In addition, from the viewpoint of wider practical application, it is important to extend the presented approach to a nonisothermal dusty plasma with negatively charged dusts, which can be performed in the framework of both nonequilibrium thermodynamics [91,92] and nonequilibrium statistical mechanics [93]. And finally, within the framework of the self-consistent model, there is a straightforward opportunity to study not only thermodynamic quantities of dusty plasmas, but transport coefficients as well, since knowledge of the interparticle interaction potentials enable direct evaluation of either the corresponding cross sections [37] or the Coulomb logarithm [94].

ACKNOWLEDGMENTS

The authors gratefully acknowledge financial support from the Ministry of Education and Science of the Republic of Kazakhstan under state Grant No. AP05132677.

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