Stationary scaling in small-scale turbulent dynamo problem

A. V. Kopyev,[1,*] A. S. Il'yn,[1,2,†] V. A. Sirota,[1,‡] and K. P. Zybin[1,2,§]
[1]P.N. Lebedev Physical Institute of RAS, 119991, Leninskij pr.53, Moscow, Russia
[2]National Research University Higher School of Economics, 101000, Myasnitskaya 20, Moscow, Russia

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We consider forced small-scale magnetic field advected by an isotropic turbulent flow. The random driving force is assumed to be distributed in a finite region with a scale smaller than the viscous scale of the flow. The two-point correlator is shown to have a stationary limit for any reasonable velocity statistics. Its spatial dependence is found to be a power law. The scaling exponent is found to be close to 3.

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I. INTRODUCTION

The origin of magnetic fields in interstellar media, as well as the origin of stellar and planetary magnetism, has been under consideration for a long time [1,2], but it still remains unknown. The common point of view is that the mechanism to generate the magnetic field is dynamo: the motions of conducting fluid stretch magnetic lines and thus amplify the field. Turbulence is a natural source for these motions [3]. Small scales corresponding to the viscous range of turbulence is one of the possibilities for intensive dynamo [1] because the smallest scales provide large velocity gradients [4] which result in fast growth of the magnetic field. In the viscous range of scales, one can simplify the problem using the Batchelor approximation for velocity flow [5].

The absence of a stationary solution is one of the problems of the dynamo approach. The usual ("standard") assumption introduced by [6,7] assumes spatial homogeneity of the initial magnetic field. In [6,8] it was shown that for homogenous initial fluctuations with correlation length less than the viscous scale, the magnetic field increases exponentially. So the way to get a stationary state was to stop the exponential growth by means of nonlinearity. This could happen when magnetic field energy density would become of the order of the kinetic energy density of the turbulent flow, to provide the feedback of magnetic field on the velocity dynamics. This approach assumed a very intensive magnetic field. The other way to avoid the exponential increase of the flux is to consider fluctuations with a correlation length much bigger than the viscous scale. The existence of stationary scaling at scales inside the inertial range[1] was shown in [9] and [10,11].

On the other hand, in [12] the evolution of a single isolated magnetic blob was considered in the Batchelor limit under some more simplifying assumptions on velocity field. It was found that after some time the exponential growth of magnetic field changed into exponential decay (the so-called antidynamo theorem).

The apparent contradiction between these two results was eliminated in [13,14], where it was shown that the infinite growth of the magnetic field in the homogeneous case can be understood as overlapping of a large amount of independent expanding blobs. In particular, for any finite initial magnetic field distribution and for any velocity statistics, the exponential increase changes into exponential decay as soon as contributions of the furthest blobs are exhausted. So the natural way to avoid the exponential growth is to consider finite initial magnetic field distribution.

This eventual free decay gives a new view on the problem of the existence of stationary solution, and a suggestion that it can be achieved for moderate fields, without a feedback. It is known that homogeneous *scalar* field decays exponentially as a function of time; this allowed [15] to construct a stationary solution for the field by adding some stochastic driving force. Just in the same way, the decay of a *quasihomogeneous vector* field allows one to propose that stationarity can be provided by some additional pumping source: the exponential decay of the magnetic field amplitude caused by the interaction with the flow (advection and diffusion) is then compensated by pumping. In this case, the statistically stationary regime in the viscous range can exist for much smaller magnetic fields, and properties of the stationary solution are determined by linear magnetohydrodynamic equation.

In this paper we investigate the stationary solution for magnetic field advected by random (generally, not Gaussian) velocity field and pumped by a Gaussian force. The diffusive scale r_d is assumed to be much smaller than the pumping correlation scale l, the region of pumping is assumed to be restricted by some scale $L \gg l$, which in turn is smaller than the Kolmogorov viscous scale of the velocity field. This last assumption allows to use the Batchelor approximation for velocity. We calculate the two-point second order correlator of magnetic flux density for scales $r \gg r_d$.

We use the method developed for freely decaying scalar and vector fields in our previous papers [13,16]. The additional difficulty in this problem is that, unlike the case of

*kopyev@lpi.ru
†asil72@mail.ru
‡sirota@lpi.ru
§zybin@lpi.ru

[1]In this case there exist regimes where the free field decays, for these regimes the stationary solution can be obtained by adding some pumping force.

forced passive scalar, extremely rare events make the most important contribution to the correlators (and even make them diverge in some cases).

It appears that the Gaussianity of the velocity field would be an important restriction because of its time-reversibility. For time-reversible flows, e.g., the mean square of magnetic flux density diverges linearly as a function of time, while for any time-irreversible statistics it has a finite limit [17]. This fact is related closely to the absence of exponential decay in the corresponding systems free of pumping [13].

The two-point correlation function converges for any velocity statistics, and the two-point correlator demonstrates scaling behavior.

The paper is organized as follows. In the next section we discuss the problem statement. In Sec. III we introduce convenient variables associated with the evolution matrix. We also recall the results of our previous papers concerning the relation between velocity statistics and statistics of these convenient variables. In Sec. IV one particular but important case of Gaussian velocity statistics is considered. In Sec. V we generalize the results for the case of arbitrary statistics. The results are summarized and discussed in Sec. VI.

II. PROBLEM STATEMENT

A. Magnetic field and pumping force

Let the magnetic field be excited by a random force $\phi(\mathbf{r}, t)$ and advected by a stochastic incompressible flow, velocity field $\mathbf{v}(\mathbf{r}, t)$ satisfying

$$\nabla \cdot \mathbf{v} = 0.$$

Then the magnetic flux density $\mathbf{B}(\mathbf{r}, t)$ obeys the equation

$$\frac{\partial \mathbf{B}}{\partial t} + (\mathbf{v}\nabla)\mathbf{B} - (\mathbf{B}\nabla)\mathbf{v} = \varkappa\Delta\mathbf{B} + \boldsymbol{\phi}. \tag{1}$$

Here $\varkappa$ is the diffusivity.

We consider the long-time evolution of magnetic field, the aim is to find statistically stationary solutions. So it is natural to consider the second order correlator

$$\beta_{ij}(\mathbf{r}_0, \mathbf{r}, t) = \langle B_i(\mathbf{r}_0, t) B_j(\mathbf{r}_0 + \mathbf{r}, t)\rangle_{\boldsymbol{\phi},\mathbf{v}}. \tag{2}$$

The average is taken over both the velocity field and the pumping force.

The pumping force is assumed to be statistically quas-homogeneous, with correlation scale l and the scale of inhomogeneity $L \gg l$. From (1) it follows that ϕ must be solenoidal; the details of its statistics are not much important; for the purposes of this paper, only the second order correlator is needed.[2] The pair correlator of the Fourier transform $\tilde{\boldsymbol{\phi}}(\mathbf{k}, t)$ can be written as [3]

$$\langle \tilde{\phi}_i(\mathbf{k}, t)\tilde{\phi}_j(\mathbf{k}', t')\rangle = \frac{1}{3\pi^2}\varepsilon_B L^3 l^5 e^{-\frac{1}{4}(\mathbf{k}+\mathbf{k}')^2 L^2} e^{-\frac{1}{4}(\mathbf{k}-\mathbf{k}')^2 l^2} \times [k_j k_i' - (\mathbf{k}\,\mathbf{k}')\delta_{ij}]\delta(t - t'). \tag{3}$$

In the physical space this corresponds to

$$\langle \phi_i(\mathbf{r}, t)\phi_j(\mathbf{r}', t')\rangle = \frac{8\pi}{3}\varepsilon_B e^{-\frac{(\mathbf{r}+\mathbf{r}')^2}{4L^2}} e^{-\frac{(\mathbf{r}-\mathbf{r}')^2}{4l^2}} \delta(t - t') \times \left[\delta_{ij} + \frac{(r - r')_i (r - r')_j - (\mathbf{r} - \mathbf{r}')^2\delta_{ij}}{4l^2} + O\left(\frac{l}{L}\right)\right].$$

Here $\varepsilon_B = \frac{1}{8\pi}\int\langle\boldsymbol{\phi}(0, t)\cdot\boldsymbol{\phi}(0, t')\rangle dt'$ is the pumping power. The scale L has the meaning of the boundary of the pumping region. In particular, $\langle\phi\phi'\rangle$ is small for $|\mathbf{r} - \mathbf{r}'| > l$, and even $\langle\phi^2\rangle$ is negligible at distances much larger than L. In the limit $L \to \infty$, the correlator depends only on $\mathbf{r} - \mathbf{r}'$; this is the homogeneous case.

We note that the velocity field and the dynamo mechanism remain the main source of energy; the role of the driving force is to provide stochastic stationary small magnetic field fluctuations.

B. Velocity field

The velocity field generally obeys the Navier-Stokes equation, the feedback of magnetic field is negligible. Here we consider velocity field to be a given stationary isotropic random process. We are interested in the scales much smaller than the Kolmogorov viscous scale r_η of the velocity field, in particular, we assume $L \ll r_\eta$. This means that the velocity field is smooth and in (quasi)Lagrangian frame [18] we get

$$\mathbf{v} = \mathbf{A}\mathbf{r}\ , \tag{4}$$

the velocity gradient tensor $\mathbf{A}(t)$ is a random process with short correlation time $\tau_c \ll t$. The incompressibility condition results in $Tr\mathbf{A} = 0$. Statistical properties of $\mathbf{A}$ will be considered later; here we only recall that the Lyapunov indices are the main characteristics of the strain tensor. These indices describe the time evolution of an infinitely small linear liquid element [19]: $\lim_{t\to\infty}\frac{1}{t}\ln|\delta\mathbf{r}| = \lambda$ (different λ_i for three different orientations of $\delta\mathbf{r}$). In this paper we use the ordering $\lambda_1 < \lambda_2 < \lambda_3$.

The diffusive scale $r_d \sim \sqrt{\varkappa/\lambda_3}$ is assumed to be small relative to l. We are interested in the scales much larger than r_d, to get nontrivial correlations, and much smaller than l, to make the details of the large-scale pumping unimportant. So, eventually the list of scales reads as

$$r_d \ll r \ll l \ll L \ll r_\eta.$$

The average over all realizations of velocity field is equivalent to the average over all $\mathbf{A}(t)$,

$$\langle\rangle_\mathbf{v} = \langle\rangle_A.$$

For simplicity, in what follows we consider the trace of β_{ij}; also, to shorten and simplify the equations, we consider the vicinity of the center, i.e., $\mathbf{r}_0 = 0$:

$$\beta = \mathrm{Tr}\boldsymbol{\beta} = \langle\mathbf{B}(0, t)\mathbf{B}(\mathbf{r}, t)\rangle_{\phi,A}. \tag{5}$$

[2] For small diffusivity, the main contribution to the whole magnetic field statistics is produced by the Gaussian part of the driving force.

[3] The specific choice of the exponential profile will be justified below by the fact that the parameters l, L do not contribute to the resulting exponents.

III. DYNAMIC SOLUTION AND AVERAGE OVER THE PUMPING

A. Evolution matrix

With account of (4), (1) can be rewritten as

$$\left(\frac{\partial}{\partial t}+A_{kj}r_j\nabla_k\right)B_i=A_{ij}B_j+\varkappa\Delta B_i+\phi_i. \qquad (6)$$

Making the Fourier transform and solving the linear equation, we get

$$\tilde{B}_p(\mathbf{k},t)=\int_0^t d\tau W_{pj}(t,\tau)\tilde{\phi}_j[\mathbf{k}\mathbf{W}(t,\tau),t-\tau] \times \mathrm{e}^{-\varkappa\mathbf{k}(\int_0^\tau \mathbf{W}(t,\tau')\mathbf{W}^T(t,\tau')\mathrm{d}\tau')\mathbf{k}^T}, \qquad (7)$$

where $\mathbf{W}(t,\tau)$ is the evolution matrix; it obeys the equation

$$\partial\mathbf{W}/\partial\tau=\mathbf{W}(t,\tau)\mathbf{A}(t-\tau), \quad \mathbf{W}(t,0)=\mathbf{1}. \qquad (8)$$

Since $\mathbf{A}$ is a random process, $\mathbf{W}$ is also random and its statistics is determined by the statistics of $\mathbf{A}$. The equations of this type have been studied by many authors beginning with [20] (the discrete version). It is convenient to make the Iwasawa decomposition for this matrix

$$\mathbf{W}=\mathbf{zdR}, \quad \mathbf{d}=\mathrm{diag}\{d_i\}, \quad d_i=e^{\rho_i}, \qquad (9)$$

where $\mathbf{z}$ is an upper triangular matrix with unities at diagonals, $\mathbf{R}$ is a rotation matrix, $\mathbf{d}$ is a diagonal matrix. The incompressibility condition implies that

$$\rho_1+\rho_2+\rho_3=0. \qquad (10)$$

According to (8) and (9), the stochastic processes ρ_i, $\mathbf{z}$, and $\mathbf{R}$ are functionals of the process $\mathbf{A}(t)$. The long-time asymptotic behavior of these three components is known to be quite different [21]: as $t\to\infty$, $\mathbf{z}(t)$ stabilizes with unitary probability at some random value that depends on the realization of the process; $\mathbf{R}(t)$ remains changing randomly, and $\rho_i(t)/t$ converge (with unitary probability) to finite limits that are called the Lyapunov indices λ_i:

$$\lambda_i=\lim_{t\to\infty}\frac{\rho_i}{t}, \quad \lambda_1\leqslant\lambda_2\leqslant\lambda_3. \qquad (11)$$

These indices are not random, they depend on statistics of $\mathbf{A}(t)$ but not on its realization.

In [22,23] statistical properties of ρ_i and $\mathbf{z}$ for large but finite time are expressed in terms of the statistical properties of $\mathbf{A}$. The details of the relation will be given in Sec. V C and Appendixes A and C. Here we note that the isotropy of $\mathbf{A}$ imposes rigorous constraints on the statistics of ρ_i. For what concerns $\mathbf{z}$, to logarithmic accuracy its nontrivial components can be expressed as functions of ρ_i at the same time and for the same realization

$$z_{12}\propto\max(1,e^{\rho_1-\rho_2}), \quad z_{13}\propto\max(1,e^{\rho_1-\rho_3},e^{\rho_2-\rho_3}),$$
$$z_{23}\propto\max(1,e^{\rho_2-\rho_3}) \qquad (12)$$

(see Appendix A for a detailed derivation).

One more important note is that, although $\rho_i(t)$ and z_{ij} have definite limits as $t\to\infty$, they may differ from these limits essentially for any finite t; in particular, (12) is the asymptotic relation both for very small and very large exponentials. We will see in what follows that highly intermittent realizations make an important contribution to the averages.

B. Averaging over the driving force

We now susbstitute the Fourier transform of (7) for each $\mathbf{B}$ in (5) and take the average over $\boldsymbol{\phi}$ by means of (3). Luckily, due to the multiplication of arguments of ϕ in (3), $\mathbf{W}(t)$ appears in $\langle\mathbf{B}(\mathbf{0})\mathbf{B}(\mathbf{r})\rangle_\phi$ only as a combination

$$\boldsymbol{\Omega}=\mathbf{W}\mathbf{W}^T=\mathbf{z}\mathbf{d}^2\mathbf{z}^T\ ,$$

so the quickly rotating matrix $\mathbf{R}$ does not affect the correlations. Thus, the average over the process $\mathbf{A}(t)$ is equivalent to the average over the processes $\rho_i(t)$ (with appropriate weight). So we get

$$\beta(\mathbf{r})=\left\langle\int d\tau\Psi[\mathbf{r},\tau,\ \boldsymbol{\rho}(\tau)]\right\rangle_A, \qquad (13)$$

where

$$\Psi=\frac{\varepsilon_B L^3 l^5}{3\pi^2}\int e^{-i\mathbf{k}'\mathbf{r}}\left(\mathbf{k}\boldsymbol{\Omega}^2\mathbf{k}'^T-\mathbf{k}\boldsymbol{\Omega}\mathbf{k}'^T\mathrm{Tr}\boldsymbol{\Omega}\right)$$
$$\times e^{-\frac14(\mathbf{k}-\mathbf{k}')\boldsymbol{\Omega}(\mathbf{k}-\mathbf{k}')^T l^2-\frac14(\mathbf{k}+\mathbf{k}')\boldsymbol{\Omega}(\mathbf{k}+\mathbf{k}')^T L^2}$$
$$\times e^{-\varkappa\mathbf{k}\int_0^\tau\boldsymbol{\Omega}d\tau'\mathbf{k}^T-\varkappa\mathbf{k}'\int_0^\tau\boldsymbol{\Omega}d\tau'\mathbf{k}'^T}d\mathbf{k}d\mathbf{k}'.$$

We first consider the "viscous" terms in the exponent. They appear in combinations $\frac14 l^2\boldsymbol{\Omega}+\varkappa\int_0^\tau\boldsymbol{\Omega}d\tau'$, $\frac14 L^2\boldsymbol{\Omega}+\varkappa\int_0^\tau\boldsymbol{\Omega}d\tau'$. We will see below that the important contribution comes from exponentially large terms, so we are only interested in logarithmic accuracy. Because of the exponential behavior of the matrix $\mathbf{d}$, each component of $\boldsymbol{\Omega}$ either grows or decreases exponentially. If it grows, for time large enough one can neglect the second terms in the sums (since $r_d\ll l\ll L$, $\int\boldsymbol{\Omega}\mathrm{d}\tau\sim\lambda_3^{-1}\boldsymbol{\Omega}$). If it decreases, after some time the first term of each sum becomes smaller than the second, so one cannot neglect the viscous term. However, in this case one can substitute a constant for $\int_0^\tau\boldsymbol{\Omega}d\tau'$. So, in any case the integral in the exponent can be replaced by a constant:

$$\frac14 l^2\boldsymbol{\Omega}+\varkappa\int_0^\tau\boldsymbol{\Omega}d\tau'\simeq\frac14 l^2\boldsymbol{\Omega}+\frac{\varkappa}{\lambda_3}\mathbf{C}, \qquad (14)$$

where $\mathbf{C}$ is a constant matrix with elements $\sim$1.

Second, we change the integration variables to $\mathbf{p}=\mathbf{kz}$, $\mathbf{p}'=\mathbf{k}'\mathbf{z}$. This is done to get rid of $\mathbf{z}$ in the exponent. Eventually, we take the Gaussian integrals over $\mathbf{p}$ and $\mathbf{p}'$.

Then, neglecting the term $\sim(l/L)^2$, we get

$$\Psi=\varepsilon_B\frac{\pi}{3}\Big\{[\mathrm{Tr}(\mathbf{z}\,\mathbf{d}^2\,\mathbf{z}^T)\mathrm{Tr}(\mathbf{d}^2\,\mathbf{g})-\mathrm{Tr}(\mathbf{z}\,\mathbf{d}^2\,\mathbf{g}\,\mathbf{d}^2\,\mathbf{z}^T)]$$
$$-\frac{1}{2l^2}[\mathrm{Tr}(\mathbf{z}\,\mathbf{d}^2\,\mathbf{z}^T)\mathrm{Tr}(\mathbf{d}^2\,\boldsymbol{\Gamma})-\mathrm{Tr}(\mathbf{z}\,\mathbf{d}^2\,\boldsymbol{\Gamma}\,\mathbf{d}^2\,\mathbf{z}^T)]\Big\}$$
$$\times\frac{\exp\left[-\frac{1}{4l^2}\mathbf{r}^T(\mathbf{z}^{-1})^T\mathbf{g}\mathbf{z}^{-1}\mathbf{r}\right]}{\sqrt{\det\mathbf{G}_l}\sqrt{\det\mathbf{G}_L}}, \qquad (15)$$

where $\mathbf{g}=\mathbf{G}_l^{-1}$, $\Gamma_{mn}=(\mathbf{g}\mathbf{z}^{-1}\mathbf{r})_m(\mathbf{g}\mathbf{z}^{-1}\mathbf{r})_n$ and the exact expressions for $\mathbf{G}_l,\mathbf{G}_L$ are given in Appendix B; with the account of (14) they can be approximated by

$$\mathbf{G}_l \simeq \mathbf{d}^2 + \left(\frac{r_d}{l}\right)^2 \mathbf{z}^{-1}\mathbf{C}\,(\,\mathbf{z}^{-1})^T,$$
$$\mathbf{G}_L \simeq \mathbf{d}^2 + \left(\frac{r_d}{L}\right)^2 \mathbf{z}^{-1}\mathbf{C}\,(\,\mathbf{z}^{-1})^T. \tag{16}$$

In what follows we choose the coordinate system in such a way that $\mathbf{r} = (x, 0, 0)$. Because of isotropy of the problem, there is no loss of generality.

C. Ensemble average for local functions

We see that Ψ can be approximated by a local functional of z and ρ (i.e., the functional that depends only on momentarily values of $\rho(\tau)$ and not on their integrals or derivatives). So the functional average in (13) can be reduced to an ordinary multiple integral; actually,

$$\beta = \left\langle \int d\tau \Psi[\mathbf{r}, \tau, \boldsymbol{\rho}(t)] \right\rangle_\rho$$
$$= \int d\tau \langle \Psi[\mathbf{r}, \tau, \boldsymbol{\rho}(\tau)]\rangle_\rho = \int d\tau \int d\boldsymbol{\rho} P(\tau, \boldsymbol{\rho})\Psi(\mathbf{r}, \boldsymbol{\rho}),$$

where P is the probability density. Now τ and ρ_i become independent variables.

Furthermore, because of the incompressibility condition (10) there are only two independent variables, so

$$\beta = \int \Psi(\mathbf{r}, \rho_1, \rho_3) f(\rho_1, \rho_3, \tau) d\rho_1 d\rho_3 d\tau, \tag{17}$$

where $f(\rho_1, \rho_3, \tau)$ is the time-dependent probability density of the two variables. We note that, according to (15), (9), and (12) Ψ depends only on ρ_1, ρ_3, but not on time. So (17) can be rewritten as

$$\beta = \int \Psi(\mathbf{r}, \rho_1, \rho_3)\Phi(\rho_1, \rho_3) d\rho_1 d\rho_3, \tag{18}$$

where

$$\Phi = \int_0^\infty f(\rho_1, \rho_3, \tau) d\tau. \tag{19}$$

IV. GAUSSIAN VELOCITY FIELD

In this section we consider the particular case of Gaussian statistics for the velocity gradient tensor. It appears to be not a typical case, but it is the easiest to calculate, and it helps to find the approach to the general situation. In addition, the conventional consideration is often restricted by only this case.

The statistics of ρ_i is then also Gaussian, with averages $\langle \rho_{1,3}\rangle = \pm\lambda t$; in terms of variables ρ_1, ρ_3 the probability density is

$$f_G(\rho_1, \rho_3, t) \propto e^{-\frac{(\rho_1+\lambda t)^2 + (\rho_1+\lambda t)(\rho_3-\lambda t) + (\rho_3-\lambda t)^2}{\lambda t}}, \tag{20}$$

so the distribution is determined by only one constant.

We will see below that important contribution to (17) comes from large $\rho = \sqrt{\rho_1^2 + \rho_3^2}$. The integral over τ in (19) is easy to calculate for these ρ:

$$\Phi_G = \int d\tau f_G(\rho_1, \rho_3, \tau)$$
$$\overset{\rho \gg 1}{\propto} \exp\left[\rho_3 - \rho_1 - 2\sqrt{\rho_1^2 + \rho_1\rho_3 + \rho_3^2}\right]. \tag{21}$$

A. Limit of zero diffusivity: Pair correlator diverges

The case $\varkappa = 0$ is the most simple. The integrand in (13) is then exactly, not only approximately, local. The matrices $\mathbf{G}_l$, $\mathbf{G}_L$, and $\mathbf{g}$ are diagonal, $\det \mathbf{G}_l = \det \mathbf{G}_L = 1$, $\Gamma_{ij} = \delta_{1i}\delta_{1j}$, and

$$\Psi \propto \mathrm{Tr}(\mathbf{z}\mathbf{d}^2\mathbf{z}^T)\left(1 - \frac{x^2}{4l^2}d_1^{-2}\right)\exp\left[-\frac{x^2}{4l^2}d_1^{-2}\right].$$

The term in the brackets is ~ 1 wherever the exponential is not negligible; with the account of (12) one can see that the trace is (to logarithmic accuracy) equal to the maximum of the three d_i^2, so

$$\Psi \propto \exp\left[2\max(\rho_1, \rho_3, -\rho_1 - \rho_3) - \frac{x^2}{4l^2}e^{-2\rho_1}\right]. \tag{22}$$

Now we have to multiply this by Φ_G and take the integral over the plane ρ_1, ρ_3. The second term in the exponent makes Ψ practically equal to zero in the semiplane

$$\rho_1 < -\ln(l/x) \Rightarrow \Psi \simeq 0. \tag{23}$$

In the other semiplane it is almost constant. So this term can be replaced by the Heaviside function θ.

To investigate the convergence of the integral, it is also convenient to consider polar coordinates (ρ, ϕ) on the ρ_1, ρ_3 plane. Then the product $\Psi\Phi$ takes the form

$$\Psi\Phi_G \propto \theta[\rho_1 + \ln(l/x)]e^{\rho\alpha(\phi)}. \tag{24}$$

From (22) it follows that α is positive for $\phi \in (\frac{\pi}{2} - \epsilon, \frac{3\pi}{2} - \epsilon')$, with ϵ, ϵ' being some constants smaller than $\pi/4$ (see Fig. 1). The most part of this range is cutoff at large ρ by the restriction (23), but in the direction $\phi \simeq \pi/2$ one has

$$\rho\alpha = 3\rho_3 - \rho_1 - 2\sqrt{\rho_1^2 + \rho_1\rho_3 + \rho_3^2} > 0\,, \tag{25}$$

and the integral over ρ diverges exponentially.

We note that the divergence appears in the sector of the ρ_1, ρ_3 plane that corresponds to $\rho_3 > \rho_1 > \rho_2$. The infinite-time limits of these variables for any realization are ordered differently, $\lambda_1 t < \lambda_2 t < \lambda_3 t$. Thus, although the maximum of the probability distribution lies in the direction $\phi = 3\pi/4$, the divergence of the integral is produced by very rare realizations with very large ρ_3 and nearly zero ρ_1 at some finite time. This is the manifestation of the intermittency of the system. Because of the multiplier $\propto\mathrm{Tr}\,\mathbf{z}\mathbf{d}^2\mathbf{z}^T \propto e^{2\rho_3}$, the intermittency in the magnetic field is much more essential than in the scalar field.

We also note that, as follows from (15) and (16), in the case of ideal conductor there is no difference between the homogeneous and quasihomogeneous ($L < \infty$) cases.

B. Nonzero diffusivity: Convergence of pair correlator

The nonzero diffusivity brings nondiagonal components to the matrices $\mathbf{G}_l, \mathbf{G}_L$ and changes the diagonal elements. Consider, for example, the "dangerous" direction $\phi \simeq \pi/2$ (that is, $\rho_3 \gg \rho_1 \gg \rho_2$). According to (12), the (1,2) component of the matrix $\mathbf{z}$ is much bigger than unity, so all the components of $\mathbf{G}_l$ except for G_{33} change significantly. However, the changes do not contribute to the main order to the most part

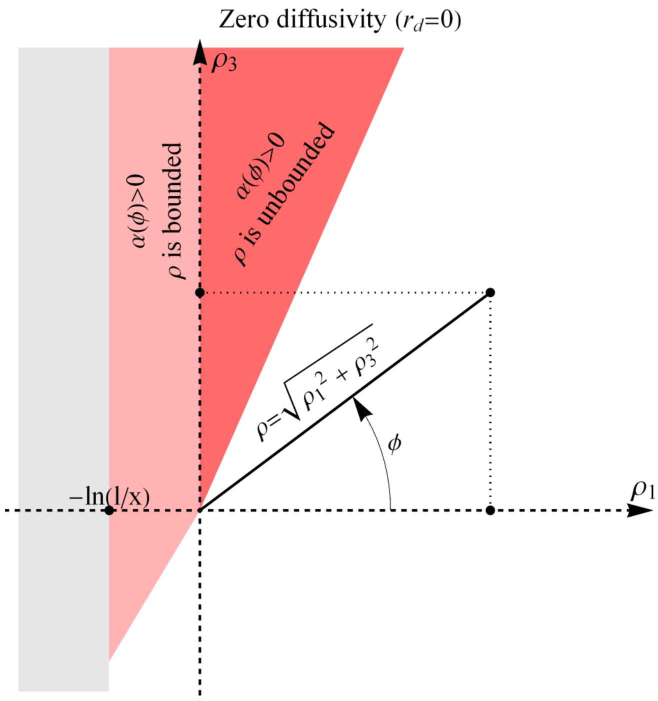

FIG. 1. Polar coordinates in the $(\rho_1;\rho_3)$ plane $\varkappa = 0$. The red zone highlights the region $\alpha(\phi) > 0$. In the lighter shaded region ρ is bounded for any given ϕ by the cutoff (23) (gray region), and in the darker shaded zone ρ is unbounded, which implies an exponential divergence of β.

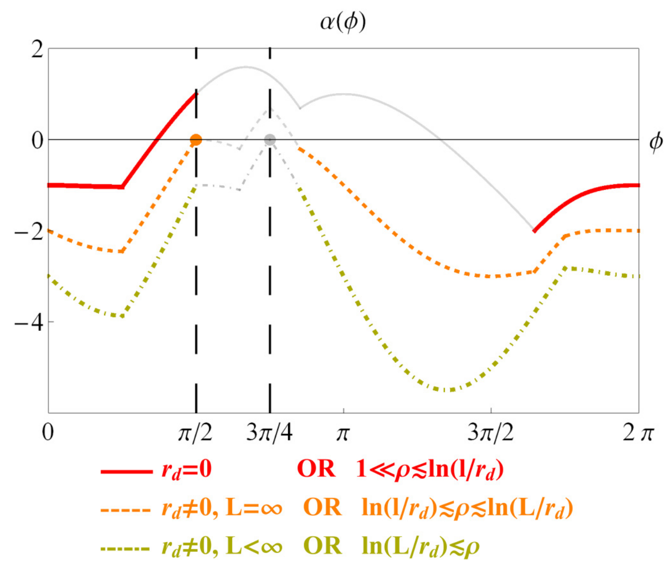

FIG. 2. $\alpha(\phi)$ dependence [see (24)] for Gaussian distribution. The gray pieces of curves mark the cutoff at large ρ in the case $r \neq 0$ (see Fig. 5 additionally). The color regions $\alpha > 0$ are responsible for the exponential growth of the integrand in (18) for $r \neq 0$, and produce exponential divergency of the two-point correlator if $\rho \to \infty$; those gray pieces where $\alpha > 0$ cause exponential dependence of the integrand if $r = 0$, and divergency of the corresponding one-point correlator [17]. The points $\alpha = 0$ give linear divergence of time integral in (17).

of (15) (see Appendix B for details), so the resulting Ψ differs from (22) only by the multiplier $\det \mathbf{G}_l \det \mathbf{G}_L$ where

$$\det\mathbf{G}_l \simeq \max\left[1, \left(\frac{r_d}{l}\right)^2 e^{2\rho_1+2\rho_3}\right],$$
$$\det\mathbf{G}_L \simeq \max\left[1, \left(\frac{r_d}{L}\right)^2 e^{2\rho_1+2\rho_3}\right]. \quad (26)$$

Thus, we have $\Psi \propto e^{\rho_3-\rho_1}$ for $l/r_d < e^{\rho_1+\rho_3} < L/r_d$ and $\Psi \propto e^{-2\rho_1}$ for $e^{\rho_1+\rho_3} > L/r_d$. As in the previous subsection, we get (24) with the same Heaviside function, and with

$$\rho\alpha = 2\rho_3 - 2\rho_1 - 2\sqrt{\rho_1^2 + \rho_1\rho_3 + \rho_3^2} \quad (27)$$

for $l/r_d < e^{\rho_1+\rho_3} < L/r_d$ and

$$\rho\alpha = \rho_3 - 3\rho_1 - 2\sqrt{\rho_1^2 + \rho_1\rho_3 + \rho_3^2}\,, \quad (28)$$

for $e^{\rho_1+\rho_3} > L/r_d$.

Now, for the homogeneous limit $L = \infty$ we have $\alpha(\pi/2) = 0$ and $\alpha(\phi < \pi/2) < 0$, so the integral over ρ still diverges at $\phi = \pi/2$ but the divergence is linear, not exponential. If L is finite, for $\rho \to \infty$ we have $\alpha < 0$ for $\phi \leqslant \pi/2$, and the integral in (18) converges.

The dependence $\alpha(\phi)$ in all directions (not only $\phi \simeq \pi/2$) is presented in Fig. 2. We see that α is negative for all the rest of the semiplane $-\pi/2 < \phi < \pi/2$. So, the integral in (18) converges in the case of nonzero diffusivity in finite spatial distribution of the pumping source; it diverges linearly (in respect to ρ) in homogeneous spatial distribution, and it diverges exponentially in the case of zero diffusivity.

Note that one-point second order correlator behaves even worse: the limit $x = 0$ destroys the Heaviside function in (24), and even the finite spatial distribution of magnetic field cannot prevent the linear divergence of the integral in the left semiplane, in the direction $\phi = 3\pi/4$ where $\alpha = 0$ (thick point in Fig. 2). We note that this direction corresponds to the maximum of the Gaussian probability density (20), $\rho_3/\rho_1 = \lambda_3/\lambda_1 = -1$.

C. Nonzero diffusivity: Scaling

We now consider the general case of nonzero diffusivity and finite pumping scale L, the aim is to find the pair correlator dependence of x.

The whole ρ_1, ρ_3 plane can be divided in three regions (Fig. 3): (I) the ideal conductor region where r_d is negligible, (II) the region where diffusivity is important but L can be set infinite (the "homogeneous" region), and (III) the region where the influence of inhomogeneity is determinative. As we have seen in the previous subsection, the influence of finite diffusion and inhomogeneity on the pair correlator is concentrated in the multiplier $(\det \mathbf{G}_l\mathbf{G}_L)^{-1/2}$ in (15). So according to (26), we find that the boundary of the first region is

$$\rho_1 + \rho_3 = -\ln l/r_d,$$

and the boundary of the second region is determined by the condition

$$\rho_1 + \rho_3 = -\ln L/r_d.$$

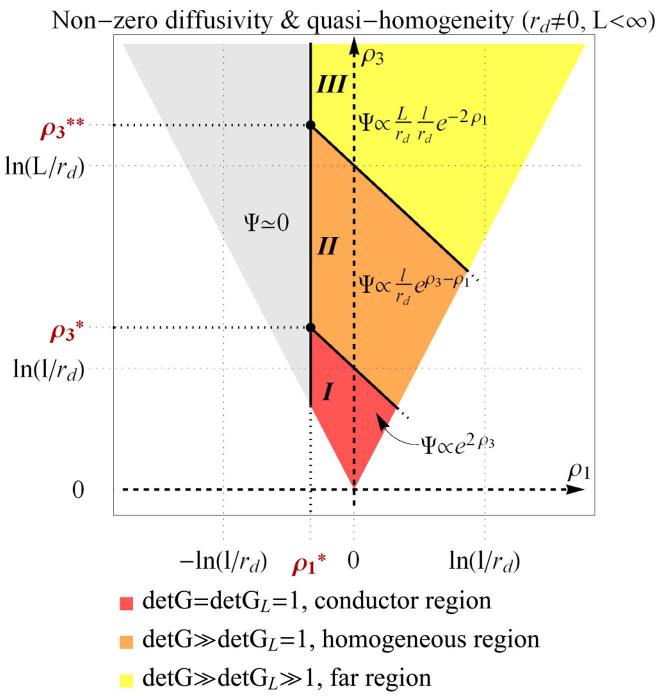

FIG. 3. The "dangerous" regions of ρ_1, ρ_3 plane in the general case of nonzero diffusivity and finite pumping scale (see Fig. 5 with the whole plane). The values of Ψ calculated to logarithmic accuracy are written in the corresponding regions (see Appendix B) and do not depend on $\Phi(\rho_1, \rho_3)$. The gray region is the region of the Heaviside filtering ($\Psi \simeq 0$).

We have to integrate $\Psi\Phi$ over these three regions, with account of the Heaviside function in (24). Since all the functions are exponential, in each of the regions the most contribution comes from the part with maximal possible $\rho\alpha$.

In the first two regions, as we have seen in (25) and (27), there are parts with positive (zero) α, which correspond to exponential (linear) increase of the integrand as a function of ρ. Since in both of them $d\alpha/d\phi > 0$, the maximum of the integrand is achieved at the upper left corner of each region (see Fig. 3):

$$\rho_1^* = -\ln l/x\,, \quad \rho_3^* = \ln l/r_d - \rho_1^*\,, \quad \rho_3^{**} = \ln L/r_d - \rho_1^*. \tag{29}$$

In the third region, $\alpha < 0$ and the integrand decreases exponentially as a function of ρ. Thus, the maximum is situated on the lower boundary.

To calculate the integral over each region to logarithmic accuracy, one has only to consider the vicinities of these extremal points. Taking into account that $\rho_1^* \ll \rho_3^*$ and expanding the square root in (25), (27), and (28) into a series up to the first order of ρ_1/ρ_3, we get

$$\begin{aligned}
&\int_I e^{\rho_3-2\rho_1} d\rho_1 d\rho_3 \propto e^{\rho_3^*-2\rho_1^*}\,;\\
&\int_{II} \frac{l}{r_d} e^{-3\rho_1} d\rho_1 d\rho_3 \propto \frac{l}{r_d}(\rho_3^{**}-\rho_3^*)e^{-3\rho_1^*}\,;\\
&\int_{III} \frac{l}{r_d}\frac{L}{r_d} e^{-\rho_3-4\rho_1} d\rho_1 d\rho_3 \propto \frac{l}{r_d}\frac{L}{r_d} e^{-\rho_3^{**}-4\rho_1^*}.
\end{aligned} \tag{30}$$

The multipliers $\frac{l}{r_d}$, $\frac{L}{r_d}$ come from the determinants (26). Substituting (29), we see that all the regions give contributions of the same order, and eventually

$$\beta_G \propto \frac{l}{r_d} \ln\frac{L}{l}\left(\frac{l}{x}\right)^3. \tag{31}$$

This is the scaling law for pair correlator of magnetic field in the case of Gaussian velocity gradient distribution.

V. ARBITRARY VELOCITY GRADIENT STATISTICS

We now proceed to calculation of β in the general case of non-Gaussian statistics of velocity gradient. We see that Ψ depends only on the point in the ρ_1, ρ_3 plane and on the relation between the scales. It does not depend on the distribution function of ρ, and it remains same as in the previous section.

The function Φ changes, but for the arbitrary distribution its maximum is still situated in some direction lying in the second quadrant of the plane [17]. With the account of the Heaviside function multiplier in Ψ, we still await the determinative contribution to (18) in the upper half-plane, near the boundary; so Fig. 3 and (29) are still valid. In this section we analyze the vicinity of the boundary. One can check (see Appendix C) that the rest of the plane does not actually contribute.

A. Calculation of Φ near the boundary

The formalism of cumulant functions [22,24] is the most convenient tool to operate with the stochastic variables $\boldsymbol{\rho}$. In particular, the probability density can be written as

$$P(\tau, \boldsymbol{\rho}) = \int d\mathbf{k} e^{-i\mathbf{k}\boldsymbol{\rho}+\tau w(i\mathbf{k})}, \tag{32}$$

where $\tau w(\boldsymbol{\eta})$ is the cumulant function of the random variable $\boldsymbol{\rho}$ at the moment τ (see [23] and Appendix C for more details). Integrating over ρ_2 we get the reduced probability density:

$$f(\rho_1, \rho_3, \tau) = \int d\sigma_1 d\sigma_3 e^{-\sigma_1\rho_1-\sigma_3\rho_3+\tau W(\sigma_1,\sigma_3)}.$$

Here

$$W(\sigma_1, \sigma_3) = w(\sigma_1, 0, \sigma_3). \tag{33}$$

For real σ_j, W is a real concave function with negative minimum, and $W(0, 0) = 0$. Other properties of the function will be considered later.

To calculate Φ, we have to take the two integrals over σ_1 and σ_3 and also take the integral over τ. We have seen in the previous section that the most contribution to the integral is made by the region

$$\rho_3 \sim \ln l/r_d \gg -\rho_1 \sim \ln l/x \gg 1.$$

This allows to take the three integrals by means of the saddle point approximation. So,

$$\Phi = e^{-\sigma_1^{\max}\rho_1-\sigma_3^{\max}\rho_3+\tau^{\max}W(\sigma_1^{\max},\sigma_3^{\max})}, \tag{34}$$

where the maximum point is determined by the conditions

$$\tau\frac{\partial W}{\partial\sigma_j} - \rho_j = 0\,, \quad W(\sigma_1, \sigma_3) = 0. \tag{35}$$

Here and below we omit the indices $^{\max}$ for brevity.

One more advantage comes from the small parameter $\rho_1 \ll \rho_3$ in the region of interest. Denote

$$\rho_1 = -\varepsilon R\,, \quad \rho_3 = R(1+\varepsilon).$$

Then $\varepsilon \ll 1$, $R \gg 1$; we present the variables as a series in ε,

$$\sigma_j = \sigma_j^{(0)} + \varepsilon\sigma_j^{(1)}\,, \quad \tau = \tau^{(0)} + \varepsilon\tau^{(1)}.$$

Expanding the conditions (35) to the zeroth order, we get

$$\left.\frac{\partial W}{\partial \sigma_1}\right|_{(0)} = 0\,, \quad \tau^{(0)}\left.\frac{\partial W}{\partial \sigma_3}\right|_{(0)} = R\,, \quad W\left(\sigma_1^{(0)}, \sigma_3^{(0)}\right) = 0.$$

To the first order, from the last equation in (35) we then obtain

$$\sigma_3^{(1)} = 0.$$

We do not need to calculate $\sigma_1^{(1)}$ since it contributes to (34) only to the second order. Eventually, we find

$$\Phi = e^{-\sigma_1^{(0)}\rho_1 - \sigma_3^{(0)}\rho_3},$$

where

$$\sigma_1^{(0)}, \sigma_3^{(0)}: \quad W = 0\,, \quad \frac{\partial W}{\partial \sigma_1} = 0\,, \quad \frac{\partial W}{\partial \sigma_3} > 0. \tag{36}$$

In Appendix C we show that there exists the unique solution to this set of conditions for any velocity gradient statistics; moreover, the solution is constrained by

$$1/2 < \sigma_1^{(0)} < 4\,, \quad 1 \leqslant \sigma_3^{(0)} < 2. \tag{37}$$

In the case of Gaussian velocity gradient distribution, $\sigma_1^{(0)} = 2$, $\sigma_3^{(0)} = 1$, and the resulting Φ coincides with that obtained in the previous section.

B. Pair correlation function

Now we multiply this Φ by Ψ found in the previous section for each of the three regions and take the integral for each of the regions just as in (30). Due to (37) the signs of the exponents confirm that the vicinity of the point (ρ_1^*, ρ_3^*), and generally the boundary $\rho_1 = \rho_1^*$, $\rho_3 > 0$ (see Fig. 3) makes the determinative contribution to the integral. So we obtain

$$\int_I \propto e^{-\sigma_1^{(0)}\rho_1^* + (2-\sigma_3^{(0)})\rho_3^*},$$

$$\int_{II} \propto \begin{cases} \frac{l}{r_d}\frac{1}{\sigma_3^{(0)}-1} e^{-(\sigma_1^{(0)}+1)\rho_1^* - (\sigma_3^{(0)}-1)\rho_3}\Big|_{\rho_3^*}^{\rho_3^{**}}, & \sigma_3^{(0)} > 1, \\ \frac{l}{r_d}\ln\frac{L}{l} e^{-(\sigma_1^{(0)}+1)\rho_1^*}, & \sigma_3^{(0)} = 1 \end{cases} \tag{38}$$

$$\int_{III} \propto \frac{l}{r_d}\frac{L}{r_d} e^{-(\sigma_1^{(0)}+2)\rho_1^* - \sigma_3^{(0)}\rho_3^{**}}.$$

We note that, as in the case of Gaussian distribution, the scaling exponent produced by all regions of integration is the same. The pre-exponents differ; two different pre-exponents in the second region appear because for $\sigma_3^{(0)} > 1$ the most contribution comes from the vicinity of the point (ρ_1^*, ρ_3^*). If $\sigma_3^{(0)} = 1$, as it happens in the Gaussian case, ρ_3^* and ρ_3^{**} (and the whole segment between them) contribute equivalently, and the pre-exponent contains $\ln L/l$. We also note that the pair correlator always converges for finite distributions and even for homogeneous distributions in the case $\sigma_3^{(0)} > 1$.

Eventually, we get the power-law dependence of the second order correlation function

$$\beta \propto \min\left\{\frac{1}{\sigma_3^{(0)}-1}, \ln\frac{L}{l}\right\}\left(\frac{l}{r_d}\right)^{2-\sigma_3^{(0)}}\left(\frac{l}{x}\right)^{\sigma_1^{(0)}+2-\sigma_3^{(0)}}. \tag{39}$$

C. Properties of the cumulant function

Statistics of the random variable $\rho(t)$ is determined by the velocity gradient tensor statistics. The relation between the cumulant functions of the processes $\rho(t)$ and $A(t)$ was found in [22]

$$w(\boldsymbol{\eta}) = w_A(\boldsymbol{\eta} + \boldsymbol{\eta}_0) - w_A(\boldsymbol{\eta}_0), \quad \boldsymbol{\eta}_0 = (-1, 0, 1), \tag{40}$$

where $w_A(\boldsymbol{\eta})$ is the “diagonal part” of the cumulant function of $A(t)$ (see Appendix C for more detailed comments). The Lyapunov spectrum can be expressed in terms of cumulant functions as

$$\lambda_j = \left.\frac{\partial w}{\partial \eta_j}\right|_{\mathbf{0}} = \left.\frac{\partial w_A}{\partial \eta_j}\right|_{\boldsymbol{\eta}_0}, \tag{41}$$

and the dispersions correspond to the second derivatives

$$D_{ij} \equiv \left\langle \frac{\rho_i - \lambda_i t}{t}\frac{\rho_j - \lambda_j t}{t}\right\rangle = \left.\frac{\partial^2 w_A}{\partial \eta_i \partial \eta_j}\right|_{\boldsymbol{\eta}_0}.$$

Proceeding to W defined in (33) we get from (40):

$$W(\sigma_1, \sigma_3) = w_A(\sigma_1 - 1, 0, \sigma_3 + 1) - w_A(-1, 0, 1). \tag{42}$$

We assume that the process $A(t)$ is statistically isotropic and that the flow is incompressible. These two claims result in essential restrictions on w_A. From isotropy it follows that w_A is a function of three symmetric combinations of η_j, namely,[4]

$$w_A(\boldsymbol{\eta}) = \tilde{w}_A\left(\sum \eta_j, \sum \eta_j^2, \sum \eta_j^3\right). \tag{43}$$

The incompressibility leads to the condition

$$\sum_j \frac{\partial w_A}{\partial \eta_j} = 0$$

for any $\boldsymbol{\eta}$. Combining these two conditions we find that w_A can be reduced to a function of only two variables:

$$\begin{aligned} w_A(\boldsymbol{\eta}) &= \tilde{w}(a, b), \\ a &= \sum \eta_j^2 - \frac{1}{3}\left(\sum \eta_j\right)^2, \\ b &= \sum \eta_j \cdot \sum \eta_j^2 - \frac{2}{9}\left(\sum \eta_j\right)^3 - \sum \eta_j^3. \end{aligned} \tag{44}$$

The set $\boldsymbol{\eta}_0 = (-1, 0, 1)$ corresponds to $a = 2$, $b = 0$. Thus, all statistical moments of the process ρ can be expressed in terms of derivatives of $\tilde{w}(a, b)$ taken at the point (2,0). For example, substituting (44) into (41) we get

$$\lambda_2 = 2\tilde{w}_b'(2, 0), \quad \lambda_3 = 2\tilde{w}_a'(2, 0) - \tilde{w}_b'(2, 0).$$

[4]To avoid misunderstanding, we note that $\boldsymbol{\eta}$ is not a vector but a set of three variables.

Second order moments imply second derivatives, e.g.,

$$D_{22} = \left(4\tilde{w}''_{bb} + \frac{4}{3}\tilde{w}'_a\right)\Big|_{(2,0)}, \text{ etc.}$$

D. Time-reversible flows

From (42) and (44) it follows that for any pair of σ_1, σ_3 corresponding to the same set $a = 2, b = 0$ we would get $W = 0$. In particular, the pair $\sigma_1 = 2, \sigma_3 = 1$ satisfies this condition; one can easily check that

$$W(2, 1) = 0$$

for *any* isotropic statistics of A. From (42) and (44) we also have

$$\frac{\partial W}{\partial \sigma_1}\Big|_{(2,1)} = 2\frac{\partial \tilde{w}_A}{\partial b}\Big|_{(a,b)=(2,0)} = \lambda_2.$$

So for $\lambda_2 = 0$ the second condition in (36) is satisfied. What about the third condition, it holds automatically since

$$\frac{\partial W}{\partial \sigma_3}\Big|_{(2,1)} = 2\frac{\partial \tilde{w}_A}{\partial a}\Big|_{(2,0)} - \frac{\partial \tilde{w}_A}{\partial b}\Big|_{(2,0)} = \lambda_3 > 0.$$

As it was shown in [25], the second Lyapunov exponent is equal to zero if and only if the flow is time-reversible. Thus, for any time-reversible isotropic flow the point $\sigma_1 = 2, \sigma_3 = 1$ is the solution to (36). From (39) we obtain for all these flows the same scaling law coinciding with that of the Gaussian case

$$\beta \propto \frac{l}{r_d} \ln\frac{L}{l}\left(\frac{l}{x}\right)^3. \tag{45}$$

In the general case $\lambda_2 \neq 0$ there is no explicit form for the solution to (36). So we consider nonzero but small λ_2, the more this corresponds to the real hydrodynamic situation [26].

E. Small deviation from time-reversibility

So let λ_2/λ_3 be a small parameter, and expand $\sigma_1^{(0)}, \sigma_3^{(0)}$ up to the first order near the point (2,1):

$$\sigma_1^{(0)} = 2 + \xi_1\lambda_2/\lambda_3, \quad \sigma_3^{(0)} = 1 + \xi_3\lambda_2/\lambda_3.$$

Substituting this into the first condition in (36) and taking into account $W(2, 1) = 0$, to the first order in λ_2/λ_3 we get

$$0 = \frac{\partial W}{\partial \sigma_1}\Big|_{(2,1)}\xi_1 + \frac{\partial W}{\partial \sigma_3}\Big|_{(2,1)}\xi_3 + O(\lambda_2/\lambda_3)$$
$$= \lambda_2\xi_1 + \lambda_3\xi_3 + O\big(\lambda_2^2/\lambda_3^2\big).$$

Thus

$$\xi_3 = O(\lambda_2/\lambda_3).$$

Expanding the second condition in (36), we obtain

$$0 = \frac{\partial W}{\partial \sigma_1}\Big|_{(2,1)} + \frac{\partial^2 W}{\partial \sigma_1^2}\Big|_{(2,1)}\xi_1\lambda_2/\lambda_3 + O\big(\lambda_2^2/\lambda_3^2\big)$$
$$= \lambda_2 + D_{22}\xi_1\lambda_2/\lambda_3 + O\big(\lambda_2^2/\lambda_3^2\big).$$

So

$$\xi_1 = -\frac{\lambda_3}{D_{22}}.$$

Expanding the first condition in (36) to the second order we get finally

$$\sigma_1^{(0)} = 2 - \frac{\lambda_2}{D_{22}}, \quad \sigma_3^{(0)} = 1 + \frac{{\lambda_2}^2}{2\lambda_3 D_{22}}.$$

Eventually, substituting this into (38), we find

$$\beta \propto \frac{\lambda_3 D_{22}}{{\lambda_2}^2}\left(1 - e^{-\frac{{\lambda_2}^2}{2\lambda_3 D_{22}}\ln\frac{L}{l}}\right)\left(\frac{l}{r_d}\right)^{1-\frac{{\lambda_2}^2}{2\lambda_3 D_{22}}}\left(\frac{l}{x}\right)^{3-\frac{\lambda_2}{D_{22}}}. \tag{46}$$

There is an interesting particular case of probability distribution close to the Gaussian:

$$w = \frac{\lambda}{2}(a - 2) + \lambda_2\delta w(a, b).$$

The first term corresponds to the Gaussian probability density (20) with $\lambda_3 = \lambda$, and the second is small addition $\sim\lambda_2/\lambda$. Then to the first order in λ_2/λ_3, $\lambda_3 \simeq \lambda$, $D_{22} \simeq \frac{2}{3}\lambda_3$. The scaling exponent in the case is equal to

$$\left(\frac{\partial \ln\beta}{\partial \ln(l/x)}\right)_{\text{almostGauss}} = 3 - \frac{3}{2}\frac{\lambda_2}{\lambda_3}. \tag{47}$$

VI. DISCUSSION

We consider the linear stage of the evolution of magnetic field advected by an isotropic turbulent flow; no feedback of the magnetic field on the flow dynamics is assumed. We restrict our consideration with the viscous range of scales where the velocity field is smooth and, therefore, locally described by the velocity gradient tensor. The exponential increase of magnetic field correlations is possible at these scales, while in the inertial range it is restricted by a power law [6,9].

In this paper we calculate the long-time asymptote of the second order correlation function of magnetic field. We find the stationary solution which obeys the scaling law: for Gaussian velocity gradient distribution (and, generally, flows with symmetric Lyapunov spectrum) the scaling exponent is -3 (45), for slightly non-Gaussian case the exponent differs slightly ((46) and (47)); in the case of general non-Gaussian velocity gradient the scaling exponent is expressed in terms of the cumulant function (39).

The existence of stationary solution may be unexpected since in previous investigations it seemed natural to consider space homogeneity and/or Gaussian statistics for the velocity gradient [8,9]. This was done, for example, in the investigation of scalar field [15] where these assumptions are not crucial and do not prevent from getting the stationary solution. On the contrary, in the case of magnetic field the order of the limits is essential: for example, taking the nondiffusive limit $r_d \to 0$ first leads to exponential divergency of the pair correlator,

TABLE I. Existence of stationary solution, linear, or exponential divergency of two-point and single-point correlation functions ($\langle \mathbf{B}(0)\mathbf{B}(r)\rangle$ and $\langle \mathbf{B}^2(0)\rangle$) for nonzero diffusivity ($r_d \neq 0$).

	$L < \infty$		$L = \infty$	
	$\langle \mathbf{B}(0)\mathbf{B}(r)\rangle$	$\langle \mathbf{B}^2(0)\rangle$	$\langle \mathbf{B}(0)\mathbf{B}(r)\rangle$	$\langle \mathbf{B}^2(0)\rangle$
$\lambda_2 \neq 0$	stationary	stationary	stationary	exponential
$\lambda_2 = 0$	stationary	linear	linear	exponential

TABLE II. Divergency of single-point and two-point scalar (θ) and vector($\mathbf{B}$) correlation functions.

	Stationary	Linear divergence	Exponential divergence
$\langle\theta^2(0)\rangle$	$r_d \neq 0, \{L, \lambda_2\} - \forall$	$r_d = 0, \{L, \lambda_2\} - \forall$	–
$\langle\theta(0)\theta(r)\rangle$	$\{r_d, L, \lambda_2\} - \forall$	–	–
$\langle\mathbf{B}^2(0)\rangle$	$r_d \neq 0, L < \infty, \lambda_2 \neq 0$	$r_d \neq 0, L < \infty, \lambda_2 = 0$	$L = \infty, \{r_d, \lambda_2\} - \forall$ $r_d = 0, \{L, \lambda_2\} - \forall$
$\langle\mathbf{B}(0)\mathbf{B}(r)\rangle$	$r_d \neq 0, L < \infty, \lambda_2 = 0$ $r_d \neq 0, \lambda_2 \neq 0, L - \forall$	$r_d \neq 0, L = \infty, \lambda_2 = 0$	$r_d = 0, \{L, \lambda_2\} - \forall$

while in the scalar field it remains stationary. In the case of finite diffusivity, the existence of the stationary solution is provided by inhomogeneity of pumping or by asymmetry of the Lyapunov spectrum of velocity gradient statistics: the pair correlator converges if either $\lambda_2 \neq 0$ or $L < \infty$ (see Table I).

It is interesting to compare the results for the pair correlator to those for the one-point second order correlator found in [17]. First, although the approach of this paper allows to determine if the one-point correlator exists or not, still (45) and (46) do not converge as $x \to 0$. This is caused by the assumption $x \gg r_d$ used in their derivation. We do not consider the case $x < r_d$ but it is evident that $\langle B^2(0)\rangle - \langle B(0)B(x)\rangle \propto x^2$ for $x \ll r_d$ in all cases in which $\langle B^2(0)\rangle$ exists.

Second, for convergence of the correlator for all x including $x = 0$, one needs both the finite size of the pumping zone and asymmetric Lyapunov spectrum (the Gaussian case does not suit). Then the complete $\beta(x)$ graph has a parabolic "hat" at small distances and power-law "tail." In other cases we get stationary scaling for the "tail" and no stationarity in the middle, so that at any large but finite time the correlator is "stabilized" outside some boundary $x > x_0(t)$, and remains increasing inside the region. A similar picture was obtained for vorticity distribution in high-Reynolds turbulent flow [27].

We stress that the difference between the Gaussian case and general velocity gradient statistics appears to be crucial. The general asymmetry of the Lyapunov spectrum not only guarantees the convergence of the pair correlator for $x \gg r_d$ even in homogeneous pumping regime; it is also necessary to provide the convergence of the one-point second order correlator. Thus the analysis of non-Gaussianity is very important. Also, the account of the finite pumping region ($L < \infty$) is determinative: it is necessary to get the stationary solution with both one-point and two-point correlators being stationary.

One more observation comes from comparison of the results for the magnetic field correlations with the corresponding correlations of scalar field advected by a turbulent flow [15], see Table II. One can see that the divergency range is much wider for magnetic field than for scalar field. This is a manifestation of much more intermittent behavior of magnetic field correlations: even for converging cases, the main contribution comes from the realizations of $\mathbf{v}(t)$ [or, more precisely, $\rho_i(t)$] that stay for a long time very far from their average value. Such a difference between the vector and scalar fields is not unexpected: the vector field is known to be more intermittent also in the inertial range of turbulence [10,28,29].

The last comment concerns the applicability range of the obtained result. In the nonstationary problem statement like that of [8,13] the inhomogeneities of magnetic field are shown to increase and stretch exponentially until the linear approximation for velocity is not valid any more; after that their scale belongs to the inertial range, and the exponential increase stops. The exponentially increasing solution does only exist for a finite time. To the contrary, in this statement one can see that, although in some realizations (those that correspond to $\rho \to \infty$) the inhomogeneities of magnetic field increase and stretch exponentially to reach the inertial range, still their contribution to the stationary correlators is insignificant. This solution is not restricted in time and does not need an additional account of the larger scales' influence.

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APPENDIX A: STATISTICAL PROPERTIES OF z

Substituting (9) into (8) we get

$$\mathbf{A} = \mathbf{R}^T\mathbf{X}\mathbf{R}, \quad \mathbf{X} = \xi + \zeta + \theta, \tag{A1}$$

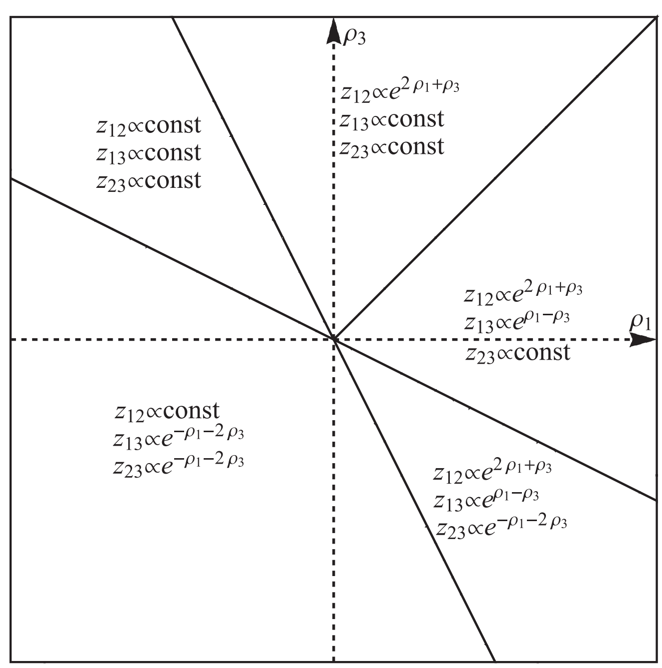

FIG. 4. Regions of different regimes for asymptotes of $\mathbf{z}$ components.

where

$$\xi = \mathbf{d}^{-1}\partial_t\mathbf{d} = \mathrm{diag}(\xi_1, \xi_2, \xi_3), \quad \boldsymbol{\zeta} = \mathbf{d}^{-1}\mathbf{z}^{-1}(\partial_t\mathbf{z})\mathbf{d},$$
$$\theta = (\partial_t\mathbf{R})\mathbf{R}^T. \tag{A2}$$

We see that the matrix ξ is diagonal, ζ is upper triangular matrix with zeros in the main diagonal, and θ is antisymmetric. Since $\mathbf{A}$ is isotropic and a stationary random process, $\mathbf{X} = \mathbf{RAR}^T$ is isotropic and stationary also. The decomposition (A1) is unique, so all its components of ξ, ζ, θ are also stationary random functions. From the first equation we then get

$$\rho_i = \int \xi_i dt. \tag{A3}$$

The second equation in (A2) gives

$$\partial_t z_{12} = \zeta_{12}\, e^{\rho_1-\rho_2},$$
$$\partial_t z_{13} - z_{12}\, \partial_t z_{23} = \zeta_{13}\, e^{\rho_1-\rho_3},$$
$$\partial_t z_{23} = \zeta_{23}\, e^{\rho_2-\rho_3}.$$

The third equation governs the dynamics of $\mathbf{R}$.

Consider the first equation in the system. Due to the stationarity of ζ_{12} there are two different regimes for z_{12}: saturation if $\rho_1 - \rho_2 \leqslant 0$ and exponential increase if $\rho_1 - \rho_2 > 0$. (The boundary case of linear growth if $\rho_1 = \rho_2$ has zero measure, and can be considered as saturation to logarithmic accuracy.) In the first case z_{12} does not depend on the current value of ρ_i due to stabilization, and can be considered as random value in $t \to \infty$ limit. Alternatively, in the second case the time dependence of the integrand is exponential, and, to logarithmic accuracy, z_{12} does not depend on the previous evolution: only the last time interval counts. So

$$z_{12} \propto \max(1, e^{\rho_1-\rho_2}).$$

Treating the two other equations analogously one can calculate the components z_{13}, z_{23} and obtain (12). This divides the plane ρ_1, ρ_3 into five parts (see Fig. 4) with different regimes for all the components.

APPENDIX B: CALCULATION OF Ψ

Taking the integrals over $\mathbf{k}$ and $\mathbf{k}'$, we obtain (15) where

$$\mathbf{G}_l = \mathbf{z}^{-1}\left(\boldsymbol{\Omega} + \frac{2\varkappa}{l^2}\int_0^\tau \boldsymbol{\Omega}' d\tau'\right)(\mathbf{z}^{-1})^T$$
$$= \mathbf{d}^2 + \frac{2\varkappa}{l^2}\,\mathbf{z}^{-1}\int_0^\tau \mathbf{z}'\mathbf{d}'^2\,\mathbf{z}'^T d\tau'\,(\mathbf{z}^{-1})^T$$

and $\mathbf{G}_L$ differs from $\mathbf{G}_l$ by replacing l with L. This relation is nonlocal; but with the account of (14) it can be to logarithmic accuracy approximated by (16).

Exact expressions for $\det\mathbf{G}_l$, $\mathbf{g}$, $\boldsymbol{\Gamma}$ are very cumbersome. However, since all the summands are exponentials of ρ_1, ρ_3, for asymptotic expressions it is enough to take only one fastest-growing summand in each case. For instance,

$$\det\mathbf{G}_l \simeq \max\left[1, \left(\frac{r_d}{l}\right)^2 e^{2\rho_1+2\rho_3}, \left(\frac{r_d}{l}\right)^4 e^{2\rho_3}, \left(\frac{r_d}{l}\right)^2 e^{-2\rho_1},\right.$$
$$\left.\left(\frac{r_d}{l}\right)^2 e^{-4\rho_1-4\rho_3}, \left(\frac{r_d}{l}\right)^2 e^{-2\rho_3}, \left(\frac{r_d}{l}\right)^4 e^{-2\rho_1}\right]. \tag{B1}$$

We note that the symmetry of $\mathbf{G}$ is very important since it results in the cancellation of some terms [in particular, the terms $\sim(\frac{r_d}{l})^4 e^{4\rho_1+4\rho_3}$ cancel in $\det\mathbf{G}_l$].

The choice of the largest term in (B1) depends on the region of the (ρ_1, ρ_3) plane. In particular, in the region of the most interest $\phi \sim \pi/2$ one has $\rho_3 > \rho_1 > \rho_2$, the two first terms are larger than the rest, and we get (26).

The matrix $\mathbf{g} = \mathbf{G}_l^{-1}$ can be found analogously; it also has different asymptotes in different zones of the (ρ_1, ρ_3) plane. For the direction $\phi \simeq \pi/2$ the asymptote reads as

$$g_{11} = e^{-2\rho_1}; \quad g_{22} = e^{2\rho_1+2\rho_3}; \quad g_{33} = e^{-2\rho_3}$$
$$g_{12} = \begin{bmatrix} (r_d/l)^2 e^{2\rho_1+3\rho_3} & \rho_1+\rho_3 < \ln(l/r_d) \\ e^{\rho_3} & \rho_1+\rho_3 > \ln(l/r_d) \end{bmatrix}\text{'}$$
$$g_{13} = \begin{bmatrix} (r_d/l)^2 e^{\rho_3} & \rho_1+\rho_3 < \ln(l/r_d) \\ e^{-2\rho_1-3\rho_3} & \rho_1+\rho_3 > \ln(l/r_d), \\ (r_d/l)^2 e^{-2\rho_1-2\rho_3} & \rho_3 > 2\ln(l/r_d) \end{bmatrix}$$
$$g_{23} = \begin{bmatrix} (r_d/l)^2 e^{2\rho_1} & \rho_1+\rho_3 < \ln(l/r_d) \\ e^{-2\rho_3} & \rho_1+\rho_3 > \ln(l/r_d). \\ (r_d/l)^2 e^{-\rho_3} & \rho_3 > 2\ln(l/r_d) \end{bmatrix}$$

The accurate calculation shows that the nondiagonal terms do not contribute to the main order in Ψ. Moreover, for $x \gg r_d$ one can ensure that, to the main order,

$$\Psi \simeq \Psi|_{r=0}\,\theta(\ln g_{11} + 2\rho_1^*). \tag{B2}$$

In Fig. 5 $\Psi|_{r=0}$ calculated to logarithmic accuracy and the region $\theta(\ln g_{11} + 2\rho_1^*) = 0$ are presented.

APPENDIX C: CUMULANT FUNCTION

By definition, the cumulant function of random variable ζ is $w(\eta)$ such that for any given η,

$$e^{w(\eta)} = \langle e^{\eta\zeta}\rangle.$$

For stationary random processes $\xi(t)$ with correlation time much shorter than τ, according to large deviations theory, there exists [30] the cumulant function such that for any given (nonrandom) number η

$$e^{\tau w(\eta)} = \left\langle e^{\eta\int_0^\tau \xi(t)dt}\right\rangle. \tag{C1}$$

(More complete consideration of cumulant functionals and functions for random processes see in [22].) From this definition it immediately follows, in particular, $w(0) = 0$, $dw/d\eta(0) = \langle\xi\rangle$.

The random quantity $\boldsymbol{\rho}(\tau)$ can be presented as $\boldsymbol{\rho}(\tau) = \int_0^\tau \boldsymbol{\xi}(t)dt$ where $\boldsymbol{\xi}$ is the stationary random process (Appendix A). The cumulant function of $\boldsymbol{\rho}(\tau)$ is

$$w_{\rho,\tau}(\boldsymbol{\eta}) = \ln\left\langle e^{\boldsymbol{\eta}\boldsymbol{\rho}(\tau)}\right\rangle = \tau w(\boldsymbol{\eta}), \quad \boldsymbol{\eta} = \{\eta_1, \eta_2, \eta_3\},$$

where w is the cumulant function of ξ. It follows, in particular, that the Lyapunov spectrum is $\lambda_i \equiv \lim_{t\to\infty}\langle\rho/\tau\rangle = \partial w/\partial\eta_i$. Taking into account that $P(\bar{\rho}, \eta) = \langle\delta(\rho - \bar{\rho})\rangle = \int dk\langle e^{ik(\rho-\bar{\rho})}\rangle$, we get (32).

The velocity gradient tensor $\mathbf{A}(t)$ and the matrix $\mathbf{X(t)}$ defined in (A1) are stationary random matrix processes, so, in accordance with (C1), their cumulant functions are defined by

$$e^{\tau w_A(\boldsymbol{\eta})} = \left\langle e^{\mathrm{Tr}(\boldsymbol{\eta}\int_0^\tau \mathbf{A}dt)}\right\rangle, \quad e^{\tau w_X(\boldsymbol{\eta})} = \left\langle e^{\mathrm{Tr}(\boldsymbol{\eta}\int_0^\tau \mathbf{X}dt)}\right\rangle,$$

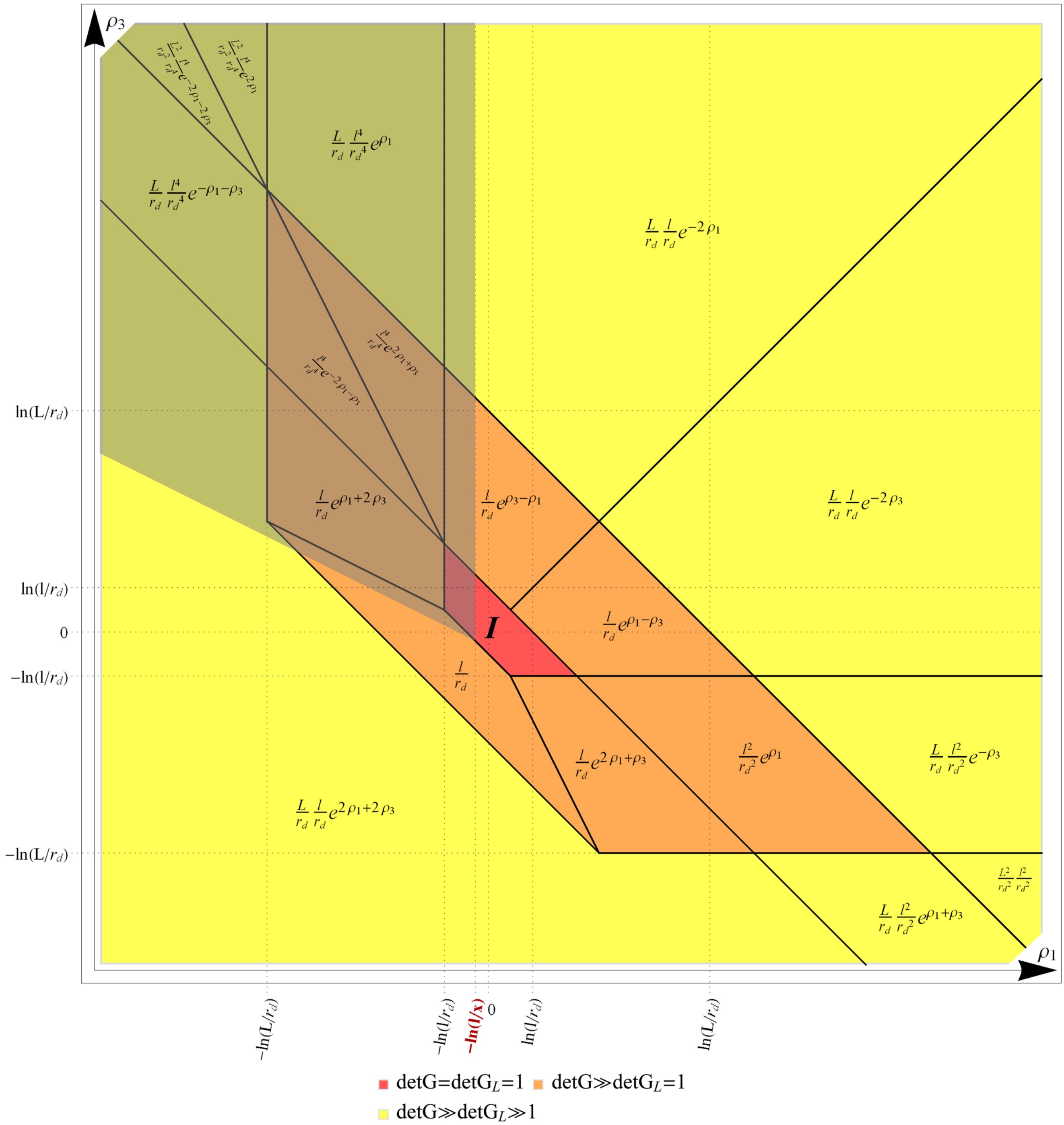

FIG. 5. The regions of ρ_1, ρ_3 plane in the general case of nonzero diffusivity and finite pumping scale. The values of $\varepsilon_B^{-1}\Psi|_{r=0}$ are placed in the corresponding regions. In the region *(I)* $\Psi|_{r=0} \propto \exp[2\max(\rho_1, \rho_3, -\rho_1-\rho_3)]$. The shaded region is the region corresponds to the filtering $\theta(\ln g_{11} + 2\rho_1^*) = 0$ in (B2). This figure incorporates Figs. 1, 2, and 3.

where $\boldsymbol{\eta}$ is a matrix. The relation between $\mathbf{A}$ and $\mathbf{X}$ is rather complicated and determined by (A1). The relation between their cumulant functions was derived in [22],

$$w_X(\boldsymbol{\eta}) = w_A(\boldsymbol{\eta}+\boldsymbol{\eta}_0) - w_A(\boldsymbol{\eta}_0), \qquad \boldsymbol{\eta}_0 = \mathrm{diag}\{-1, 0, 1\}.$$

To extract ξ from $\mathbf{X}$, one should take its diagonal part; thus, to find the cumulant function of ξ, one has to take a diagonal matrix $\boldsymbol{\eta}$:

$$w(\eta_1, \eta_2, \eta_3) = w_X(\boldsymbol{\eta})|_{\eta_{i\neq j}=0}.$$

This brings us to (40) where $\boldsymbol{\eta}$ is a set of three variables, and to (42) where W is a function of two variables σ_1, σ_3.

From the isotropy of the process $\mathbf{A}$ it follows that its cumulant function $w_A(\boldsymbol{\eta})$ possesses permutation symmetry:

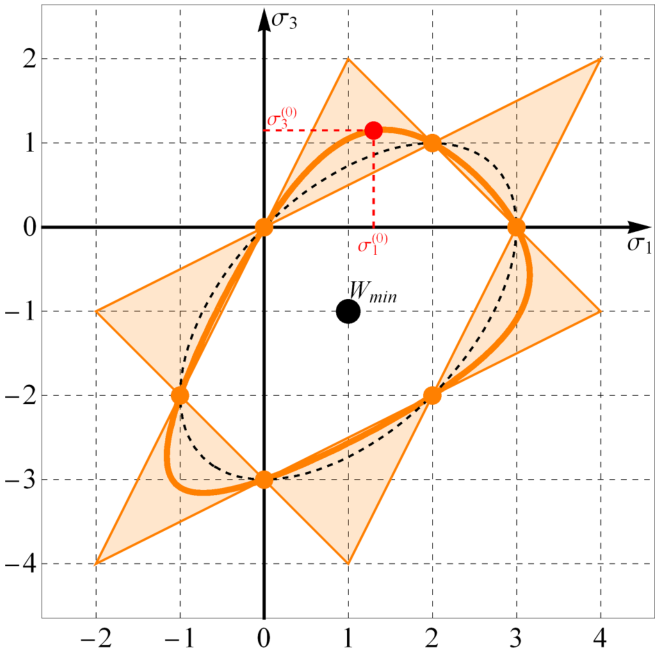

FIG. 6. Illustration for the level line $W = 0$, six universal points, and the point $\boldsymbol{\sigma}_0$ (36). The shaded region contains all the points that can belong to the line $W = 0$. The dashed line corresponds to the Gaussian case. The universal point of minimum is also depicted.

according to (43), all η_i contribute equivalently. In particular, since $\langle A_i \rangle = \frac{\partial w_A}{\partial \eta_i}\big|_{\boldsymbol{\eta}=0} = 0$, and the dispersion tensor $\langle A_i A_j \rangle$ is positively defined, the point $\boldsymbol{\eta} = \mathbf{0}$ is the minimum of w_A. Moreover, from the Cauchy inequality w_A is known to be concave, $\partial^2 w_A/\partial\eta_i\partial\eta_j > 0 \forall \boldsymbol{\eta}$. Since (40) is nothing but a shift of the function w_A by $\boldsymbol{\eta}_0$ and its overall decrease by $w_A(\boldsymbol{\eta}_0)$, and (33) is just a section of w_A by the plane $\eta_2 = 0$, the function $W(\sigma_1, \sigma_3)$ is also concave and has a minimum at the point $(1, -1)$ (which corresponds to $\eta = -\eta_0$). The level line $W = 0$ is shown schematically in Fig. 6. In the particular case of Gaussian distribution they are ellipses, in general case they are convex curves.

One more general property of these level lines comes from isotropy and incompressibility. According to (44) and (42), all pairs (σ_1, σ_3) that correspond to the same (a, b) produce the same W. In particular, six points $(0, 0), (2, 1), (3, 0), (2, -2), (0 - 3), (-2, -1)$ in the σ plane correspond to $(a, b) = (2, 0)$, and for all of them $W = 0$ generally. The rest of the line $W = 0$ must (because of convexity) lie inside the triangles in Fig. 6.

The point $\sigma^{(0)}$ introduced in (36) is the upper point of the level line $W = 0$ in Fig. 6. It does evidently exist, and it satisfies the restrictions (37). Actually, the lower boundary of $\sigma_3^{(0)}$ is restricted by the point (2,1). In the case of Gaussian distribution, this point is the required $\sigma^{(0)}$ point; in other distributions $\sigma^{(0)}$ is situated in one of two upper triangles shown in Fig. 6.

To find (or at least to restrict) the exponent $\sigma_1\rho_1 + \sigma_3\rho_3$ for any given (ρ_1, ρ_3), one can also use the graphic representation of (35) in Fig. 6 (considering the tangent to the line $W = 0$ in a given direction). This restrictions together with Fig. 5 allow to compare contributions of every (ρ_1, ρ_3) in (18). This allows one to check that in the case $x \gg r_d$ the "dangerous" direction considered in Sec. V gives the largest contribution.

[1] H. K. Moffatt, *Magnetic Field Generation in Electrically Conducting Fluids* (Cambridge University Press, Cambridge, England, 1978).
[2] E. N. Parker, *Cosmic Magnetic Fields, Their Origin and Activity* (Clarendon, Oxford, 1979).
[3] R. Kraichnan and S. Nagarajan, Phys. Fluids **10**, 859 (1967).
[4] R. Kulsrud and S. Anderson, Astrophys. J. **396**, 606 (1992).
[5] G. Batchelor, J. Fluid. Mech. **5**, 113 (1959).
[6] A. P. Kazantsev, Sov. Phys JETP **26**, 1031 (1968).
[7] R. H. Kraichnan, Phys. Fluids. **11**, 945 (1968).
[8] M. Chertkov, G. Falkovich, I. Kolokolov, and M. Vergassola, Phys. Rev. Lett. **83**, 4065 (1999).
[9] M. Vergassola, Phys. Rev. E **53**, R3021 (1996).
[10] E. Jurcisinova and M. Jurcisin, J. Phys. A: Math. Theor. **45**, 485501 (2012).
[11] L. T. Adzhemyan, A. N. Vasilev, and M. Gnatich, Theore. Math. Phys. **58**, 1 (1984).
[12] Y. B. Zel'dovich, A. A. Ruzmaikin, S. A. Molchanov, and D. D. Sokoloff, J. Fluid Mech. **144**, 1 (1984).
[13] A. S. Il'yn, V. A. Sirota, and K. P. Zybin, Europhys. Lett. **121**, 34002 (2018).
[14] A. S. Il'yn, V. A. Sirota, and K. P. Zybin, Phys. Scr. **94**, 064001 (2019).
[15] M. Chertkov, G. Falkovich, I. Kolokolov, and V. Lebedev, Phys. Rev. E **51**, 5609 (1995).
[16] A. S. Il'yn, V. A. Sirota, and K. P. Zybin, Phys. Rev. E **96**, 013117 (2017).
[17] A. S. Il'yn, A. V. Kopyev, V. A. Sirota, and K. P. Zybin (unpublished).
[18] V. I. Belinicher and V. S. L'vov, Sov. Phys. JETP **66**, 303 (1987) [Zh. Eksp. Teor. Fiz. 93, 533 (1987)].
[19] V. I. Oseledets, Trans. Moscow Math. Soc. **19**, 197 (1968) [Moscov. Mat. Obsch. 19, 179 (1968)].
[20] H. Furstenberg, Trans. Amer. Math. Soc. **108**, 377 (1963).
[21] A. V. Letchikov, Russian Math. Surveys **51**, 49 (1996).
[22] A. S. Il'yn, V. A. Sirota, and K. P. Zybin, J. Stat. Phys. **163**, 765 (2016).
[23] A. S. Il'yn, V. A. Sirota, and K. P. Zybin, J. Stat. Phys. **166**, 24 (2017).
[24] V. I. Klyatskin, *Dynamics of Stochastic Systems* (Elsevier, New York, 2005).
[25] A. S. Il'yn and K. P. Zybin, Phys. Lett. A **379**, 650 (2015).
[26] S.Girimaji and S.Pope J., Fluid Mech. **220**, 427 (1990).
[27] K. P. Zybin and V. A. Sirota, Sov. Phys. Usp. **58**, 556 (2015).
[28] G. Falkovich, K. Gawedzki, and M. Vergassola, Rev. Mod. Phys. **73**, 913 (2001).
[29] K. Gawedzki and A. Kupiainen, Phys. Rev. Lett. **75**, 3834 (1995); I. Rogachevskii and N. Kleeorin, Phys. Rev. E **56**, 417 (1997).
[30] S. R. S. Varadhan, Ann. Probab. **36**, 397 (2008).