Eigenstate thermalization and ensemble equivalence in few-body fermionic systems

Ph. Jacquod
*Department of Quantum Matter Physics, University of Geneva, CH-1211 Geneva, Switzerland
and School of Engineering, University of Applied Sciences of Western Switzerland HES-SO, CH-1951 Sion, Switzerland*

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We investigate eigenstate thermalization from the point of view of vanishing particle and heat currents between a few-body fermionic Hamiltonian prepared in one of its eigenstates and an external, weakly coupled Fermi-Dirac gas. The latter acts as a thermometric probe, with its temperature and chemical potential set so that there is neither particle nor heat current between the two subsystems. We argue that the probe temperature can be attributed to the few-fermion eigenstate in the sense that (i) it varies smoothly with energy from eigenstate to eigenstate, (ii) it is equal to the temperature obtained from a thermodynamic relation in a wide energy range, (iii) it is independent of details of the coupling between the two systems in a finite parameter range, (iv) it satisfies the transitivity condition underlying the zeroth law of thermodynamics, and (v) it is consistent with Carnot's theorem. For the spinless fermion model considered here, these conditions are essentially independent of the interaction strength. When the latter is weak, however, orbital occupancies in the few-fermion system differ from the Fermi-Dirac distribution so that partial currents from or to the probe will eventually change its state. We find that (vi) above a certain critical interaction strength, orbital occupancies become close to the Fermi-Dirac distribution, leading to a true equilibrium between the few-fermion system and the probe. In that case, the coupling between the Fermi-Dirac gas and few-fermion system does not modify the state of the latter, which justifies our approach *a posteriori*. From these results, we conjecture that for few-body systems with sufficiently strong interaction, the eigenstate thermalization hypothesis is complemented by ensemble equivalence: individual many-body eigenstates define a microcanonical ensemble that is equivalent to a canonical ensemble with grand canonical orbital occupancies.

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I. INTRODUCTION

In equilibrium statistical mechanics, observable properties of macroscopic systems are given by ensemble averages over microscopic states. When considering the microcanonical ensemble, the average is taken over states of similar energy. All states in that average contribute with equal weight—this is the postulate of equal *a priori* probabilities [1]. The process by which almost all initial out-of-equilibrium conditions evolve into equilibrium states that are well represented by the microcanonical ensemble, is called thermalization. Within classical mechanics, thermalization and the emergence of equilibrium statistical mechanics is standardly explained at a microscopic level by dynamical complexity [2]: regardless of their origin, almost all classical trajectories of chaotic dynamical systems eventually look the same as they ergodically explore the constant energy hypersurface in phase space. It is usually accepted that the macroscopic laws of classical statistical mechanics emerge from this ergodicity [3].

The situation is different in quantum mechanics, whose time evolution cannot by itself lead to the uniform covering of the constant energy manifold typical of the microcanonical measure. Instead, it has been postulated that thermalization occurs at the level of individual eigenstates [4,5], in such a way that expectation values of almost all observables taken over almost any such state in a narrow energy interval give the same result in the thermodynamic limit, tending moreover to the microcanonical average [4–6]. For recent reviews of and details on this *eigenstate thermalization hypothesis* (ETH), see Refs. [7,8].

A question that naturally arises is whether the ETH is accompanied by an ensemble equivalence similar to the one prevailing in statistical mechanics [1]. In the spirit of the standard construction of the canonical ensemble from the microcanonical one, it has been shown that tracing over some of the degrees of freedom of a pure quantum state gives, for almost all pure states in a narrow energy interval, a reduced density matrix corresponding to the canonical ensemble [9–11]. A similar procedure showed that typical many-body eigenstates have consistently defined thermodynamic entropies [12]. Taken together with the ETH, this can be interpreted as ensemble equivalence at the level of individual many-body eigenstates. This equivalence requires, however, a bipartitioning of quantal eigenstates and one may wonder if a temperature can be attributed to individual eigenstates without partitioning. This question was asked in Ref. [13,14], where an *ad hoc* microcanonical partition function was constructed from the energy profile of many-body eigenstates and shown to bear similarities with the canonical partition function. In particular, the resulting occupation number $f(E)$ of single-particle orbitals was found to become a smooth, monotonously decreasing function of the energy E for sufficiently strong particle-particle interactions [13–15]. The standard approach to attribute a temperature to few-body eigenstates is to fit $f(E)$, or the expectation value of some other

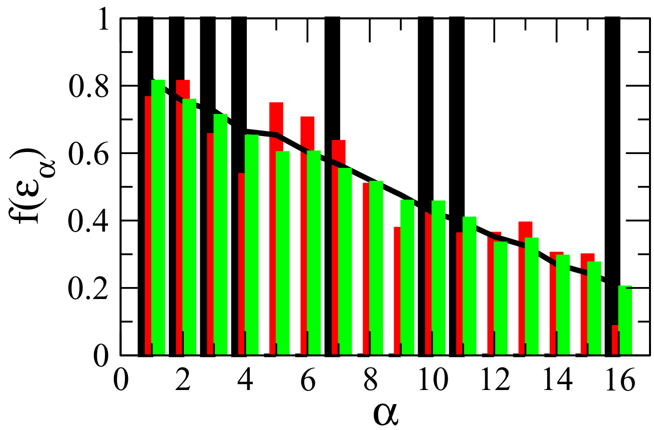

FIG. 1. Occupation number for the 1001st many-body eigenstate of $\mathcal{H}_{\rm sys}$ (3) with $m = 16$ orbitals and $n = 8$ particles, for $U/\Delta = 0$ [black histogram, with temperature and chemical potential $(T/\Delta, \mu/\Delta) = (5.41, 0.0141)$], 0.1 [red (dark gray), $(T/\Delta, \mu/\Delta) = (5.43, 0.0139)$], and 0.2 [green (light gray), $(T/\Delta, \mu/\Delta) = (5.47, 0.0135)$]. The black solid line is the Fermi function corresponding to $(T/\Delta, \mu/\Delta) = (5.47, 0.0135)$. It is slightly unsmooth because the one-body spectrum of (3) is not equidistant.

observable, to a theoretical, temperature-dependent function, treating the temperature as a fitting parameter.

One of our motivations in this paper is to go beyond the just-described operational temperature definition. Inspired by the experimental technique of scanning thermal microscopy [16], we show in this manuscript that a temperature that is consistent with a number of standard thermodynamic criteria can indeed be attributed to individual many-body eigenstates $|A\rangle$ of few-fermion systems, by coupling them weakly to an external Fermi-Dirac gas. The coupling allows for particle exchange between the two systems, and the external Fermi-Dirac gas acts as a probe [17–22]: its temperature T_A and chemical potential μ_A are set so that particle and heat currents between the few-fermion system, prepared in its eigenstate $|A\rangle$, and the probe vanish. We find that T_A varies smoothly with energy from eigenstate to eigenstate, regardless of the strength of the particle-particle interaction, despite the fact that the latter strongly influences the structure of the many-body eigenstates (see Fig. 1). In a large energy range, the obtained temperature is furthermore well approximated by the thermodynamic relation [1]

$$T = (\partial S/\partial E)_X^{-1}, \tag{1}$$

with the subscript X indicating the set of thermodynamic variables kept constant, and the entropy S determined by the density of states $\rho(E)$ through a quantum version of Boltzmann's formula,

$$S(E) = k_{\rm B} \ln[\rho(E)\delta], \tag{2}$$

where δ is a typical energy scale in the system, for instance, a single-particle level spacing.

It is tempting to attribute the probe temperature T_A to the many-body eigenstate of the few-fermion system to which the probe is connected. Below, we argue that this thermometric definition of eigenstate temperature is indeed consistent with standard thermodynamic definitions in the sense that (i) it varies smoothly from eigenstate to eigenstate and is monotonously increasing with energy, (ii) it is equal to a temperature independently obtained from the thermodynamic relation of Eqs. (1) and (2) in a large energy range, (iii) it is independent of details of the system-probe coupling in a finite parameter range, (iv) it satisfies the transitivity condition underlying the zeroth law of thermodynamics, and (v) it is consistent with Carnot's theorem. Quite interestingly, these consistency conditions are valid regardless of the particle-particle interaction in the few-fermion system. We argue that ensemble equivalence is, however, achieved only once a further condition is imposed, that (vi) all partial currents between single-particle orbitals in the few-fermion system and the Fermi-Dirac gas vanish. This latter condition ensures that the few-fermion state does not eventually change over time, and guarantees that a true equilibrium exists between the two subsystems. In particular, it is only under that latter condition that our approach is fully justified because then the structure of the few-fermion eigenstates is not altered by its coupling with the external Fermi-Dirac gas. We find that condition (vi) is achieved for sufficiently strong particle-particle interaction, $U \gtrsim U_c \sim n^{-3}$, with the number n of fermions, in systems with fixed particle density.

Our numerical results are based on exact diagonalization of few-particle systems, and are therefore limited to systems with up to $n = 8$ fermions. They suggest that the threshold interaction strength U_c above which (vi) is valid is parametrically similar to the many-body quantum chaos threshold derived in Refs. [23,24], giving a parametric critical interaction strength going down algebraically with the system size and the number of fermions. Taken together, the conditions (i)–(vi) suggest that small fermionic systems with sufficient but not too strong interaction have many-body eigenstates that not only satisfy the ETH—this was already known—but that each of them defines both a microcanonical and a canonical ensemble which are equivalent to each other, and correspond to a temperature in standard thermodynamic senses.

II. THE MODEL

We consider systems of n interacting, spinless fermions with the Hamiltonian given by the two-body random ensemble [25],

$$\mathcal{H}_{\rm sys} = \sum \epsilon_\alpha c_\alpha^\dagger c_\alpha + \sum U_{\alpha,\beta}^{\gamma,\delta} c_\alpha^\dagger c_\beta^\dagger c_\gamma c_\delta. \tag{3}$$

Here, $c_\alpha^\dagger$ and c_α are creation and annihilation operators obeying fermionic anticommutation relations and $\epsilon_\alpha \in [-m\Delta/2, m\Delta/2]$ are m single-particle orbital energies with average spacing Δ. We take ϵ_α as eigenvalues of a $m \times m$ random matrix of the Gaussian orthogonal ensemble [25]. Fermions occupying these single-particle energies interact via a two-body interaction with matrix elements randomly distributed as $U_{\alpha,\beta}^{\gamma,\delta} \in [-U, U]$. The total number of many-body eigenstates of $\mathcal{H}_{\rm sys}$ is $N = m!/n!(m-n)!$ and the corresponding eigenvalues are distributed over a bandwidth $B \simeq n(m-n)\Delta$. In this manuscript, we focus on systems at half filling with $n = m/2$ fermions. Many-body eigenstates of $\mathcal{H}_{\rm sys}$

are linear combinations of totally antisymmetric n-body wave functions over m orbitals. Each such many-body eigenstate $|A\rangle$ has single-orbital occupancies $f_A(\epsilon_\alpha) \in [0, 1]$. Occupancies for many-body states corresponding to three different interaction strengths are shown in Fig. 1.

We weakly couple that system to an external noninteracting fermionic gas with Hamiltonian

$$\mathcal{H}_{\rm FD} = \sum E_i d_i^\dagger d_i. \tag{4}$$

In this gedanken experiment, we assume that n consecutive energy eigenvalues E_i, $i = i_0, i_0 + 1, \ldots, i_0 + n - 1$, are equal to the n single-particle energies ϵ_α, $\alpha = 1, 2, \ldots, n$. Furthermore, that external fermionic gas is thermalized in the standard textbook way, being, e.g., connected to an infinite external reservoir at a tunable temperature T. Consequently, we assume that it is in a mixed state where its single-particle occupancies obey a Fermi-Dirac distribution,

$$f_{\rm FD}(E_i, \mu, T) = \frac{1}{1 + \exp[(E_i - \mu)/k_{\rm B}T]}, \tag{5}$$

with chemical potential μ. This Fermi-Dirac gas (FDG) is the probe which will attribute a temperature to each many-body eigenstates of the few interacting fermion system of Eq. (3).

We assume that the tunneling amplitude t between the FDG probe and the few interacting fermion system is small, constant, and strictly energy conserving. We write the tunneling Hamiltonian as

$$\mathcal{H}_{\rm T} = t \sum_{\alpha,i} (c_\alpha^\dagger d_i + d_i^\dagger c_\alpha)\, \delta(\epsilon_\alpha - E_i). \tag{6}$$

The few-fermion system is prepared in one of its many-body eigenstates $|A\rangle$. Under our assumptions, the probe-system coupling induces particle (I_A) and heat (J_A) currents between the two subsystems. We approximate them as sums over energy-conserving partial currents,

$$I_A = \frac{2\pi t^2}{\Delta} \sum_\alpha [f_A(\epsilon_\alpha) - f_{\rm FD}(E_i = \epsilon_\alpha, \mu, T)], \tag{7a}$$

$$J_A = \frac{2\pi t^2}{\Delta} \sum_\alpha (\epsilon_\alpha - \mu)[f_A(\epsilon_\alpha) - f_{\rm FD}(E_i = \epsilon_\alpha, \mu, T)]. \tag{7b}$$

For each many-body eigenstate $|A\rangle$, I_A and J_A are functions of T and μ. We then tune the temperature $T \to T_A$ and chemical potential $\mu \to \mu_A$ of the FDG to ensure that $I_A(\mu_A, T_A) = J_A(\mu_A, T_A) \equiv 0$. If a temperature can at all be attributed to $|A\rangle$, then this temperature is T_A [1]. It is important to understand that with this Hamiltonian, the coupled few-fermion eigenstate will in general be modified over time. Even with very weak coupling t, the external Fermi-Dirac gas will not act as a true probe—while measuring the few-fermion system, it will, in general, alter it. We will find below, however, that the few-fermion eigenstates are not modified by the coupling, and that the Fermi-Dirac gas acts as a true probe when the eigenstates are truly thermalized so that neither global nor partial currents flow between the two systems. This occurs when the few-fermion systems have sufficiently strong interactions. In that case, we conclude that ensemble equivalence accompanies eigenstate thermalization. Only in that case is our approach retroactively justified.

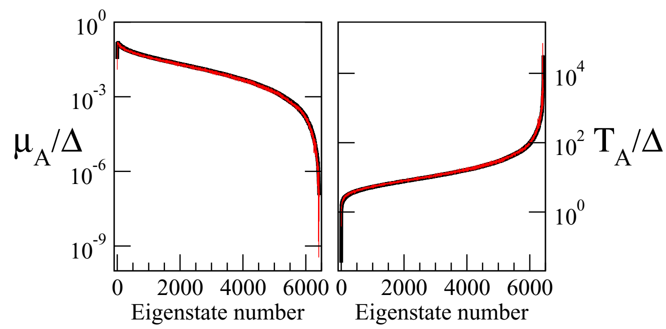

FIG. 2. Chemical potential (left) and temperature (right) vs many-body eigenstate number $A \in [1, N/2]$ for a single realization of the two-body randomly interacting fermion model (3) with $m = 16$ orbitals and $n = 8$ fermions and, thus, $N = 12\,870$ many-body states. Black dots are for $U/\Delta = 0$ and red (gray) dots are for $U/\Delta = 0.2$.

III. PROBE TEMPERATURE AND CHEMICAL POTENTIAL

Our numerical procedure is the following. We diagonalize exactly the Hamiltonian (3) and calculate its full set of many-body eigenvectors and the corresponding eigenvalues. We calculate the single-particle occupancies of each eigenstate and use a Newton-Raphson algorithm to obtain the values T_A and μ_A defined by $I_A = J_A = 0$.

Figure 2 shows results for a single realization of (3) with $n = 8$ fermions on $m = 16$ orbitals, for $U/\Delta = 0$ and $U/\Delta = 0.2$. We see that, first, both temperature and chemical potential vary smoothly from eigenstate to eigenstate in both cases. Second, turning on the interaction has only a small effect on both temperature and chemical potential, despite the fact that it changes the structure of individual many-body eigenstates very significantly, as illustrated in Fig. 1.

We next investigate to what extent the obtained probe temperature corresponds to the thermodynamic relation for the few-fermion system given in Eq. (1). For weak interaction, $U \ll \Delta$, the density of states is Gaussian, $\rho(E) = \rho_0 \exp[-E^2/2\sigma^2]$, with a variance $\sigma^2 \propto n(m - n)$ for dilute systems $n \ll m$ [25]. With Eqs. (1) and (2), this density of states gives a theoretical system temperature (see, also, Ref. [15]),

$$T_{\rm th}(E) = -\sigma^2/k_{\rm B}E. \tag{8}$$

This temperature diverges in the middle of the spectrum and becomes negative at higher temperature, as standardly happens in systems with nonmonotonous density of states [26]. We focus on the lower half of the spectrum, corresponding to positive temperatures.

Figure 3(a) confirms that the density of states is Gaussian and that it does not change much as a moderate particle-particle interaction is introduced. Figures 3(b) and 3(c) further show that in the noninteracting case, the probe temperature is very close to $T_{\rm th}$ of Eq. (8), except in the tail of the many-body density of states. This is due to well-known deviations from Gaussianity there [25]. For the case $m = 16$ and $n = 8$

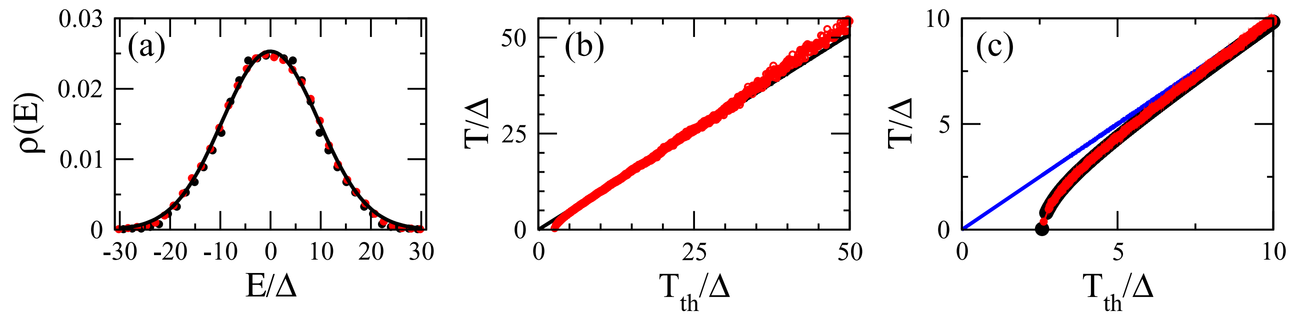

FIG. 3. (a) Many-body density of states. (b) Numerically obtained eigenstate temperature T vs theoretical temperature $T_{\rm th}$ of Eq. (8) for $0 < T_{\rm th}/\Delta \leqslant 50$. The interval contains about 90% of the positive temperature eigenstates. (c) Same as in (b) for a restricted range $0 < T_{\rm th}/\Delta \leqslant 10$, showing deviations at low energy due to non-Gaussian tails in the density of states [25]. All panels correspond to a single realization of the two-body randomly interacting fermion model (3) with $m = 16$ orbitals, $n = 8$ fermions, and $N = 12\,870$ many-body states. Black dots are for $U = 0$ and red (gray) dots are for $U = 0.2\Delta$.

shown in Fig. 3, the probe temperature differs from $T_{\rm th}$ by less than 10% for $T_{\rm th}/\Delta \gtrsim 5.5$ for $U/\Delta = 0$, corresponding to about 90% of the many-body eigenstates. A moderate interaction has very little impact, except in the middle of the spectrum, where large energy eigenvalue fluctuations increase the discrepancy between the probe temperature and $T_{\rm th}$. We found that for $U/\Delta = 0.2$, the difference between the two remains below 10% for $50 \gtrsim T_{\rm th}/\Delta \gtrsim 5.5$, corresponding to about 80% of the spectrum. These results do not significantly change upon further increase of the interaction strength as long as $U/\Delta \ll 1$.

IV. FROM PROBE TO EIGENSTATE TEMPERATURE

The results presented so far seem to indicate that the probe temperature gives more than just an operational temperature definition for individual many-body eigenstates. We now argue that this definition satisfies further properties expected of an absolute temperature [1]. First, it is clear from the currents definition, given by Eqs. (7), that the temperature definition does not depend on the magnitude of the probe-system coupling amplitude t. This is so only as long as t is small enough, so that Eqs. (7) are valid, but still leaves a sizable range of parameter t.

Second, the probe temperature is defined by the vanishing of the particle and heat currents between the probe and the system, $I_A(\mu_A, T_A) = J_A(\mu_A, T_A) \equiv 0$. Let us introduce a second many-body state $|B\rangle$ and couple it to the probe while keeping the latter's temperature and chemical potential the same. Let us further suppose then that there is no current between $|B\rangle$ and the probe, $I_B(\mu_A, T_A) = J_B(\mu_A, T_A) \equiv 0$. From Eqs. (7), one straightforwardly concludes that there would also be no current between $|A\rangle$ and $|B\rangle$, $I_{AB} = t^2 \sum_\alpha [f_A(\epsilon_\alpha) - f_B(\epsilon_\alpha)] = 0$, $J_{AB} = t^2 \sum_\alpha (\epsilon_\alpha - \mu)[f_A(\epsilon_\alpha) - f_B(\epsilon_\alpha))] = 0$. The transitivity condition of the zeroth law of thermodynamics is thus satisfied by the probe definition of the temperature.

Third, in the regime of validity of Eqs. (7), the system may temporarily work as a heat engine when the probe is biased away from the equilibrium condition, i.e., $T_A \to T_A + \delta T$ and $\mu_A \to \mu_A + \delta\mu$. Assume that all fermions carry an electric charge q. With this bias, both a heat and an electric current flow, with the latter being accompanied by electrical work. The efficiency of the resulting heat engine is given by $\eta = -I_A \delta\mu / J_A$ [27] with, from Eqs. (7),

$$I_A(\delta\mu, \delta T) = -\frac{2\pi t^2}{\Delta} \sum_\alpha [\partial_\mu f_{\rm FD} \delta\mu + \partial_T f_{\rm FD} \delta T], \tag{9a}$$

$$J_A(\delta\mu, \delta T) = -\frac{2\pi t^2}{\Delta} \sum_\alpha (\epsilon_\alpha - \mu)[\partial_\mu f_{\rm FD} \delta\mu + \partial_T f_{\rm FD} \delta T], \tag{9b}$$

where, in both expressions, $f_{\rm FD} = f_{\rm FD}(E_i = \epsilon_\alpha, \mu, T)$ and both δT and $\delta\mu$ are assumed very small to justify the linearization of the Fermi-Dirac distributions. A straightforward calculation gives that the maximal efficiency of the engine is given by

$$\eta_{\rm max} = \left(\frac{\sqrt{1 + \mathcal{ZT}} - 1}{\sqrt{1 + \mathcal{ZT}} + 1} \right) \frac{|\delta T|}{T_A}. \tag{10}$$

The dimensionless figure of merit is given by $\mathcal{ZT}^{-1} = \mathcal{L}^{(0)} \mathcal{L}^{(2)} / (\mathcal{L}^{(1)})^2 - 1$, with $\mathcal{L}^{(0)} = \sum_\alpha \partial_\mu f_{\rm FD}$, $\mathcal{L}^{(1)} = \sum_\alpha \partial_T f_{\rm FD} T_A$, and $\mathcal{L}^{(2)} = \sum_\alpha (\epsilon_\alpha - \mu_A) \partial_T f_{\rm FD} T_A$. Equation (10) defines a relative temperature scale in that $\eta_{\rm max}$ is a function of the temperature difference δT between system and probe, in agreement with Carnot's theorem.

V. MANY-BODY EIGENSTATE TEMPERATURE

There seem to be good reasons to take the probe temperature as a definition of the many-body eigenstate temperature. Nevertheless, a further condition needs to be satisfied before this is done. In general, $I_B(\mu_A, T_A) = J_B(mu_A, T_A) \equiv 0$ still allows for partial currents $[f_A(\epsilon_\alpha) - f_{\rm FD}(E_i = \epsilon_\alpha, \mu, T)] \neq 0$ and $(\epsilon_\alpha - \mu)[f_A(\epsilon_\alpha) - f_{\rm FD}(E_i = \epsilon_\alpha, \mu, T)] \neq 0$ in Eqs. (7). Therefore, the state of the few-fermion system will eventually change, even with a weak, finite system-probe coupling, unless detailed balance conditions are satisfied,

$$[f_A(\epsilon_\alpha) - f_{\rm FD}(E_i = \epsilon_\alpha, \mu, T)] = 0\,, \quad \forall\alpha. \tag{11}$$

This, of course, means that particle occupancies in few-fermion states are given by the Fermi-Dirac distribution. Looking at Fig. 1, we see that this may occur for sufficiently interacting few-fermion systems. To quantify the rate at which Fermi-Dirac-like occupancies emerge as the interaction

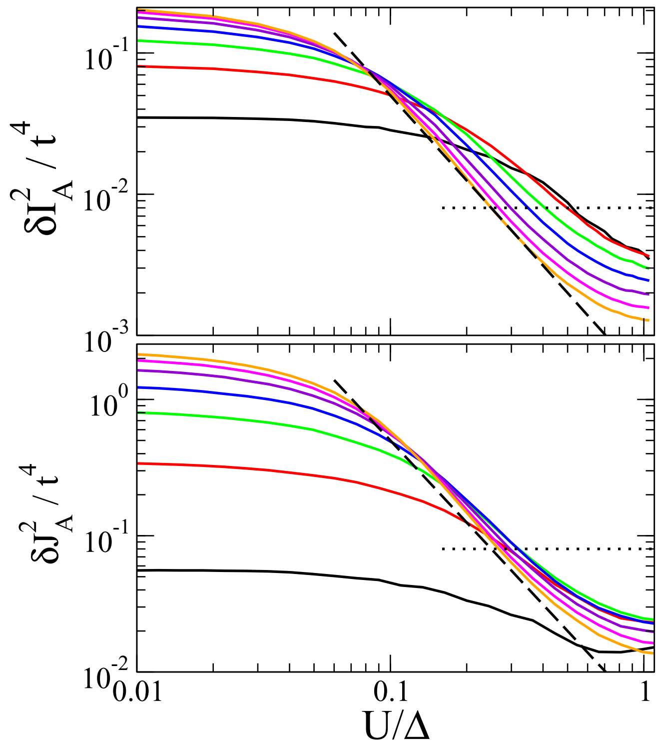

FIG. 4. Deviation from detailed balance for the particle current (top panel) and the heat current (bottom panel) as a function of the interaction strength U/Δ for 1000 realizations of the two-body randomly interacting fermion model (3) with $m = 12$ orbitals and $n = 6$ fermions and thus $N = 924$ many-body states. Different curves correspond to different excitations energies above the ground state, starting at $\epsilon = 2\Delta$ and increasing in steps of $\delta\epsilon = 2\Delta$ from black to red, green, blue, violet, magenta, and orange (from bottom to top, at $U/\Delta = 0.01$). The dotted line indicates the arbitrarily chosen threshold $\delta I_A^2/t^4 = 8 \times 10^{-3}$ and $\delta J_A^2/t^4 = 8 \times 10^{-2}$ used to define critical interaction strengths U_{c1} and U_{c2}.

strength increases, we investigate the variance of the partial currents making up the particle and heat currents of Eq. (7),

$$\delta I_A^2 = \frac{4\pi^2 t^4}{\Delta^2} \sum_\alpha [f_A(\epsilon_\alpha) - f_{\rm FD}(E_i = \epsilon_\alpha, \mu_A, T_A)]^2,$$

$$\delta J_A^2 = \frac{4\pi^2 t^4}{\Delta^2} \sum_\alpha (\epsilon_\alpha - \mu_A)^2 [f_A(\epsilon_\alpha) - f_{\rm FD}(E_i = \epsilon_\alpha, \mu_A, T_A)]^2. \quad (12)$$

These current variances vanish only when the detailed balance conditions of Eq. (11) are satisfied. When this is the case, orbital occupancies in the few-fermion system are given by the Fermi-Dirac distribution, partial currents accordingly vanish and the many-body eigenstate $|A\rangle$ is at equilibrium with the thermometric Fermi-Dirac gas in the usual sense. In particular, the weak coupling between the two subsystems does not change the state of the few-fermion system.

Figure 4 shows the particle and heat current variances as a function of the normalized interaction strength U/Δ. It is seen that as U/Δ increases, both variances decrease with rates that depend on the excitation energy ϵ above the

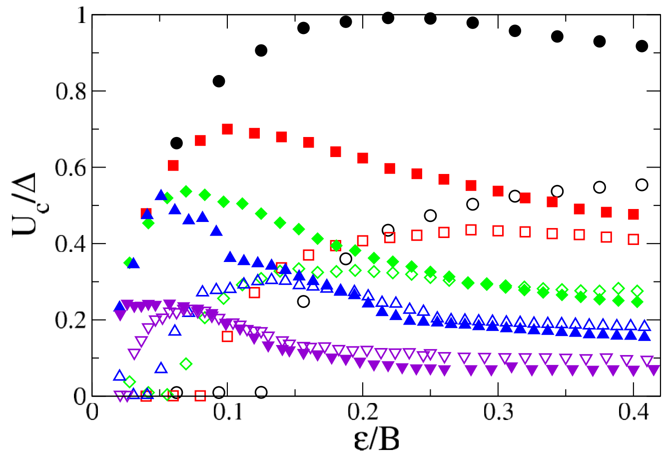

FIG. 5. Critical interaction strength U_c vs normalized excitation energy ϵ/B with the many-body bandwidth $B \simeq n(m-n)\Delta$, for $m = 8$, $n = 4$ (black circles), $m = 10$, $n = 5$ (red squares), $m = 12$, $n = 6$ (green diamonds), $m = 14$, $n = 7$ (blue triangles, pointing up), and $m = 16$, $n = 8$ (violet triangles, pointing down). Solid circles give U_{c1} and empty circles give U_{c2} (see text).

few-fermion ground state. At high enough excitation energy—corresponding to few level spacing Δ above the ground state—we find that they behave as $\delta I_A^2, \delta J_A^2 \propto (U/\Delta)^{-2}$ (indicated by dashed lines in Fig. 4) until they saturate at a value depending on both m and n. We have seen this behavior for other values of m and n, not shown in Fig. 4.

We want to extract the parametric rate at which the current variances vanish. To that end, we define critical interaction strengths U_{c1} and U_{c2} with $\delta I_A^2(U_{c1}) = 8 \times 10^{-3}\, t^4$ and $\delta J_A^2(U_{c2}) = 8 \times 10^{-2}\, t^4$. We chose these values, somehow arbitrary, because they correspond to interaction strengths with significantly reduced current variances, but well before their large-U, finite-size saturation for all cases considered, $m/n = 2$ and $n = 4, 5, \dots, 8$. Figure 5 shows the obtained values of U_{c1} (solid circles) and U_{c2} (empty circles) as a function of the excitation energy ϵ/B normalized by the many-body bandwidth B. At small excitation energy close to the ground-state energy, both U_{c1} and U_{c2} increase with ϵ/B. This reflects the fact that at low excitation energies, close to the ground state, the number of available single-particle orbitals is restricted, which induces a faster transition to a steplike Fermi-Dirac distribution. Furthermore, quasiparticles with small excitation energies carry very little heat current, and hence δJ_A is always small because J_A is. This explains why U_{c2} is always very small at small excitation energies. For higher excitation energy, but already above $\epsilon/\Delta \simeq 2$–3, both U_{c1} and U_{c2} are monotonously decreasing functions of the number of particles and orbitals. We find that the transition from U_c increasing to decreasing with respect to ϵ occurs at an excitation energy ϵ/Δ that also decreases with the number of particles.

Because of the restricted range of variation of n reachable by exact diagonalization, it is hard to extract a parametric dependence of U_c. Nevertheless, the data shown in Fig. 5 at half filling, $n = m/2$, seem to indicate a behavior $U_{c1} \propto n^{-3}$, consistent with the emergence of quantum chaos reported in Ref. [24]. They also suggest that for sufficiently large systems, both particle and heat current based critical interactions become the same. With these data, we conjecture that eigenstate thermalization is accompanied by ensemble equivalence, where each individual few-fermion eigenstate exhibits a Fermi-Dirac occupancy distribution, and accordingly defines a canonical ensemble with grand canonical occupancies, above a critical interaction strength $U_c/\Delta \propto n^{-3}$ for systems at half filling.

VI. CONCLUSION

Few-fermion systems have eigenstates with a Fermi-Dirac occupancy of single-particle orbitals, provided they have a sufficiently strong interaction. Our results indicate that above an excitation energy $\epsilon \simeq 2-3\Delta$ above the ground state, the critical interaction strength scales parametrically as $U_c \propto n^{-3}$ in systems with constant filling factor. In particular, $U_c \simeq 0.1\Delta$ with the single-particle orbital spacing Δ, for $n = 8$ fermions on $m = 16$ orbitals and excitation energies $\epsilon \gtrsim 5-6\Delta$. This indicates that for still small systems with, say, $n = 20$, the critical interaction strength is only a fraction of this single-particle orbital spacing. Ensemble equivalence at the level of individual few-body eigenstates is therefore achieved very fast in the number n of fermions, and requires only a weak interaction.

Perhaps the main result reported in this paper is that eigenstate thermometry allows one to define a temperature for individual eigenstates in few-body fermionic systems in a way that is consistent with standard approaches of thermodynamics and, in particular, with the zeroth law and Carnot's theorem.

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