Non-Markovian dynamics of quantum systems coupled with several mixed fermionic-bosonic heat baths

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For the fermionic or bosonic oscillator fully coupled to several heat baths with mixed statistics, the analytical expressions for the occupation numbers are derived within the non-Markovian quantum Langevin approach. Employing two or three heat baths and the Ohmic dissipation with Lorenzian cutoffs, the role of statistics of the system and heat baths in the dynamics of the system is studied.

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I. INTRODUCTION

The environmental effect on a quantum system is an important subject of the theory of open quantum systems [1–22]. In practice, a quantum system is often coupled to several reservoirs, like in the case of cavity quantum electrodynamics [23], Jaynes-Cummings lattices [24], photon-ion interfaces [25], ion chain systems [26], and phonon-induced spin squeezing [27]. An example of collective motion in a bosonic quantum system coupled with two bosonic reservoirs [28,29] is the attenuation of single-mode field inside the resonator which has two different partially transparent mirrors [28]. Because the losses in these mirrors are different, one can consider them as two reservoirs. They consist of a large number of phonon-type modes excited in mirrors. An example of a fermionic system is the single-quantum-dot in fermionic environments [30]. It can be in the strong Coulomb blockage regime, so that only one electron is allowed therein. Two electric leads act as two fermionic heat baths. It might be that the quantum system and reservoirs have different statistics, fermionic or bosonic. An example of collective motion in surrounding of several thermostats with different statistics is the fusion of nuclei in a stellar medium. Fusion occurs in the relevant collective coordinate. One thermostat is connected with the nucleon degrees of freedom of the nuclei. Other thermostats can be associated with electromagnetic and pressure fields. This description allows us to take effectively into consideration the effects associated with the internal structure of interacting nuclei and with the properties of medium. Another example is the formation of atomic (nuclear) molecule in the external electromagnetic field. The effects of the electron (nucleon) degrees of freedom and the electromagnetic field can be effectively described by means of appropriate thermostats. So, there are many physical problems which can be solved by dividing the total system into the open system coupled to several heat baths.

In Ref. [31], we considered the case of FC ("fully coupled") oscillator modeling fermionic (bosonic) collective system coupled with one bosonic (fermionic) heat-bath. We have used the quadratic mixed fermionic-bosonic Hamiltonians for the collective and internal systems with a linear coupling, Ohmic dissipation with Lorenzian cutoffs, and presented the detailed analysis of the role of the fermionic statistics (in the comparison with the bosonic one) in the collective motion. For the fermionic collective system plus fermionic bath (shortly, fermionic-fermionic), fermionic-bosonic, bosonic-bosonic, and bosonic-fermionic systems, the master equations and the analytical expressions for the collective occupation numbers were derived. As shown, at large coupling strengths or low temperatures the asymptotic occupation numbers for the fermionic, bosonic, and mixed systems noticeably deviate from the corresponding Fermi-Dirac or Bose-Einstein values. For the systems with bosonic bath, the friction and diffusion coefficients oscillate in time [31]. In Ref. [29], for the collective bosonic (fermionic) oscillator and several internal bosonic (fermionic) heat baths, i.e., b-b-...-b (f-f-...-f), coupled linearly (FC couplings), the analytical expression for the collective occupation number was derived with the non-Markovian quantum Langevin approach. The asymptotes of the fermionic and bosonic occupation numbers were found. The time-dependent transport coefficients of the master equations for the collective occupation number were obtained as well. In the case of Ohmic dissipation with Lorenzian cutoffs, the possibility of reduction of the system with several heat baths to the system with one heat bath was analytically demonstrated. In bosonic and fermionic cases, the statistics of a particle in the effective bath differs from the original ones and can not be taken anymore as either bosonic or fermionic. In the Marokovian limit, the coupling to several bosonic baths is reduced to the coupling to one bosonic bath in which the effective temperature is a weighted average of the temperature of the original baths. As well as for the fermionic case, the inverse temperature is the weighted average of the inverse temperatures of different original baths.

The aim of the present work is to extend the results of Refs. [29,31] and consider the case of FC oscillator modeling fermionic (bosonic) quantum system coupled with several bosonic and/or fermionic heat baths. Note that the systems with mixed statistics of baths were not considered in Refs. [28–30]. In the contrast to the present work and Ref. [29], the rotating wave approximation (RWA) couplings were employed in Refs. [28,30]. In the present work, we use the quadratic mixed fermionic-bosonic Hamiltonians for the system and heat baths with a linear FC, and we present a detailed analysis of the role of fermionic and bosonic statistics in the dynamics of system. The FC contains the resonant and nonresonant terms [2]. The fluctuation-dissipation relation is satisfied for the FC-oscillator. Note that the interest in considering fermionic baths is growing up due to the possibility to create and manipulate fermionic systems in condensed matter, atomic, and nuclear physics [17,18]. For the system fully coupled to several baths, we will check if it might always reach a stationary asymptotic limit. Our aim is to find the condition for the oscillation of asymptotic occupation number. This interesting asymptotic behavior can be proposed for practical use, for example, to increase the channels and speed of communication lines and to control some states for recording data in quantum computers.

In Sec. II, the formalism is presented and the expressions for the master equations and the occupation numbers are obtained. To our knowledge, it is shown for the first time that the baths of different statistics can be used to obtain nonstationary asymptotic occupation numbers. The asymptotic occupation numbers are discussed. For the systems with two baths with the same and mixed statistics and with three baths with the same statistics, the illustrative numerical calculations of diffusion and friction coefficients and level populations are performed in Sec. III. A summary is given in Sec. IV.

II. FORMALISM

A. Hamiltonian

The Hamiltonian of the total system (the quantum system plus several heat baths "λ," $\lambda = 1, \cdots, N_b$) is written as

$$H = H_c + \sum_{\lambda=1}^{N_b} H_\lambda + \sum_{\lambda=1}^{N_b} H_{c,\lambda}, \tag{1}$$

where

$$H_c = \hbar\omega \mathrm{a}^\dagger \mathrm{a} \tag{2}$$

is the Hamiltonian of the isolated system being either fermionic or bosonic oscillator with frequency ω,

$$H_\lambda = \sum_i \hbar\omega_{\lambda,i} c^\dagger_{\lambda,i} c_{\lambda,i}$$

are the Hamiltonians of the thermal baths. The value of N_b is the number of heat baths. Each heat bath "λ" is modeled by the assembly of independent fermionic or bosonic oscillators labelled in both cases by "i" with frequencies $\omega_{\lambda,i}$. For the FC coupling between the system and heat baths, the interaction Hamiltonians $H_{c,\lambda}$ are

$$H_{c,\lambda} = \sum_i \alpha_{\lambda,i}(\mathrm{a}^\dagger + \mathrm{a})(c^\dagger_{\lambda,i} + c_{\lambda,i}). \tag{3}$$

The real constants $\alpha_{\lambda,i}$ determine the coupling strengths. These couplings are linear in the system and baths operators. They have important consequences on the dynamics of the system by altering the effective collective potential and by allowing energy to be exchanged with the thermal reservoirs, thereby allowing the system to attain some equilibrium with the heat baths.

Here, the system and heat baths have the fermionic or bosonic statistics. So, the creation and annihilation operators of the system and heat baths satisfy the commutation or anticommutation relations,

$$\begin{aligned}
&\mathrm{a}\mathrm{a}^\dagger - \varepsilon_{\mathrm{a}} \mathrm{a}^\dagger \mathrm{a} = 1, \quad \mathrm{a}^\dagger \mathrm{a}^\dagger - \varepsilon_{\mathrm{a}} \mathrm{a}^\dagger \mathrm{a}^\dagger = \mathrm{a}\mathrm{a} - \varepsilon_{\mathrm{a}} \mathrm{a}\mathrm{a} = 0,\\
&c_{\lambda,i} c^\dagger_{\lambda,i} - \varepsilon_\lambda c^\dagger_{\lambda,i} c_{\lambda,i} = 1,\\
&c^\dagger_{\lambda,i} c^\dagger_{\lambda,i} - \varepsilon_\lambda c^\dagger_{\lambda,i} c^\dagger_{\lambda,i} = c_{\lambda,i} c_{\lambda,i} - \varepsilon_\lambda c_{\lambda,i} c_{\lambda,i} = 0,
\end{aligned} \tag{4}$$

where ε_{a} and ε_λ are equal to 1 (-1) for the bosonic (fermionic) system and bosonic (fermionic) heat baths, respectively.

B. Master equation for occupation number of quantum system

Employing the Hamiltonian Eq. (1) for the fermionic and bosonic systems, we deduce the equations of motion for the occupation number

$$\begin{aligned}
\frac{d\mathrm{a}^\dagger(t)\mathrm{a}(t)}{dt} &= \frac{i}{\hbar} \sum_{\lambda,i} \alpha_{\lambda,i} [\mathrm{a}(t) - \mathrm{a}^\dagger(t)][c^\dagger_{\lambda,i}(t) + c_{\lambda,i}(t)]\\
&= \frac{i}{\hbar} \sum_{\lambda,i} \alpha_{\lambda,i} [c^\dagger_{\lambda,i}(t)\mathrm{a}(t) - \mathrm{a}^\dagger(t) c_{\lambda,i}(t)\\
&\quad + \mathrm{a}(t) c_{\lambda,i}(t) - \mathrm{a}^\dagger(t) c^\dagger_{\lambda,i}(t)].
\end{aligned} \tag{5}$$

For the operators $c^\dagger_{\lambda,i}(t)\mathrm{a}(t)$ and $\mathrm{a}(t)c_{\lambda,i}$ in Eq. (5), one can derive the following equations:

$$\begin{aligned}
\frac{dc^\dagger_{\lambda,i}\mathrm{a}}{dt} &= i(\omega_{\lambda,i} - \omega) c^\dagger_{\lambda,i}\mathrm{a} + \frac{i}{\hbar}\alpha_{\lambda,i}[\mathrm{a}^\dagger\mathrm{a} + \mathrm{a}\mathrm{a}]\\
&\quad \times [1 - (1 - \varepsilon_\lambda) c^\dagger_{\lambda,i} c_{\lambda,i}]\\
&\quad - \frac{i}{\hbar}\left[\sum_{\lambda',i'} \alpha_{\lambda',i'} (c^\dagger_{\lambda,i} c^\dagger_{\lambda',i'} + c^\dagger_{\lambda,i} c_{\lambda',i'})\right]\\
&\quad \times [1 - (1 - \varepsilon_{\mathrm{a}})\mathrm{a}^\dagger\mathrm{a}],
\end{aligned} \tag{6}$$

$$\begin{aligned}
\frac{d\mathrm{a}c_{\lambda,i}}{dt} &= -i(\omega_{\lambda,i} + \omega)\mathrm{a}c_{\lambda,i} - \frac{i}{\hbar}\alpha_{\lambda,i}[\mathrm{a}^\dagger\mathrm{a} + \mathrm{a}\mathrm{a}]\\
&\quad \times [1 - (1 - \varepsilon_\lambda) c^\dagger_{\lambda,i} c_{\lambda,i}]\\
&\quad - \frac{i}{\hbar}\left[\sum_{\lambda',i'} \alpha_{\lambda',i'} (c_{\lambda,i} c^\dagger_{\lambda',i'} + c_{\lambda,i} c_{\lambda',i'})\right]\\
&\quad \times [1 - (1 - \varepsilon_{\mathrm{a}})\mathrm{a}^\dagger\mathrm{a}].
\end{aligned} \tag{7}$$

Substituting the formal solutions of Eqs. (6) and (7) in Eq. (5) and averaging over the heat baths and oscillator, we obtain the master equation for the occupation number $n_{\mathrm{a}} = \langle \mathrm{a}^\dagger \mathrm{a} \rangle$ of the oscillator (a = f and a = b for fermionic and bosonic systems, respectively),

$$\frac{dn_{\rm a}(t)}{dt} = \sum_{\lambda,i}\int_0^t ds\{W^-_{\lambda,i}(t-s)[\bar{n}_{\rm a}(s)n_{\lambda,{\rm i}}(s) - n_{\rm a}(s)\bar{n}_{\lambda,{\rm i}}(s)] + W^+_{\lambda,i}(t-s)[\bar{n}_{\rm a}(s)\bar{n}_{\lambda,{\rm i}}(s) - n_{\rm a}(s)n_{\lambda,{\rm i}}(s)]\}, \quad (8)$$

where

$$W^-_{\lambda,i} = \frac{2\alpha^2_{\lambda,i}}{\hbar^2}\cos([\omega-\omega_{\lambda,i}][t-s]),$$
$$W^+_{\lambda,i} = \frac{2\alpha^2_{\lambda,i}}{\hbar^2}\cos([\omega+\omega_{\lambda,i}][t-s]). \quad (9)$$

Here, $\bar{n}_{\rm a}(t) = 1+\varepsilon_{\rm a}\langle {\rm a}^\dagger {\rm a}\rangle$, $n_{\lambda,{\rm i}}(t) = \langle c^\dagger_{\lambda,i}c_{\lambda,i}\rangle$, $\bar{n}_{\lambda,{\rm i}}(t) = 1 + \varepsilon_\lambda\langle c^\dagger_{\lambda,i}c_{\lambda,i}\rangle$. The symbol $\langle ...\rangle$ denotes the expectation value over the whole system of heat baths and oscillator. To obtain Eq. (8), we assume $\langle {\rm a}^\dagger {\rm a} c^\dagger_{\lambda,i}c_{\lambda,i}\rangle = n_{\rm a}(t)n_{\lambda,{\rm i}}(t)$, $\langle {\rm a}^2\rangle = \langle({\rm a}^\dagger)^2\rangle = 0$ (for the bosonic subsystem), and $\langle c^\dagger_{\lambda,i}c^\dagger_{\lambda',i'}\rangle = \langle c_{\lambda',i'}c_{\lambda,i}\rangle = 0$, $\langle c^\dagger_{\lambda,i}c_{\lambda',i'}\rangle = n_{\lambda,{\rm i}}(t)\delta_{\lambda,\lambda'}\delta_{i,i'}$ (the heat baths consist of independent oscillators).

One can rewrite Eq. (8) as

$$\frac{dn_{\rm a}}{dt} = \int_0^t d\tau\{W_+(t-\tau)\bar{n}_{\rm a}(\tau) - W_-(t-\tau)n_{\rm a}(\tau)\}, \quad (10)$$

where

$$W_+ = \sum_\lambda W_+^{(\lambda)} = \sum_{\lambda,i}[W^-_{\lambda,i}(t-\tau)n_{\lambda,{\rm i}}(\tau) + W^+_{\lambda,i}(t-\tau)\bar{n}_{\lambda,{\rm i}}(\tau)],$$
$$W_- = \sum_\lambda W_-^{(\lambda)} = \sum_{\lambda,i}[W^-_{\lambda,i}(t-\tau)\bar{n}_{\lambda,{\rm i}}(\tau) + W^+_{\lambda,i}(t-\tau)n_{\lambda,{\rm i}}(\tau)]. \quad (11)$$

The physical meaning of the coefficients W_+ and W_- becomes clear from the master equation Eq. (10). The coefficient W_+ (W_-) defines the rate of occupation (leaving) of the state "a" in the open quantum system. The ratio between the W_+ and W_- characterizes the rate of equilibrium. The occupation number reaches the equilibrium value if the ratio of W_+ and W_- has asymptotic at $t\to\infty$.

At equilibrium ($\frac{dn_{\rm a}}{dt}|_{t\to\infty} = 0$), we have the general relationship

$$n_{\rm a}(t\to\infty) = \frac{W_+(t\to\infty)}{W_-(t\to\infty) - \varepsilon_{\rm a}W_+(t\to\infty)} = \frac{W_+(t\to\infty)}{W(t\to\infty) + \sum_\lambda[\varepsilon_\lambda - \varepsilon_{\rm a}]W_+^{(\lambda)}(t\to\infty)}, \quad (12)$$

where the asymptotics $W_+(\infty)$, $W_-(\infty)$, and $W(\infty)$ are related as

$$W = \sum_\lambda W^{(\lambda)} = \sum_{\lambda,i}[W^-_{\lambda,i} - \varepsilon_\lambda W^+_{\lambda,i}].$$

To study the possibility of reaching an equilibrium, we consider the bosonic and fermionic oscillators coupled with the baths of various statistics.

C. Bosonic (Fermionic) oscillator coupled with bosonic (fermionic) heat baths

Let us consider the case when all N_b heat baths and system oscillator are either all bosonic or all fermionic. For these systems, the details of the procedure for obtaining the occupation number of system are given in Ref. [29]. Here, we directly write the final expression for the time dependence of occupation number

$$n_{\rm a}(t) = n_{\rm a}(0)|A(t)|^2 + [1+\varepsilon_{\rm a}n_{\rm a}(0)]|B(t)|^2 + I_{\rm a}(t), \quad (13)$$

where $I_{\rm a}(t) = \sum_\lambda I_{\rm a}^{(\lambda)}(t)$ and

$$I_{\rm a}^{(\lambda)}(t) = \frac{\alpha_\lambda\gamma_\lambda^2}{\pi}\int_0^\infty dw\frac{w}{\gamma_\lambda^2+w^2}[n^{(\lambda)}(w)|M(w,t)|^2 + [1+\varepsilon_\lambda n^{(\lambda)}(w)]|N(w,t)|^2]. \quad (14)$$

In Eq. (14), $n^{(\lambda)}(w) = (\exp[\hbar w/(kT_\lambda)] - \varepsilon_\lambda)^{-1}$ are equilibrium Fermi-Dirac (Bose-Einstein) distributions of the fermionic (bosonic) heat baths "λ." The T_λ is the initial thermodynamic temperature of the corresponding heat bath. Here, we introduce the spectral density $\rho_\lambda(w)$ of the heat-bath excitations, which allows us to replace the sum over i by integral over the frequency w: $\sum_i ... \to \int_0^\infty dw\rho_\lambda(w)....$ For all baths, we consider the following spectral function [2]:

$$\frac{\alpha^2_{\lambda,i}}{\hbar^2 w_{\lambda,i}} \to \frac{\rho_\lambda(w)\alpha^2_{\lambda,w}}{\hbar^2 w} = \frac{1}{\pi}\alpha_\lambda\frac{\gamma_\lambda^2}{\gamma_\lambda^2+w^2}, \quad (15)$$

where the memory time γ_λ^{-1} of dissipation is inverse to the bandwidth of the heat-bath excitations which are coupled to the collective system. This is the Ohmic dissipation with the Lorenzian cutoff (Drude dissipation). We consider the case when the memory time of dissipation much less than the characteristic collective time, i.e., $\gamma_\lambda \gg \omega$. For the time-dependent coefficients $A(t)$, $B(t)$, $M(w,t)$, and $N(w,t)$, the analytical expressions are presented in Appendix A (see also Ref. [29]). The similarity of expressions for the occupation numbers for fermionic and bosonic systems results from the similarity of the equations of motion for creation and annihilation operators [18–21].

Making derivative of Eq. (13) in t, the following decomposition $|B(t)|^2 = \sum_\lambda J^{(\lambda)}(t)$, and simple but tedious algebra, we derive the differential equation for the occupation number:

$$\frac{dn_{\rm a}(t)}{dt} = -2\lambda_{\rm a}(t)n_{\rm a}(t) + 2D_{\rm a}(t), \quad (16)$$

where

$$\lambda_{\rm a}(t) = -\frac{1}{2}\frac{d}{dt}\ln[|A(t)| + \varepsilon_{\rm a}|B(t)|^2] \quad (17)$$

and

$$D_{\rm a}(t) = \sum_\lambda D_{\rm a}^{(\lambda)}(t) = \lambda_{\rm a}(t)[|B(t)|^2 + I_{\rm a}(t)] + \frac{1}{2}\frac{d}{dt}[|B(t)|^2 + I_{\rm a}(t)],$$

$$D_{\mathrm{a}}^{(\lambda)}(t) = \lambda_{\mathrm{a}}(t)\left[J^{(\lambda)}(t) + I_{\mathrm{a}}^{(\lambda)}(t)\right] + \frac{1}{2}\frac{d}{dt}\left[J^{(\lambda)}(t) + I_{\mathrm{a}}^{(\lambda)}(t)\right] \quad (18)$$

are the time-dependent friction and diffusion coefficients, respectively. Here, $\lambda_{\mathrm{a}}(t=0) = D_{\mathrm{a}}(t=0) = 0$. Therefore, we have obtained the local in time Eq. (16) for the $n_{\mathrm{a}}(t)$. In the case of constant friction and diffusion coefficients, this equation describes Markovian dynamics, i.e. the evolution of $n_{\mathrm{a}}(t)$ is independent of the past. In Eq. (16), the transport coefficients explicitly depend on time and the non-Markovian effects are taken into consideration through this time dependence [4,8]. The non-Markovian feature of Eq. (16) is well seen at $D_{\mathrm{a}} = 0$. In this case, $n_{\mathrm{a}}(t) \sim \exp(-2\int_0^t \lambda_{\mathrm{a}}(\tau)d\tau)$, i.e., the occupation number depends on the time dependence of $\lambda_{\mathrm{a}}(t)$. Because $A = B = 0$ [29] and $D_{\mathrm{a}} = \lambda_{\mathrm{a}} I_{\mathrm{a}}$ at $t \to \infty$, the appropriate asymptotic equilibrium distribution ($\frac{dn_{\mathrm{a}}(t)}{dt} = 0$)

$$n_{\mathrm{a}}(\infty) = \lim_{t\to\infty} \frac{D_{\mathrm{a}}(t)}{\lambda_{\mathrm{a}}(t)} = I_{\mathrm{a}}(\infty) = \sum_{\lambda} I_{\mathrm{a}}^{(\lambda)}(\infty) \quad (19)$$

is achieved [see Eqs. (13) and (16)]. Using the asymptotic values of $|M(w,t)|^2$ and $|N(w,t)|^2$, we obtain from Eq. (14)

$$I_{\mathrm{a}}^{(\lambda)}(t\to\infty) = \frac{\alpha_\lambda \gamma_\lambda^2}{\pi}\int_0^\infty dw \frac{w}{\gamma_\lambda^2 + w^2}\{[\omega + w]^2 n^{(\lambda)}(w) + [\omega - w]^2[1 + \varepsilon_\lambda n^{(\lambda)}(w)]\} \times \frac{\prod_{\mu=1}^{N_b}(\gamma_\mu^2 + w^2)}{\prod_{k=1}^{N_0}(s_k^2 + w^2)}. \quad (20)$$

Here, s_k $(k = 1, \ldots, N_0)$ are the roots of of the $N_0 = N_b + 2$-order polynomial

$$\left[s^2 + \omega^2 - 2\omega \sum_{\lambda=1}^{N_b} \alpha_\lambda \gamma_\lambda^2/(s + \gamma_\lambda)\right] \prod_{\mu=1}^{N_b}(s + \gamma_\mu) = 0. \quad (21)$$

The specific quantum nature of the baths enter into the diffusion coefficient through the appearance of occupation probabilities. The asymptotic diffusion and friction coefficients are related by the well-known fluctuation-dissipation relations connecting diffusion and damping constants. Fulfillment of the fluctuation-dissipation relations means that we have correctly defined the dissipative kernels in the non-Markovian equations of motion.

In the Markovian limit, the asymptotic occupation number is [29]

$$n_{\mathrm{a}}(\infty) = \frac{1}{g_0}\sum_\lambda \alpha_\lambda n^{(\lambda)}(\omega),$$

where $g_0 = \sum_\lambda \alpha_\lambda$. In an asymptotic equilibrium, the occupation number differs from the Fermi-Dirac or Bose-Einstein occupation number if the heat baths have different temperatures. If $T_\lambda = T$, then $n^{(\lambda)}(\omega) = n_{\mathrm{a}}^{(\mathrm{eq})}(\omega) = (\exp[\hbar\omega/(kT)] - \varepsilon_{\mathrm{a}})^{-1}$ and

$$n_{\mathrm{a}}(\infty) = n_{\mathrm{a}}^{(\mathrm{eq})}(\omega)$$

has the usual form of Fermi-Dirac or Bose-Einstein distribution (a thermal equilibrium) [29].

Rewriting Eq. (16) as

$$\begin{aligned}\frac{d}{dt}n_{\mathrm{a}}(t) &= \int_0^t d\tau \frac{d}{d\tau}\{-2\lambda_{\mathrm{a}}(\tau)n_{\mathrm{a}}(\tau) + 2D_{\mathrm{a}}(\tau)\} \\ &= \int_0^t d\tau\{[4\lambda_{\mathrm{a}}^2(\tau) - 2\dot{\lambda}_{\mathrm{a}}(\tau)]n_{\mathrm{a}}(\tau) + 2\dot{D}_{\mathrm{a}}(\tau) \\ &\quad - 4\lambda_{\mathrm{a}}(\tau)D_{\mathrm{a}}(\tau)\}, \end{aligned} \quad (22)$$

and comparing with Eq. (10), we obtain the following relationships:

$$\begin{aligned} W &= 2\dot{\lambda}_{\mathrm{a}}(t) - 4\lambda_{\mathrm{a}}(t)\lambda_{\mathrm{a}}(t), \\ W_+ &= 2\dot{D}_{\mathrm{a}}(t) - 4\lambda_{\mathrm{a}}(t)D_{\mathrm{a}}(t). \end{aligned} \quad (23)$$

Here, $\dot{\lambda}_{\mathrm{a}}(t) = \frac{d\lambda_{\mathrm{a}}(t)}{dt}$ and $\dot{D}_{\mathrm{a}}(t) = \frac{dD_{\mathrm{a}}(t)}{dt}$.

D. Fermionic (Bosonic) oscillator coupled with bosonic (fermionic) heat baths

Let us consider the case when the N_b heat baths are bosonic (fermionic) and the system oscillator is fermionic (bosonic). By the analogy with the fermion or boson system, one can introduce an equation similar to Eq. (16),

$$\begin{aligned}\frac{d}{dt}n_{\mathrm{a}}(t) &= -2\tilde{\lambda}(t)n_{\mathrm{a}}(t) + 2\tilde{D}(t) \\ &= \int_0^t d\tau\{[4\tilde{\lambda}^2(\tau) - 2\dot{\tilde{\lambda}}(\tau)]n_{\mathrm{a}}(\tau) + 2\dot{\tilde{D}}(\tau) \\ &\quad - 4\tilde{\lambda}(\tau)\tilde{D}(\tau)\},\end{aligned} \quad (24)$$

and compare it to the master equation Eq. (10). As a result, we obtain the following relationships between the time-dependent coefficients of these equations:

$$\begin{aligned} W - 2\varepsilon_{\mathrm{a}}W_+ &= 2\dot{\tilde{\lambda}}(t) - 4\tilde{\lambda}(t)\tilde{\lambda}(t), \\ W_+ &= 2\dot{\tilde{D}}(t) - 4\tilde{\lambda}(t)\tilde{D}(t). \end{aligned} \quad (25)$$

To respect Eqs. (23) for W and W_+, the friction and diffusion coefficients have to be taken as

$$\begin{aligned} \tilde{\lambda}(t) &= \lambda_{\bar{\mathrm{a}}}(t) - 2\varepsilon_{\mathrm{a}}D_{\bar{\mathrm{a}}}(t), \\ \tilde{D}(t) &= D_{\bar{\mathrm{a}}}(t), \end{aligned} \quad (26)$$

where if $\mathrm{a} = \mathrm{f}$ ($\mathrm{a} = \mathrm{b}$), then $\bar{\mathrm{a}} = \mathrm{b}$ ($\bar{\mathrm{a}} = \mathrm{f}$). Here, the $\lambda_{\bar{\mathrm{a}}}(t)$ and $D_{\bar{\mathrm{a}}}(t)$ are calculated with Eqs. (17) and (18) for the bosonic or fermionic system. Using Eqs. (25) and (26), we obtain

$$\begin{aligned} W &= 2\dot{\lambda}_{\bar{\mathrm{a}}}(t) - 4\tilde{\lambda}(t)\lambda_{\bar{\mathrm{a}}}(t), \\ W_+ &= 2\dot{D}_{\bar{\mathrm{a}}}(t) - 4\tilde{\lambda}(t)D_{\bar{\mathrm{a}}}(t). \end{aligned} \quad (27)$$

Note that the friction coefficient $\tilde{\lambda}(t)$ depends on temperature through $D_{\bar{\mathrm{a}}}(t)$. This dependence also occurs in the case of nonlinear coupling between the system and thermal baths when they are fermionic or bosonic [2].

Using $\tilde{D}(t)$ and $\tilde{\lambda}(t)$ from Eqs. (26) and the solution

$$n_{\mathrm{a}}(t) = e^{-2\int_0^t d\tau\tilde{\lambda}(\tau)}\left\{n_{\mathrm{a}}(0) + 2\int_0^t d\tau \tilde{D}(\tau)e^{2\int_0^\tau d\tau'\tilde{\lambda}(\tau')}\right\} \quad (28)$$

of Eq. (24), one can numerically calculate the time-dependent occupation number of the quantum system.

If the temperatures and bandwidths of all baths are the same, then one can show that the occupation number evolution is equivalent to the one for a system coupled to one bath with an effective coupling coupling strength $g_0 = \sum_\lambda \alpha_\lambda$.

Using Eqs. (18) and (26), and the fact that $B(\infty) = 0$, the asymptotic occupation number is obtained as

$$n_{\rm a}(\infty) = \frac{\sum_\lambda I_{\bar{\rm a}}^{(\lambda)}(\infty)}{1 - 2\varepsilon_{\rm a}\sum_\lambda I_{\bar{\rm a}}^{(\lambda)}(\infty)}. \tag{29}$$

So, if all reservoirs have the same quantum nature [ā = b(f)], which differs from the one of the system [a = f(b)], then the asymptotic occupation number is always stationary. In the Markovian limit, we obtain

$$n_{\rm a}(\infty) = \frac{\sum_\lambda \alpha_\lambda n^{(\lambda)}(\omega)}{g_0 - 2\varepsilon_{\rm a}\sum_\lambda \alpha_\lambda n^{(\lambda)}(\omega)}. \tag{30}$$

If $T_\lambda = T$, then $n^{(\lambda)}(\omega) = n_{\bar{\rm a}}^{({\rm eq})}(\omega)$ and the asymptotic occupation number

$$n_{\rm a}(\infty) = \frac{n_{\bar{\rm a}}^{({\rm eq})}(\omega)}{1 - 2\varepsilon_{\rm a} n_{\bar{\rm a}}^{({\rm eq})}(\omega)} = n_{\rm a}^{({\rm eq})}(\omega) \tag{31}$$

is the subject to the Fermi-Dirac or Bose-Einstein distribution.

E. Bosonic (Fermionic) oscillator coupled with mixed fermionic (bosonic) and bosonic (fermionic) heat baths

Let us consider the system where the oscillator is bosonic or fermionic, the N_b^f heat baths are fermionic, and N_b^b heat baths are bosonic subsystems ($N_b = N_b^f + N_b^b$), respectively. In this case we obtain the following relationships between the time-dependent coefficients:

$$W - 2\varepsilon_{\rm a}\sum_{\lambda=1}^{N_b^{\bar{\rm a}}} W_+^{(\lambda)} = 2\dot{\tilde{\lambda}}(t) - 4\tilde{\lambda}(t)\tilde{\lambda}(t),$$
$$\sum_{\lambda=1}^{N_b^{\bar{\rm a}}} W_+^{(\lambda)} + \sum_{\lambda=N_b^{\bar{\rm a}}+1}^{N_b} W_+^{(\lambda)} = 2\dot{\tilde{D}}(t) - 4\tilde{\lambda}(t)\tilde{D}(t). \tag{32}$$

One can similarly to the systems discussed above derive the friction and diffusion coefficients

$$\tilde{\lambda}(t) = \lambda_0(t) - 2\varepsilon_{\rm a}\sum_{\lambda=1}^{N_b^{\bar{\rm a}}} D_{\bar{\rm a}}^{(\lambda)}(t),$$
$$\lambda_0(t) = p\lambda_{\bar{\rm a}}(t) + (1-p)\lambda_{\rm a}(t),$$
$$\tilde{D}(t) = \sum_{\lambda=1}^{N_b^{\bar{\rm a}}} D_{\bar{\rm a}}^{(\lambda)}(t) + \sum_{\lambda=N_b^{\bar{\rm a}}+1}^{N_b} D_{\rm a}^{(\lambda)}(t), \tag{33}$$

where $p = \sum_{\lambda=1}^{N_b^{\bar{\rm a}}} \alpha_\lambda / \sum_{\lambda=1}^{N_b} \alpha_\lambda$. The $\lambda_{\rm a}(t)$, $\lambda_{\bar{\rm a}}(t)$, $D_{\rm a}^{(\lambda)}(t)$, and $D_{\bar{\rm a}}^{(\lambda)}(t)$ are calculated with Eqs. (17) and (18). The friction $\lambda_0(t)$ is defined from the condition to obtain the Bose-Einstein or Fermi-Dirac distribution in the Markovian limit. Using Eqs. (33), we obtain

$$W = 2\dot{\lambda}_0(t) - 4\tilde{\lambda}(t)\lambda_0(t),$$
$$\sum_{\lambda=1}^{N_b^{\bar{\rm a}}} W_+^{(\lambda)} = \sum_{\lambda=1}^{N_b^{\bar{\rm a}}}\left[2\dot{D}_{\bar{\rm a}}^{(\lambda)}(t) - 4\tilde{\lambda}(t)D_{\bar{\rm a}}^{(\lambda)}(t)\right],$$
$$\sum_{\lambda=N_b^{\bar{\rm a}}+1}^{N_b} W_+^{(\lambda)} = \sum_{\lambda=N_b^{\bar{\rm a}}+1}^{N_b}\left[2\dot{D}_{\rm a}^{(\lambda)}(t) - 4\tilde{\lambda}(t)D_{\rm a}^{(\lambda)}(t)\right]. \tag{34}$$

In contrast to the systems discussed above, there is limitation on the set of coupling strengths, bandwidths, and temperatures, because the friction coefficient $\lambda_{\rm b}(t)$ does not converge to a stationary value as $t \to +\infty$. An asymptotic stationary value of occupation number can only be reached if

$$\frac{1}{1-p}\sum_{\lambda=N_b^{\bar{\rm a}}+1}^{N_b} I_{\rm a}^{(\lambda)}(\infty) = \frac{\frac{1}{p}\sum_{\lambda=1}^{N_b^{\bar{\rm a}}} I_{\bar{\rm a}}^{(\lambda)}(\infty)}{1 - \frac{2\varepsilon_{\rm a}}{p}\sum_{\lambda=1}^{N_b^{\bar{\rm a}}} I_{\bar{\rm a}}^{(\lambda)}(\infty)}. \tag{35}$$

In the case $\alpha_1 = \alpha_2 = \ldots = \alpha_{N_b^{\bar{\rm a}}} = \alpha_{\bar{\rm a}}$ and $\alpha_{N_b^{\bar{\rm a}}+1} = \alpha_{N_b^{\bar{\rm a}}+2} = \ldots = \alpha_{N_a} = \alpha_{\rm a}$, the conditions Eq. (35) result in the same bandwidths $\gamma_\lambda = \gamma$ and temperatures $T_\lambda = T$, and relate the values of $\alpha_{\bar{\rm a}}$ and $\alpha_{\rm a}$. Note that the possible absence of asymptotic stationary limit is not seen if we make the RWA couplings $H_{c,\lambda} = \sum_i \alpha_{\lambda,i}({\rm a}^\dagger c_{\lambda,i} + {\rm a}c_{\lambda,i}^\dagger)$ between the system and heat baths ($W_{\lambda,i}^+ = 0$), because both friction coefficients $\lambda_{\rm f,b}(t)$ have the asymptotic stationary values.

In fusion of two nuclei, we deal with the bosonic collective system coupled with the internal fermionic bath. If the fusion occurs in the star, then there is additional bosonic baths. As follows from our results, these bosonic baths can considerably modify the fusion process. So, the results on the sub-barrier fusion obtained with the heavy-ion accelerator can not definitely related to the nuclear fusion in the star.

In the Markovian weak-coupling, the condition Eq. (35) is satisfied and the system has an asymptotic equilibrium. In particular, the asymptotic occupation number might differ from the Fermi-Dirac or Bose-Einstein occupation number if the heat baths have different temperatures. If the temperatures $T_\lambda = T$ of all baths are the same, then $n_{\rm a}(\infty) = n_{\rm a}^{({\rm eq})}(\omega)$ and the system has a thermal equilibrium (the Bose-Einstein or Fermi-Dirac distribution).

III. CALCULATED RESULTS

One of the examples of collective motion with several heat baths is the fusion of nuclei in a stellar medium or in some external field. The effects of the nucleon and field degrees of freedom can be effectively modeled with appropriate heat baths. In this paper we use the realistic frequencies and friction coefficients for the case of fusion of atomic nuclei along the coordinate of relative distance between the centers of mass of nuclei. Because all calculated results are shown in dimensionless units, they can be generalized and applied to other processes and systems. The calculations are performed for the systems with several heat baths in the case of Ohmic dissipation with Lorenzian cutoffs.

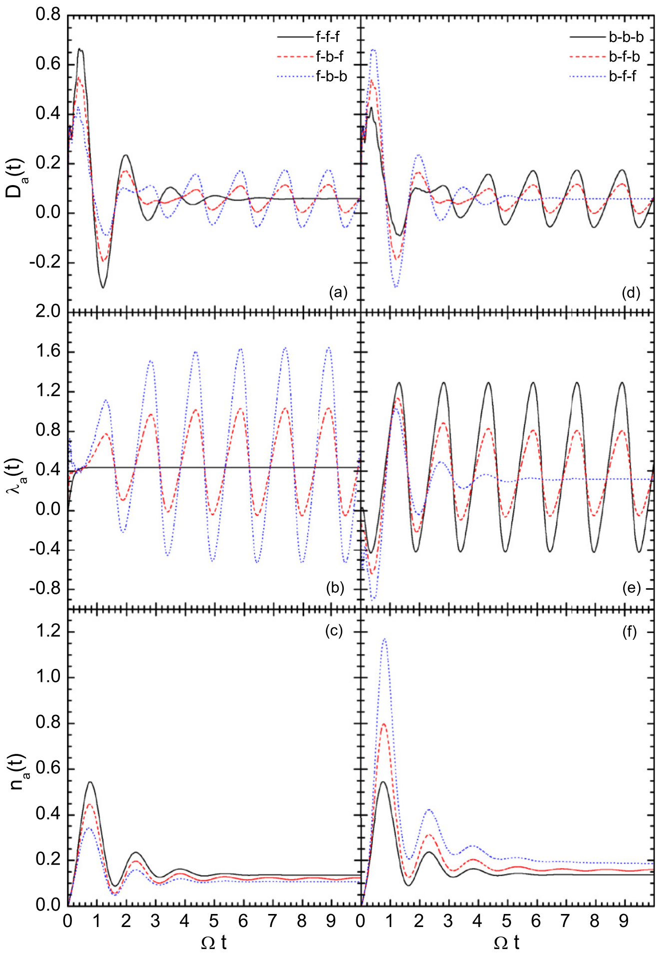

FIG. 1. The calculated time-dependent diffusion (a, d) and friction (b, e) coefficients and average occupation numbers (c, f) for the systems f-f-f (solid lines), f-b-b (dotted lines), f-b-f (dashed lines) (a–c, respectively) and b-b-b (solid lines), b-f-f (dotted lines), b-f-b (dashed lines) (d–f, respectively) at coupling constants $\alpha_1 = \alpha_2 = 0.05$, inverse memory times $\gamma_1/\Omega = 20$ and $\gamma_2/\Omega = 12$, and temperatures $kT_1/(\hbar\Omega) = kT_2/(\hbar\Omega) = 0.1$. The plots in panels (c) and (f) correspond to initially unoccupied, $n_a(t = 0) = 0$, oscillator states.

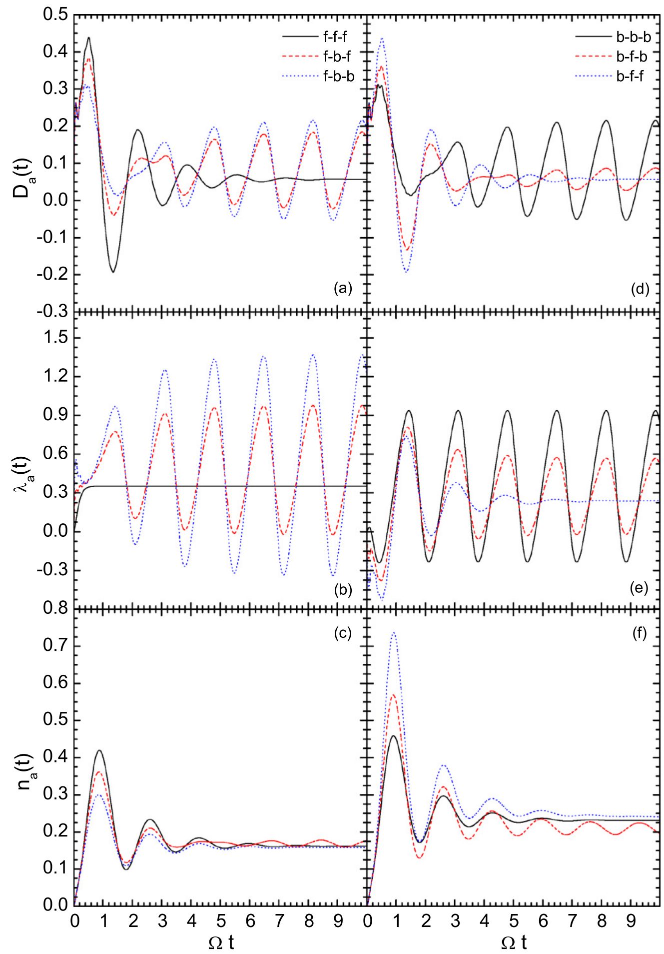

FIG. 2. The same as in Fig. 1, but for the coupling constants $\alpha_1 = \alpha_2 = 0.05$, inverse memory times $\gamma_1/\Omega = \gamma_2/\Omega = 12$, and temperatures $kT_1/(\hbar\Omega) = 1$ and $kT_2/(\hbar\Omega) = 0.1$.

A. Fermionic or bosonic oscillator coupled with two heat baths

The occupation numbers, diffusion, and friction coefficients depend on the values of ω, α_1, α_2, and $\gamma_{1,2}$ (see Appendix B). The values of α_1 and α_2 are chosen to have the realistic values of friction coefficients which are known from the microscopic calculations. Indeed, these coupling strengths provide almost the same friction coefficient for relative motion of two nuclei like in Refs. [32,33]. The values of $\gamma_{1,2}$ should be taken to hold the conditions $\gamma_{1,2} \gg \Omega$. We set $\gamma_{1,2}/\Omega \geqslant 12$. As shown in Ref. [29], in the case of $g_0 = \alpha_1 + \alpha_2$, $\gamma_1 = \gamma_2$, and $T_1 = T_2$, the fermionic and bosonic systems with two baths of the same statistics is reduced to the system with one bath and the coupling strength g_0. As an example of bosonic system, the atomic or nuclear molecular state can be considered. The bound or quasibound particle (electron in the trap or nucleon in the isomeric state) can be taken as an example of fermionic system. The electromagnetic field and phonon bath can be treated as the bosonic baths. Free and bound electrons and impurities in sample can act as the fermionic baths.

1. Time-dependent diffusion and friction coefficients

For the fermionic-fermionic-fermionic (f-f-f), bosonic-bosonic-bosonic (b-b-b), the mixed fermionic-bosonic-bosonic (f-b-b), bosonic-fermionic-fermionic (b-f-f), fermionic-bosonic-fermionic (f-b-f), and bosonic-fermionic-bosonic (b-f-b) systems, the time-dependent friction and diffusion coefficients are shown in Fig. 1 at the same coupling strengths $\alpha_1 = \alpha_2$ and temperatures $T_1 = T_2$ but different inverse memory times $\gamma_{1,2}$. The two independent baths modify the friction and diffusion coefficients in nonadditive manner. The diffusion and friction coefficients are equal to zero at initial time. As seen, the time dependencies of these coefficients are not the same for the fermionic and bosonic heat baths. For the f-f-f and b-f-f systems, the friction $\lambda_a(t)$ and diffusion $D_a(t)$ coefficients relatively fast reach their asymptotic values (the transient time for the friction is quite short), whereas in the case of b-b-b, f-b-b, f-b-f, and b-f-b systems they oscillate. This means that the quasibound electron coupled with the electron gas and inclusions in the compound is almost Markovian system. The coupling with

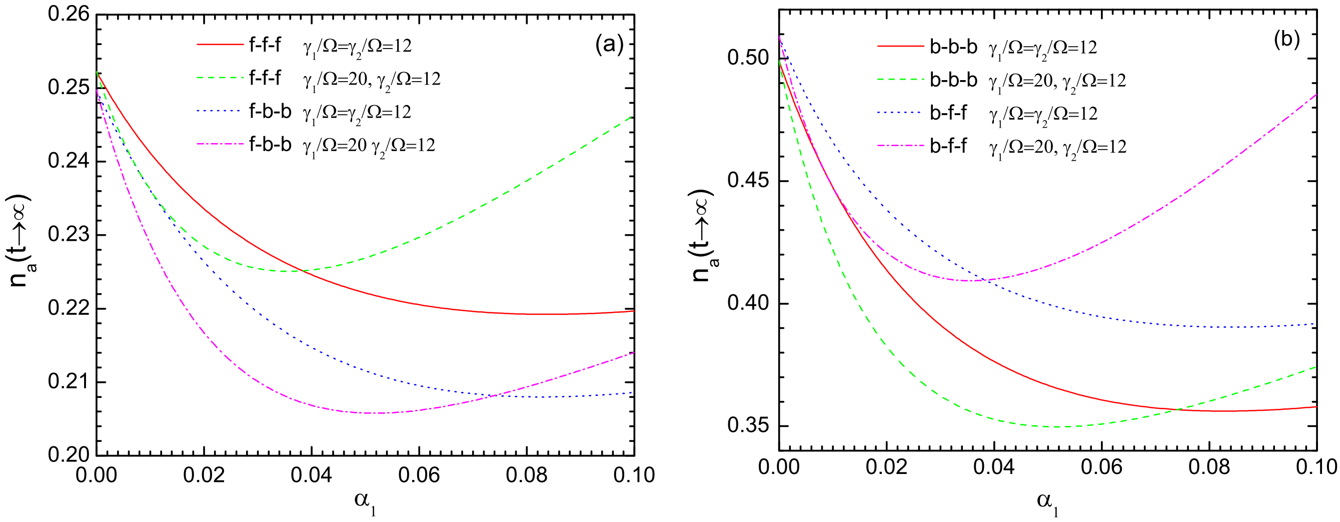

FIG. 3. For the f-f-f, mixed f-b-b [(a)] and the b-b-b, mixed b-f-f [(b)] systems, the calculated dependencies of the asymptotic occupation numbers on coupling constant α_1 at $\alpha_2 = 0.01$ and $kT_{1,2}/(\hbar\Omega) = 1$. The calculations are performed at $\gamma_1/\Omega = \gamma_2/\Omega = 12$ and $\gamma_1/\Omega = 12$, $\gamma_2/\Omega = 20$.

phonon field would elongate a time-dependence of friction and diffusion coefficients and result in non-Markovian behavior. The amplitudes of oscillations of $\lambda_a(t)$ and $D_a(t)$ for the systems with two bosonic baths ($\gamma_1 \neq \gamma_2$) are larger than those for the systems with one bosonic bath ($\gamma_1 = \gamma_2$). As a result, the occupation number oscillates with larger amplitude in the case of two bosonic (fermionic) baths (Fig. 1) than one bosonic (fermionic) bath. The relaxation times of the systems with two bosonic (fermionic) baths and one bosonic (fermionic) bath are almost identical and mainly depend on the coupling constants. The molecular state coupled with phonon and temperature baths would demonstrate more oscillations and non-Markovian behavior than that coupled with the electron gas. A good example is an electron cooling of heavy-ion beam that is effective tool for reducing collective fluctuations in the beam. The coupling of relative motion of two colliding nuclei with their internal degrees of freedom (fermionic bath) creates strong dissipation. As known [34], this process is well described by the phenomenological Markovian diffusion equations.

For the f-f-f, f-b-b, f-b-f and b-b-b, b-f-f, b-f-b systems, the time dependencies of the friction and diffusion coefficients are shown in Fig. 2 at the same coupling strengths $\alpha_1 = \alpha_2$ and inverse memory times $\gamma_1 = \gamma_2$ but different temperatures $T_{1,2}$ ($T_1 > T_2$). As seen, in general, the behavior of friction and diffusion coefficients is similar to the behavior of these coefficients in Fig. 1. For example, the amplitudes of oscillations of the friction and diffusion coefficients for the systems with two bosonic baths are larger than those for other systems. Compared to a system with one bosonic bath ($T = T_1$), a system with two bosonic baths ($T_1 > T_2$) has small amplitudes of oscillations of the $\lambda_a(t)$ and $D_a(t)$. So, the non-Markovian

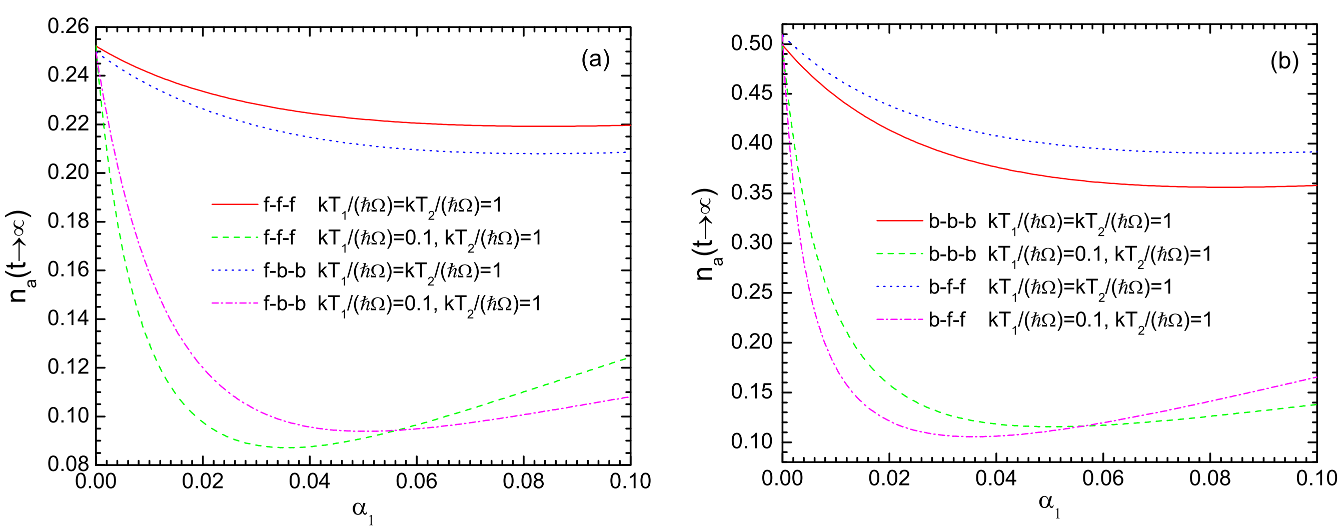

FIG. 4. For the f-f-f, mixed f-b-b [(a)] and the b-b-b, mixed b-f-f [(b)] systems, the calculated dependencies of the asymptotic occupation numbers on coupling constant α_1 at $\alpha_2 = 0.01$ and $\gamma_1/\Omega = \gamma_2/\Omega = 12$. The calculations are performed at temperatures $kT_1/(\hbar\Omega) = kT_2/(\hbar\Omega) = 1$ and $kT_1/(\hbar\Omega)$=0.1, $kT_2/(\hbar\Omega)$=1.

features can be reduced by adding the bosonic bath of other temperature.

2. Time-dependent occupation numbers

For the f-f-f, f-b-b, b-b-b, and b-f-f systems, the friction and diffusion coefficients oscillate in phase at large times and, as a result, the occupation numbers $n_a(t)$ reach asymptotic limits after oscillations during the transient time (Figs. 1 and 2). In the contrast, for the systems f-b-f and b-f-b, where fermionic bath coexists with bosonic one, the occupation numbers oscillate around certain average values at large times. Because the condition Eq. (35) is not satisfied, the systems f-b-f and b-f-b have no asymptotic limits. In this case, at large times the influence of the thermostats is minimal and reversible—it takes energy from the system and gives the same amount of energy back. As a result, the population of the excited state(s) decreases and then increases on the same level independent of the environment. In the case of two mixed baths with ($T_1 \neq T_2$, $\gamma_1 = \gamma_2$), the occupation number for the fermionic bosonic oscillator oscillates stronger with the larger amplitude than one for the fermionic oscillator.

As seen in Figs. 1 and 2, the time evolution of occupation numbers is affected by the nature of the bathes and oscillator. Two independent non-Markovian baths modify in nonadditive manner the dynamics of system. If the systems have the same characteristics, then n_b(b-f-f)$>$ n_f(f-f-f)$>$ n_f(f-b-b) and n_b(b-f-f)$>$ n_b(b-b-b)$>$ n_f(f-b-b) at large time. So, the fermionic (bosonic) oscillator interacting with two fermionic (bosonic) baths has larger (smaller) value of the occupation number than that interacting with bosonic-bosonic (fermionic-fermionic) baths. The occupation number at large time depends on the friction which is related to the energy exchange rate between the system and baths. In the f-b-b case, the value of $\lambda_f(t)$ in average larger than that in the f-f-f case. The larger friction results in smaller occupation number. In the b-b-b system, λ_b oscillates around average value which is larger than λ_b in the b-f-f case. So, n_b(b-f-f) $>$ n_b(b-b-b). At the same coupling strength between the system and heat baths the bosonic baths result in larger friction. For example, if the molecular fermionic state is coupled with two fermionic baths, then the initially unoccupied state is more populated then in the case of coupling with two phonon baths.

For the b-b-b (f-f-f) system with two bosonic (fermionic) baths ($\gamma_1 \neq \gamma_2$, $T_1 = T_2$), the occupation number at large time is close to that for the b-b (f-f) system with one bosonic (fermionic) bath ($\gamma_1 = \gamma_2$, $T_1 = T_2$, $g_0 = \alpha_1 + \alpha_2$). In the case of two bosonic (fermionic) baths with ($T_1 \neq T_2$, $\gamma_1 = \gamma_2$), the occupation number n_b(b-b-b) [n_f(f-f-f)] at large time is smaller than that for the b-b (f-f) system with one bosonic (fermionic) bath. The difference of the bath temperatures is important for the dynamics because causes the energy exchange through the oscillator. One can use the baths of different temperatures to affect the occupation numbers. The occupation number is less sensitive to the structure of the additional bath.

3. Asymptotic occupation numbers

In Fig. 3, the dependencies of asymptotic occupation numbers on the coupling strength α_1 for the f-f-f, f-b-b and b-b-b, b-f-f systems are shown at different $\gamma_{1,2}$, fixed $\alpha_2 = 0.01$, and $kT_1/(\hbar\Omega) = kT_2/(\hbar\Omega) = 1$. As seen, the change of occupation number with α_1 is stronger for the bosonic systems than for the fermionic ones. In the case of $\gamma_1 = \gamma_2$ (one thermal bath with coupling strength $g_0 = \alpha_1 + \alpha_2$), the asymptotic values of n_f(f-f-f), n_f(f-b-b), n_b(b-b-b), and n_b(b-f-f) monotonically decrease with increasing α_1. In the case of $\gamma_1 \neq \gamma_2$, their dependencies on α_1 are more complicated. The values of n_f(f-f-f) [n_b(b-f-f)] and n_f(f-b-b) [n_b(b-b-b)] decrease with increasing α_1 up to $\alpha_1 \approx 0.03$ and $\alpha_1 \approx 0.05$, respectively, and then start to increase. This behavior of occupation number is due to the contributions of the terms $I^{(1),(2)}_{b,f}(t \to \infty)$ which have different dependencies on α_1. Note that in the case of $\gamma_1 \neq \gamma_2$, the asymptotic occupation numbers satisfy the following inequalities: n_f(f-f-f)$>$ n_f(f-b-b) and n_b(b-b-b)$<$ n_b(b-f-f). For

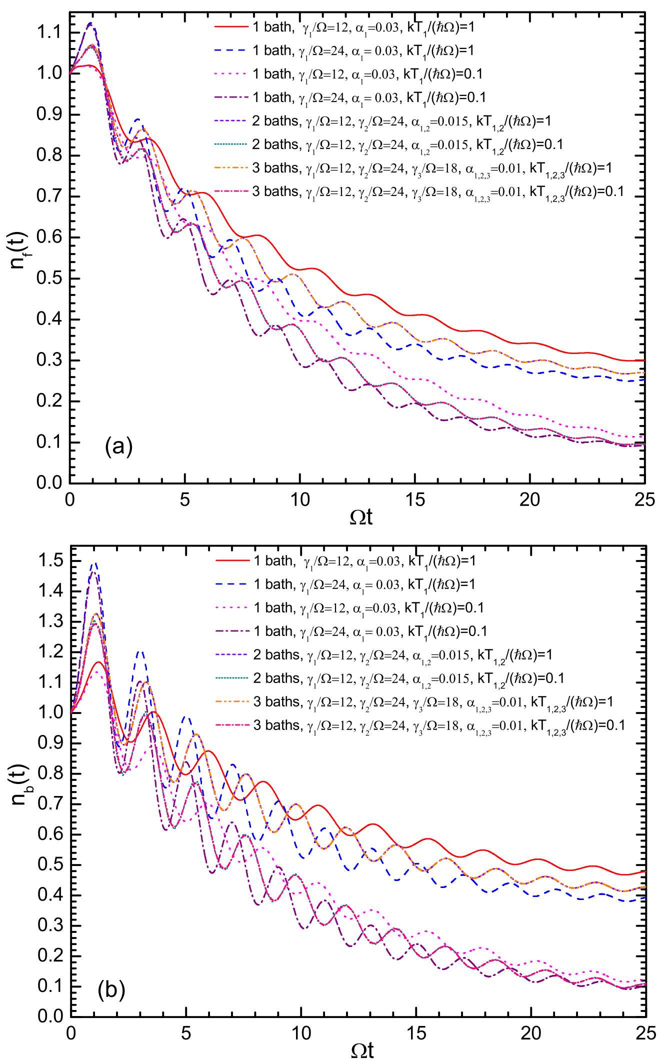

FIG. 5. For the f-f ($\alpha_1 = 0.03$, γ_1/Ω =12 or 24), f-f-f ($\alpha_{1,2} = 0.015$, γ_1/Ω =12, γ_2/Ω =24), f-f-f-f ($\alpha_{1,2,3} = 0.01$, γ_1/Ω =12, γ_2/Ω =24, γ_3/Ω =18) (a) and b-b ($\alpha_1 = 0.03$, γ_1/Ω =12 or 24), b-b-b ($\alpha_{1,2} = 0.015$, γ_1/Ω =12, γ_2/Ω =24), b-b-b-b ($\alpha_{1,2,3} = 0.01$, γ_1/Ω =12, γ_2/Ω =24, γ_3/Ω =18) (b) systems, the calculated dependencies of average occupation numbers on time t at total coupling constant $g_0 = 0.03$ and temperatures $kT_{1,2,3}/(\hbar\Omega)$ =0.1 and 1. In the plots in panels (a) and (b), the occupation numbers for the systems with two and three baths almost coincide. The plots correspond to initially occupied, $n_a(t = 0)$=1, oscillator state.

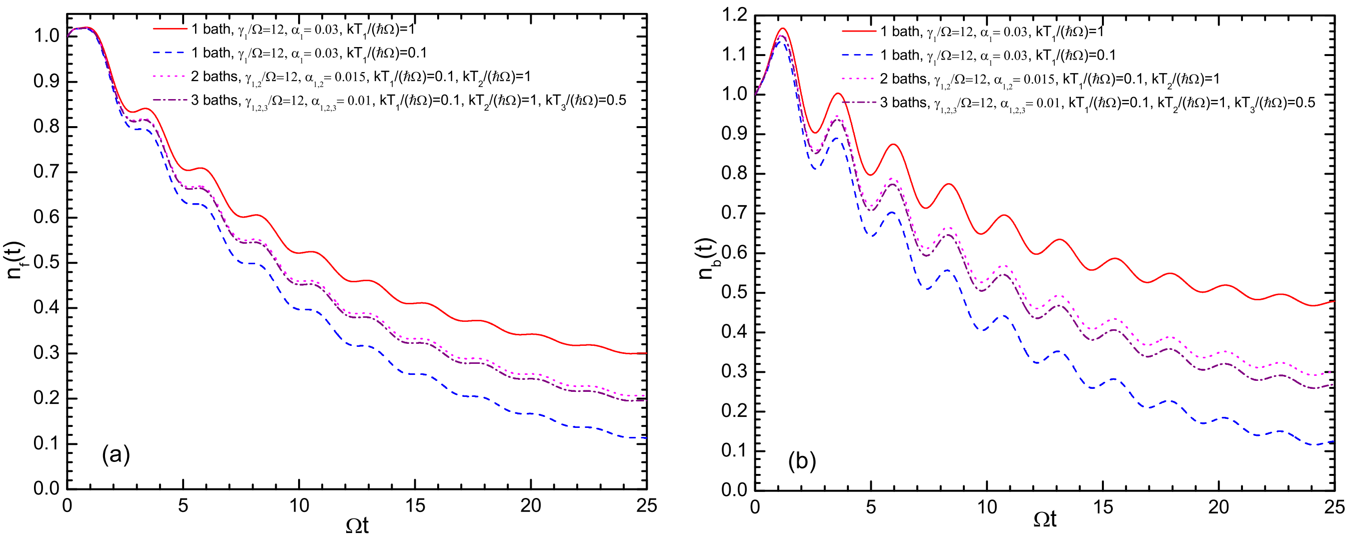

FIG. 6. For the f-f ($\alpha_1 = 0.03$, $kT_1/(\hbar\Omega)$ =0.1 or 1), f-f-f ($\alpha_{1,2} = 0.015$, $kT_1/(\hbar\Omega)$ =0.1, $kT_2/(\hbar\Omega)$ =1), f-f-f-f ($\alpha_{1,2,3} = 0.01$, $kT_1/(\hbar\Omega)$ =0.1, $kT_2/(\hbar\Omega)$ =1, $kT_3/(\hbar\Omega)$ =0.5) (a) and b-b ($\alpha_1 = 0.03$, $kT_1/(\hbar\Omega)$ =0.1 or 1), b-b-b ($\alpha_{1,2} = 0.015$, $kT_1/(\hbar\Omega)$ =0.1, $kT_2/(\hbar\Omega)$ =1), b-b-b-b ($\alpha_{1,2,3} = 0.01$, $kT_1/(\hbar\Omega)$ =0.1, $kT_2/(\hbar\Omega)$ =1, $kT_3/(\hbar\Omega)$ =0.5) (b) systems, the calculated dependencies of average occupation numbers on time t at total coupling constant $g_0 = 0.03$ and inverse memory times $\gamma_{1,2,3}/\Omega$ =12. The plots correspond to initially occupied, $n_a(t = 0)$=1, oscillator state.

the f-f-f [Fig. 3(a), dashed line] and b-f-f [Fig. 3(b), dash-dotted line] systems, at α_1 approaching 1 the values of $n_a(\infty)$ approaches those corresponding to $\alpha_1 = 0$. So, there are cases in which the second bath weakly influences the asymptotic occupation number even at large coupling strength.

As seen in Fig. 4, with increasing coupling strength α_1 of one bath at fixed α_2, $\gamma_1 = \gamma_2$, and $T_1 \neq T_2$, the calculated fermionic and bosonic asymptotic occupation numbers first decrease and then increase at $\alpha_1 \geqslant 0.03$–0.06. So, the contributions of increasing and decreasing terms $I_{b,f}^{(1),(2)}(t \to \infty)$ provide the complicated dependence of the asymptotic occupation number on the coupling strength of one of the baths. As seen, adding the second bath of the same statistics and structure but other temperature, one can cause a larger change of asymptotic occupation number. This provides the sensitive control of the bosonic system by the electromagnetic field considered as the cold bath. The electron in the trap can be controlled by introducing some inclusions in its surrounding. Indeed, these inclusions can be considered as the fermion bath of small temperature.

B. Fermionic (Bosonic) oscillator coupled with three fermionic (bosonic) heat baths

For the fermionic-fermionic-fermionic-fermionic (f-f-f-f) and bosonic-bosonic-bosonic-bosonic (b-b-b-b) systems, the time-dependent fermionic and bosonic occupation numbers are calculated at the same coupling strengths $\alpha_1 = \alpha_2 = \alpha_3$ and temperatures $T_1 = T_2 = T_3$ but different inverse memory times $\gamma_{1,2,3}$ (Fig. 5) and at the same coupling strengths $\alpha_1 = \alpha_2 = \alpha_3$ and inverse memory times $\gamma_1 = \gamma_2 = \gamma_3$ but different temperatures $T_{1,2,3}$ (Fig. 6). In the cases of bosonic and fermionic systems with one, two, and three heat baths, the occupation numbers oscillate and reach their asymptotic values with decreasing the amplitude of oscillations. The values of $n_a(t)$ in the bosonic systems oscillate with a larger amplitude than those in the case fermionic systems. However, the relaxation times in these systems are almost independent of the number of heat baths. For both statistics, the results in Figs. 5 and 6 clearly show that the values of $n_a(t)$ of the systems with two and three heat baths and with the same total coupling strengths $g_0 = \sum_\lambda \alpha_\lambda$ are very close to each other. For the bosonic (fermionic) systems with two and three heat baths, the values of occupation number at large times are between the values of occupation number of the bosonic (fermionic) systems with one bath at the minimum and maximum γ_λ (Fig. 5) or T_λ (Fig. 6).

IV. CONCLUSIONS

For the bosonic or fermionic FC-oscillator and internal independent bosonic or fermionic or mixed bosonic-fermionic heat baths coupled linearly (full coupling, the Ohmic dissipation with Lorenzian cutoff) to the system, the analytical expressions for the occupation number were derived under the physical assumption that at initial time $t = 0$ the heat baths are in thermal equilibrium in the absence of the quantum system. We proposed here the approach to treat open quantum system coupled with several heat baths of different quantum statistics. The asymptotes of the fermionic and bosonic occupation numbers were derived. The time-dependent transport coefficients of the master equations for the occupation number were obtained as well.

The results of illustrative numerical calculations of diffusion and friction coefficients and level populations were presented. As shown, the dissipative and diffusion aspects depend on the statistics of the independent heat baths. After some transient time, the friction and diffusion coefficients of the systems coupled with fermionic baths reach asymptotic values. For the systems with bosonic baths, the friction and diffusion coefficients oscillate in time. The coupling of system with bosonic baths would elongate a time-dependence of friction and diffusion coefficients and result in non-Markovian behavior. So, the non-Markovian features can be changed by manipulating with several baths. The independent baths modify the friction and diffusion coefficients in nonadditive manner. We found that there is energy exchange between the baths, especially with different temperatures, through the oscillator that accelerates the energy dissipation. For the systems with two bosonic or fermionic thermal baths, the oscillation of occupation numbers occurs before reaching the asymptotes. The time dependencies of occupation numbers are affected by the nature of the heat baths and oscillator as well as the coupling strengths. The relaxation times of the systems with two and three bosonic (fermionic) baths and one bosonic (fermionic) bath are almost identical and mainly depend on the coupling constants. For the f-f-f and b-f-f systems with $\gamma_1 \neq \gamma_2$ and $T_1 = T_2$, we found that the second bath could weakly affect the asymptotic occupation number even at large coupling strength. Adding the second bath of the same statistics and structure but other temperature, one can cause larger change of asymptotic occupation number. For the bosonic (fermionic) systems with two and three heat baths, the values of asymptotic occupation numbers are between those of the bosonic (fermionic) systems with one bath at the minimum and maximum γ_λ or T_λ. We finally have shown that the independent non-Markovian baths modify in a nonadditive manner the dynamics of system leading to nontrivial asymptotic behavior.

Starting from a system fully coupled to several baths of different statistics, we have illustrated that the system might never reach a stationary asymptotic limit. This absence of equilibrium is at variance with what is usually found in simple quantum open systems coupled to complex environments and are predicted to only happen when fermionic baths coexist with bosonic baths. We found that in the system with mixed statistics baths the occupation number does reach an asymptotic equilibrium limit if the condition Eq. (35) is fulfilled. Note that the possible absence of asymptotic stationary limit is not seen if we make the RWA couplings between the system and heat baths. With the recent progresses of preparing and manipulating open quantum systems, we hope that the absence of equilibrium can be confronted to future experimental probes. The dependence of asymptotic occupation number on time is nonmonotonic. That is, over time, the population of the excited level can decrease and then increase independent of the environment. This behavior can be used, for example, to increase the throughput of a quantum channels (communication line).

ACKNOWLEDGMENTS

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APPENDIX A: TIME-DEPENDENT COEFFICIENTS $A(t)$, $B(t)$, $M(w,t)$, and $N(w,t)$

The time-dependent coefficients $A(t)$, $B(t)$, $M(w,t)$, and $N(w,t)$ used in Eqs. (13) and (14) are obtained in Ref. [29] as

$$\begin{aligned}
A(t) &= \frac{1}{2}\sum_{k=1}^{N_0}\xi_k e^{s_k t}(s_k - s_0)\left\{2s_k - i[\Omega+\omega] - 2is_k\sum_{\lambda=1}^{N_b}\alpha_\lambda\gamma_\lambda/(s_k+\gamma_\lambda)\right\}\prod_{\mu=1}^{N_b}(s_k+\gamma_\mu),\\
&= i\sum_{\lambda=1}^{N_b}\alpha_\lambda\gamma_\lambda^2\sum_{k=1}^{N_0}\xi_k e^{s_k t}(s_k-s_0)(s_k-i\omega)(s_k+i\omega)^{-1}\prod_{\mu=1,\mu\neq\lambda}^{N_b}(s_k+\gamma_\mu),\\
B(t) &= \frac{i}{2}\sum_{k=1}^{N_0}\xi_k e^{s_k t}(s_k - s_0)\left\{\Omega-\omega+2s_k\sum_{\lambda=1}^{N_b}\alpha_\lambda\gamma_\lambda/(s_k+\gamma_\lambda)\right\}\prod_{\mu=1}^{N_b}(s_k+\gamma_\mu),\\
&= i\sum_{\lambda=1}^{N_b}\alpha_\lambda\gamma_\lambda^2\sum_{k=1}^{N_0}\xi_k e^{s_k t}(s_0-s_k)\prod_{\mu=1,\mu\neq\lambda}^{N_b}(s_k+\gamma_\mu),\\
N(w,t) &= \sum_{k=0}^{N_0}\xi_k e^{s_k t}(is_k-\omega)\prod_{\mu=1}^{N_b}(s_k+\gamma_\mu),\\
M(w,t) &= -\sum_{k=0}^{N_0}\xi_k e^{s_k t}(is_k+\omega)\prod_{\mu=1}^{N_b}(s_k+\gamma_\mu),
\end{aligned}\tag{A1}$$

where

$$\xi_k = \prod_{i=0,\, i\neq k}^{N_0}\frac{1}{s_k - s_i},\tag{A2}$$

with $s_0 = -iw$ and the roots s_k, $k = 1, ..., N_0$ of Eq. (21) or

$$\left[s^2 + \omega\Omega + 2s\omega \sum_{\lambda=1}^{N_b} \alpha_\lambda\gamma_\lambda/(s+\gamma_\lambda)\right] \prod_{\mu=1}^{N_b} (s+\gamma_\mu) = 0 \tag{A3}$$

and

$$\Omega = \omega - 2\sum_{\lambda=1}^{N_b} \alpha_\lambda\gamma_\lambda. \tag{A4}$$

APPENDIX B: EXPLICIT EXPRESSIONS FOR FRICTION AND DIFFUSION COEFFICIENTS OF SYSTEMS WITH TWO BATHS

Here, we present the explicit expressions for $A(t)$, $B(t)$, $N(w,t)$, $M(w,t)$, and $I_{\rm a}^{(\lambda)}(t)$ used to calculate $n_{\rm a}(t)$ and $\lambda_{\rm a}(t)$ with Eqs. (13) and (17) in the case of two baths.

1. Bosonic-bosonic-bosonic and fermionic-fermionic-fermonic systems

In the case of fermionic or bosonic system with two heat baths ($\varepsilon_{\rm a} = \varepsilon_1 = \varepsilon_2 = \varepsilon$) [29],

$$I_{\rm a}(t) = I_{\rm a}^{(1)}(t) + I_{\rm a}^{(2)}(t) \tag{B1}$$

and

$$I_{\rm a}^{(1)}(t) = \frac{\alpha_1\gamma_1^2}{\pi}\int_0^\infty dw \frac{w}{\gamma_1^2 + w^2}\{n^{(1)}(w)|M(w,t)|^2 + [1+\varepsilon n^{(1)}(w)]|N(w,t)|^2\}, \tag{B2}$$

$$I_{\rm a}^{(2)}(t) = \frac{\alpha_2\gamma_2^2}{\pi}\int_0^\infty dw \frac{w}{\gamma_2^2 + w^2}\{n^{(2)}(w)|M(w,t)|^2 + [1+\varepsilon n^{(2)}(w)]|N(w,t)|^2\}. \tag{B3}$$

For the time-dependent coefficients $A(t)$, $B(t)$, $M(w,t)$, $N(w,t)$, the following expressions are obtained:

$$\begin{aligned}
A(t) &= \frac{1}{2}\sum_{k=1}^{4} \xi_k e^{s_k t}(s_k - s_0)[(2s_k - i[\omega+\Omega])(s_k+\gamma_1)(s_k+\gamma_2) - 2is_k(\alpha_1\gamma_1(s_k+\gamma_2) + \alpha_2\gamma_2(s_k+\gamma_1))],\\
&= i\sum_{k=1}^{4} \xi_k e^{s_k t}(s_k - s_0)(s_k - i\omega)(s_k + i\omega)^{-1}\big[\alpha_1\gamma_1^2(s_k+\gamma_2) + \alpha_2\gamma_2^2(s_k+\gamma_1)\big],\\
B(t) &= \frac{i}{2}\sum_{k=1}^{4} \xi_k e^{s_k t}(s_k - s_0)[(\Omega-\omega)(s_k+\gamma_1)(s_k+\gamma_2) + 2s_k(\alpha_1\gamma_1(s_k+\gamma_2) + \alpha_2\gamma_2(s_k+\gamma_1))],\\
&= i\sum_{k=1}^{4} \xi_k e^{s_k t}(s_0 - s_k)\big[\alpha_1\gamma_1^2(s_k+\gamma_2) + \alpha_2\gamma_2^2(s_k+\gamma_1)\big],\\
N(w,t) &= \sum_{k=0}^{4} \xi_k e^{s_k t}(is_k - \omega)(s_k+\gamma_1)(s_k+\gamma_2),\\
M(w,t) &= -\sum_{k=0}^{4} \xi_k e^{s_k t}(is_k + \omega)(s_k+\gamma_1)(s_k+\gamma_2),
\end{aligned} \tag{B4}$$

where

$$\xi_k = \prod_{i=0,\, i\neq k}^{4} \frac{1}{s_k - s_i}, \tag{B5}$$

$s_0 = -iw$, and s_1, s_2, s_3, s_4 are the roots of the following equation:

$$(s^2 + \omega\Omega)(s+\gamma_1)(s+\gamma_2) + 2s\omega[\alpha_1\gamma_1(s+\gamma_2) + \alpha_2\gamma_2(s+\gamma_1)] = 0 \tag{B6}$$

or

$$(s^2 + \omega^2)(s+\gamma_1)(s+\gamma_2) - 2\omega\big[\alpha_1\gamma_1^2(s+\gamma_2) + \alpha_2\gamma_2^2(s+\gamma_1)\big] = 0. \tag{B7}$$

In Eq. (B6), $\Omega = \omega - 2\alpha_1\gamma_1 - 2\alpha_2\gamma_2$.

The asymptotic occupation numbers for the bosonic and fermionic systems are defined only by the integral terms $I_{\rm a}^{(1)}(t\to\infty)$ and $I_{\rm a}^{(2)}(t\to\infty)$ [29]:

$$n_{\rm a}(t\to\infty)=I_{\rm a}^{(1)}(t\to\infty)+I_{\rm a}^{(2)}(t\to\infty), \tag{B8}$$

where

$$I_{\rm a}^{(1)}(t\to\infty)=\frac{\alpha_1\gamma_1^2}{\pi}\int_0^\infty dw\frac{w(\gamma_2^2+w^2)\{[\omega+w]^2n^{(1)}(w)+[\omega-w]^2[1+\varepsilon n^{(1)}(w)]\}}{(s_1^2+w^2)(s_2^2+w^2)(s_3^2+w^2)(s_4^2+w^2)}, \tag{B9}$$

$$I_{\rm a}^{(2)}(t\to\infty)=\frac{\alpha_2\gamma_2^2}{\pi}\int_0^\infty dw\frac{w(\gamma_1^2+w^2)\{[\omega+w]^2n^{(2)}(w)+[\omega-w]^2[1+\varepsilon n^{(2)}(w)]\}}{(s_1^2+w^2)(s_2^2+w^2)(s_3^2+w^2)(s_4^2+w^2)}. \tag{B10}$$

So, these expressions are used in Eq. (19). In the Markovian weak-coupling limit, we obtain at $T_1\neq T_2$

$$n_{\rm a}(t\to\infty)=\frac{\alpha_1}{\alpha_1+\alpha_2}n^{(1)}(\omega)+\frac{\alpha_2}{\alpha_1+\alpha_2}n^{(2)}(\omega). \tag{B11}$$

If $T_1=T_2=T$, then $n_{\rm a}(t\to\infty)=n_{\rm a}^{\rm eq}(\omega)$.

In Eqs. (17) and (18) for the time-dependent friction and diffusion coefficients we use expressions [29]:

$$J^{(1)}(t)=|B_1(t)|^2+\tfrac{1}{2}[B_1(t)B_2^*(t)+B_1^*(t)B_2(t)],$$
$$J^{(2)}(t)=|B_2(t)|^2+\tfrac{1}{2}[B_1(t)B_2^*(t)+B_1^*(t)B_2(t)],$$

which result from the decomposition of the coefficient $B(t)$:

$$B(t)=B_1(t)+B_2(t),\quad B_1(t)=i\alpha_1\gamma_1^2\sum_{k=1}^4\xi_ke^{s_kt}(s_0-s_k)(s_k+\gamma_2),$$
$$B_2(t)=i\alpha_2\gamma_2^2\sum_{k=1}^4\xi_ke^{s_kt}(s_0-s_k)(s_k+\gamma_1),\quad |B(t)|^2=J^{(1)}(t)+J^{(2)}(t). \tag{B12}$$

2. Fermionic-bosonic-bosonic system

If two heat baths are bosonic and the oscillator is fermionic, then the friction and diffusion coefficients

$$\tilde\lambda(t)=\lambda_{\rm b}(t)+2D_{\rm b}(t),\quad \tilde D(t)=D_{\rm b}(t), \tag{B13}$$

are derived. Here, the $\lambda_{\rm b}(t)$ and $D_{\rm b}(t)$ are calculated with Eqs. (17) and (18) for the total bosonic system, respectively.

3. Bosonic-fermionic-fermionic system

For the bosonic system with two fermionic heat baths, one can similarly derive the equation for $n_{\rm b}(t)$, where

$$\tilde\lambda(t)=\lambda_{\rm f}(t)-2D_{\rm f}(t),\quad \tilde D(t)=D_{\rm f}(t). \tag{B14}$$

The $\lambda_{\rm f}(t)$ and $D_{\rm f}(t)$ are calculated with Eqs. (17) and (18) for the fermionic system, respectively.

4. Bosonic-fermionic-bosonic system

Let us consider the bosonic-fermionic-bosonic (b-f-b) system, where the quantum oscillator is bosonic, the heat baths "1" and "2" are fermionic and bosonic, respectively. Using Eqs. (25) and (26), we obtain

$$W-2W_+^{(1)}=2\dot{\tilde\lambda}(t)-4\tilde\lambda(t)\tilde\lambda(t),\quad W_+^{(1)}+W_+^{(2)}=2\dot{\tilde D}(t)-4\tilde\lambda(t)\tilde D(t),$$
$$\tilde\lambda(t)=\lambda_0(t)-2D_{\rm f}^{(1)}(t),\quad \lambda_0(t)=\frac{\alpha_1}{\alpha_1+\alpha_2}\lambda_{\rm f}(t)+\frac{\alpha_2}{\alpha_1+\alpha_2}\lambda_{\rm b}(t),$$
$$\tilde D(t)=D_{\rm f}^{(1)}(t)+D_{\rm b}^{(2)}(t),\quad W=2\dot\lambda_0(t)-4\tilde\lambda(t)\lambda_0(t),$$
$$W_+^{(1)}=2\dot D_{\rm f}^{(1)}(t)-4\tilde\lambda(t)D_{\rm f}^{(1)}(t),\quad W_+^{(2)}=2\dot D_{\rm b}^{(2)}(t)-4\tilde\lambda(t)D_{\rm b}^{(2)}(t). \tag{B15}$$

The values $\lambda_{\rm f}(t)$, $\lambda_{\rm b}(t)$, $D_{\rm b}^{(2)}(t)$, and $D_{\rm f}^{(1)}(t)$ are calculated with Eqs. (17) and (18) for the fermion and boson systems, respectively. The friction $\lambda_0(t)$ is defined in such a way as to obtain the Bose-Einstein distribution

$$n_{\rm b}(\infty)=n_{\rm b}^{\rm (eq)}(\omega). \tag{B16}$$

in the Markovian weak-coupling and $T_1=T_2$ limits.

In contrast to the b-f-f and f-b-b systems, there is limitation on the set of coupling strengths α_1 and α_2, bandwidths $\gamma_{1,2}$, and temperatures $T_{1,2}$ to obtain the asymptotic $n_{\mathrm{b}}(\infty)$ because the friction coefficient $\lambda_{\mathrm{b}}(t)$ has no the asymptotic limit. The occupation number has the asymptotic limit if

$$\frac{1}{\alpha_2} I_{\mathrm{b}}^{(2)}(\infty) = \frac{\frac{1}{\alpha_1} I_{\mathrm{f}}^{(1)}(\infty)}{1 - \frac{2(\alpha_1+\alpha_2)}{\alpha_1} I_{\mathrm{f}}^{(1)}(\infty)}. \tag{B17}$$

This equation relates the coupling strengths α_1 and α_2. In the Markovian weak-coupling and $T_1 = T_2$ limits, there is no restriction imposed on the coupling strengths α_1 and α_2.

5. Fermionic-fermionic-bosonic system

In the system, where the oscillator is fermionic, the heat baths “1” and “2” are fermionic and bosonic, respectively, one can similarly derive the equation for $n_{\mathrm{f}}(t)$, where

$$\tilde{\lambda}(t) = \frac{\alpha_1}{\alpha_1+\alpha_2}\lambda_{\mathrm{f}}(t) + \frac{\alpha_2}{\alpha_1+\alpha_2}\lambda_{\mathrm{b}}(t) + 2D_{\mathrm{b}}^{(2)}(t), \quad \tilde{D}(t) = D_{\mathrm{f}}^{(1)}(t) + D_{\mathrm{b}}^{(2)}(t), \tag{B18}$$

and the $\lambda_{\mathrm{b}}(t)$, $\lambda_{\mathrm{f}}(t)$, $D_{\mathrm{f}}^{(1)}(t)$, and $D_{\mathrm{b}}^{(2)}(t)$ are calculated with Eqs. (17) and (18) for the bosonic and fermionic systems, respectively.

As in the case of the b-f-b system, there is limitation on the set of coupling strengths α_1 and α_2, bandwidths $\gamma_{1,2}$, and temperatures $T_{1,2}$ to reach the asymptotic value of n_{f}, because the friction coefficient $\lambda_{\mathrm{b}}(t)$ has no asymptotic limit. Analogous to the b-f-b system, the occupation number has the asymptotic limit if the condition Eq. (B17) is satisfied.

In the Markovian weak-coupling and $T_1 = T_2 = T$ limits, there is no restriction imposed on the coupling strengths α_1 and α_2, and the asymptotic occupation number obeys the Fermi-Dirac distribution

$$n_{\mathrm{f}}(\infty) = n_{\mathrm{f}}^{(\mathrm{eq})}(\omega). \tag{B19}$$

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