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Rapid Communications

Thermodynamics from first principles: Correlations and nonextensivity

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The standard formulation of thermostatistics, being based on the Boltzmann-Gibbs distribution and logarithmic Shannon entropy, describes idealized uncorrelated systems with extensive energies and short-range interactions. In this Rapid Communication, we use the fundamental principles of ergodicity (via Liouville's theorem), the self-similarity of correlations, and the existence of the thermodynamic limit to derive generalized forms of the equilibrium distribution for long-range-interacting systems. Significantly, our formalism provides a justification for the well-studied nonextensive thermostatistics characterized by the Tsallis distribution, which it includes as a special case. We also give the complementary maximum entropy derivation of the same distributions by constrained maximization of the Gibbs-Shannon entropy. The consistency between the ergodic and maximum entropy approaches clarifies the use of the latter in the study of correlations and nonextensive thermodynamics.

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Introduction. The ability to describe the statistical state of a macroscopic system is central to many areas of physics [1–4]. In thermostatistics, the statistical state of a system of N particles in equilibrium is described by the distribution function $w_{\mathbf{z}}$ over $\mathbf{z}$ where $\mathbf{z} = (\{q_1, \dots, q_N\}, \{p_1, \dots, p_N\})$ defines a point (the microstate) in the concomitant $6N$-dimensional phase space. The central question addressed in this Rapid Communication is, *what is the generalized form of* $w_{\mathbf{z}}$ *for a composite system at thermodynamic equilibrium that features correlated subsystems?*

This question has been the subject of intense research for more than a century [3–23]. Correlations and nonextensive energies are associated with long-range-interacting systems, which are at the focus of much of the effort (in particular, see [13,18–20,22]). The most widespread approach to finding $w_{\mathbf{z}}$ is the maximum entropy (MaxEnt) principle introduced by Jaynes [6,7] on the basis of information theory. The principle entails making the *least-biased* statistical inferences about a physical system consistent with prior expected values of a set of its quantities $\{\bar{f}^{(1)}, \bar{f}^{(2)}, \dots, \bar{f}^{(l)}\}$. It requires the distribution $w_{\mathbf{z}}$ to maximize the Gibbs-Shannon (GS) logarithmic entropy functional $S^{\mathrm{GS}}(\{w_{\mathbf{z}}\})$ subject to constraints $\sum_{\mathbf{z}} f_{\mathbf{z}}^{(i)} w_{\mathbf{z}} = \bar{f}^{(i)}$. Here, $S^{\mathrm{GS}}(\{w_{\mathbf{z}}\}) = -k\sum_{\mathbf{z}} w_{\mathbf{z}} \ln(w_{\mathbf{z}})$ for constant $k > 0$. Despite outstanding success [24,25] in capturing the thermodynamics of weakly interacting gases, the principle—in its original form—does not describe correlated systems. Attempts have been made to generalize the principle; however, as there is no accepted method for doing so, controversy has ensued [26–28].

One approach is based on the extension of the MaxEnt principle by Shore and Johnson [29], and entails generalizing the way knowledge of the system is represented by constraints [25,29]. Information about correlations are incorporated, e.g., by modifying the partition function [30] or the structure of the microstates [27]. Another widely used approach is to generalize the MaxEnt principle to apply to a different entropy functional in place of $S^{\mathrm{GS}}(\{w_{\mathbf{z}}\})$. At the forefront of this effort is the so-called q thermostatistics based on Tsallis' entropy $S_q^{\mathrm{Ts}}(\{w_{\mathbf{z}}\}) \propto (1 - \sum_{\mathbf{z}} w_{\mathbf{z}}^q)/(q-1)$ and expressing the constraints as averages with respect to escort probabilities $\{w_{\mathbf{z}}^q\}$ [4,8]. q thermostatistics is used to describe a wide range of physical scenarios [3,13,20,23,31–46], including high-T_c superconductivity, long-range-interacting Ising magnets, turbulent pure-electron plasmas, N-body self-gravitating stellar systems, high-energy hadronic collisions, and low-dimensional chaotic maps. The approach has also been refined and extended [3,47–50].

A contentious issue, however, is that the Tsallis entropy does not satisfy Shore and Johnson's system-independence axiom [26–29]. Although Jizba and Korbel [22] recently made some headway towards a resolution, objections remain [51], and the generalization of the MaxEnt principle continues to be controversial. This brings into focus the need for an independent approach to our central question.

We propose an answer by introducing a general formalism based on ergodicity [1] for deriving equilibrium distributions, including ones for correlated systems. Previously this derivation was thought impossible as correlations have been linked with nonergodicity (see, e.g., [4], pp. 68 and 320). However, we circumvent these difficulties by showing how the self-similarity of correlations can be invoked to derive the distribution $w_{\mathbf{z}}$ under well-defined criteria. Then we show how to employ the MaxEnt principle consistently with correlations encoded as a self-similarity constraint function. After comparing our results with previous works, we present a numerical example for completeness and end with a conclusion.

Key ideas. Our approach rests on two key ideas.

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(i) Liouville's theorem for equilibrium systems. Consider a generic, classical, dynamical system described by Hamiltonian H and phase-space distribution $w(\mathbf{z};t)$. Being a Hamiltonian system guarantees the incompressibility of phase-space flows, which is represented via Liouville's equation [1,5] by w being a constant of motion along a trajectory, i.e.,

$$\frac{\partial w}{\partial t} + \dot{\mathbf{z}} \cdot \nabla w = \frac{\partial w}{\partial t} + \{w, H\} = \frac{dw}{dt} = 0, \quad (1)$$

where { , } denotes the Poisson bracket. Imposing the equilibrium condition $\partial w/\partial t = 0$ implies

$$\partial w/\partial t = -\{w, H\} = 0, \quad (2)$$

i.e., the existence of a steady state, $w(\mathbf{z};t) = w_{\mathbf{z}}\,\forall t$. Any Hamiltonian *ergodic* system at equilibrium will obey this condition. A possible solution of Eq. (2) is given by $w_{\mathbf{z}} = \mathfrak{a} f(\mathfrak{b} H_{\mathbf{z}} + \mathfrak{c})$, where $f(\cdot)$ is any differentiable function, for macrostate-defining and normalization constants $\mathfrak{a}$, $\mathfrak{b}$, and $\mathfrak{c}$ (see, e.g., [1,52]). We only consider solutions of this form, which is equivalent to invoking the fundamental postulate of equal *a priori* probabilities for accessible microstates [1]. For brevity, we shall write the solution as

$$w_{\mathbf{z}} = \mathcal{G}_X(H_{\mathbf{z}}) \quad (3)$$

and leave the dependence on the parameters $\mathfrak{a}$, $\mathfrak{b}$, and $\mathfrak{c}$ as being implicit in the label X.

(ii) Deriving equilibrium distributions. Consider the equilibrium distributions w^A, w^B and Hamiltonians H^A, H^B of two isolated, conservative, short-range-interacting systems labeled A and B where

$$H^{AB}_{\mathbf{z}_{AB}} = H^A_{\mathbf{z}_A} + H^B_{\mathbf{z}_B}, \quad (4)$$

$$w^{AB}_{\mathbf{z}_{AB}} = w^A_{\mathbf{z}_A} w^B_{\mathbf{z}_B} \quad (5)$$

are the total Hamiltonian and joint distributions, respectively, for the isolated, composite system AB at equilibrium, and $\mathbf{z}_{AB} \equiv (\mathbf{z}_A, \mathbf{z}_B)$. From (3), each distribution is a function of its respective Hamiltonian. Taken together, Eqs. (3)–(5) imply the general solution $w^X_{\mathbf{z}_X}$ is the Boltzmann-Gibbs (BG) distribution $\mathcal{G}_X(H^X_{\mathbf{z}_X}) = \mathfrak{a} e^{\mathfrak{b} H_{\mathbf{z}_X}}$ for macrostate-dependent constants $\mathfrak{a}$, $\mathfrak{b}$ and $X = A, B$ and AB.

This well-known result can easily be generalized. For example, replacing Eq. (5) with

$$w^{AB}_{\mathbf{z}_{AB}} = w^A_{\mathbf{z}_A} \otimes_q w^B_{\mathbf{z}_B}, \quad (6)$$

where $\otimes_q$ is the q product [15], correspondingly implies that the general solution is given by the Tsallis distribution $\mathcal{G}^X(H_{\mathbf{z}_X}) = \mathfrak{a} e_q^{\mathfrak{b} H_{\mathbf{z}_X}}$ where e_q^x is the q exponential of x provided due care is taken with respect to applying the q algebra [15,53] and normalization [52]. Note that each $w^X_{\mathbf{z}_X}$ is the equilibrium distribution for system X in isolation, and Eq. (6) represents a correlated state of A and B, where $w^A_{\mathbf{z}_A}$ and $w^B_{\mathbf{z}_B}$ are *not* the marginals of $w^{AB}_{\mathbf{z}_{AB}}$ for $q \neq 1$. As this result has previously been regarded [4] as incompatible with Eq. (2), it shows that Liouville's theorem has an underappreciated application for describing highly correlated systems.

Finding a generalized distribution. With these ideas in mind, we derive our main results for a composite, self-similar, classical Hamiltonian system in thermodynamic equilibrium. For brevity, we explicitly treat a composite system AB composed of two subsystems A and B, although our results are easily extendable to compositions involving an arbitrary number of macroscopic subsystems. Let the tuples $(w^{AB}_{\mathbf{z}_{AB}}, H^{AB}_{\mathbf{z}_{AB}})$, $(w^A_{\mathbf{z}_A}, H^A_{\mathbf{z}_A})$, and $(w^B_{\mathbf{z}_B}, H^B_{\mathbf{z}_B})$ denote the composite and isolated equilibrium distributions and Hamiltonians of the composite AB, and separate A and B subsystems, respectively; $w^X_{\mathbf{z}_X}$ is the equilibrium probability that system X is in phase space point $\mathbf{z}_X$. The following criteria encapsulate properties of the system required for subsequent work. They immediately lead to two key theorems, which generalize thermostatistics.

Criterion I—Thermodynamic limit. Consider a sequence of systems $A_1, A_2, \ldots$ for which the solution Eq. (3) for the nth term is given by $w^{A_n}_{\mathbf{z}_{A_n}} = \mathcal{G}^{(n)}_{A_n}(H^{A_n}_{\mathbf{z}_{A_n}})$. A sequence that increases in size is said to have a *thermodynamic limit* if $\mathcal{G}^{(n)}_{A_n}$ attains a limiting parametrized form as A_n becomes macroscopic, i.e., if $\mathcal{G}^{(n)}_{A_n} \to \mathcal{G}_A$ as $n \to \infty$. The distribution $w^A_{\mathbf{z}_A} = \mathcal{G}_A(H^A_{\mathbf{z}_A})$, where the dependence on system, macrostate, and normalization constants is implicit in the label A on $\mathcal{G}_A$, is said to represent the thermostatistical properties of the physical material comprising A in the thermodynamic limit.

Examples of limiting forms include the BG distribution $\mathcal{G}(H_{\mathbf{z}}) = \mathfrak{a} e^{-\mathfrak{b} H_{\mathbf{z}}}$ and the Tsallis distribution $\mathcal{G}(H_{\mathbf{z}}) = \mathfrak{a} e_q^{-\mathfrak{b} H_{\mathbf{z}}}$ for macrostate-dependent parameter $\mathfrak{b}$ and normalization constant $\mathfrak{a}$.

Criterion II—Compositional self-similarity. We define a system as having compositional self-similarity if there exist mapping functions $\mathbb{C}$ and $\mathbb{H}$ such that the composite equilibrium distribution and energy of macroscopic AB are related to the isolated equilibrium distribution and energy of macroscopic A and B by the following relations:

$$H^{AB}_{\mathbf{z}_{AB}} = \mathbb{H}\big(H^A_{\mathbf{z}_A}, H^B_{\mathbf{z}_B}\big), \quad (7a)$$

$$w^{AB}_{\mathbf{z}_{AB}} = \mathbb{C}\big(w^A_{\mathbf{z}_A}, w^B_{\mathbf{z}_B}\big), \quad 0 \leqslant \mathbb{C} \leqslant 1, \quad (7b)$$

for all $\mathbf{z}_{AB}$, where $\mathbb{H}$ embodies the nature of the interactions, and $\mathbb{C}$ embodies the nature of the correlations. For example, short-range interactions are well approximated by $H^{AB}_{\mathbf{z}_{AB}} = H^A_{\mathbf{z}_A} + H^B_{\mathbf{z}_B}$ and $w^{AB}_{\mathbf{z}_{AB}} = w^A_{\mathbf{z}_A} w^B_{\mathbf{z}_B}$, whereas the Tsallis distribution in Eq. (6) has been applied to a wide range of physical situations [3,13,20,23,31–46] exhibiting strong correlations and long-range interactions. Other relations hold in general, as shown in Table I [52]. For brevity we will henceforth use "self-similar" to refer to compositional self-similarity.

Theorem 1. For systems satisfying compositional self-similarity in the thermodynamic limit, the equilibrium distribution is given by $w^X_{\mathbf{z}_X} = \mathcal{G}_X(H^X_{\mathbf{z}_X})$ where the function $\mathcal{G}_X$ satisfies

$$\mathbb{C}\big(\mathcal{G}_A\big(H^A_{\mathbf{z}_A}\big), \mathcal{G}_B\big(H^B_{\mathbf{z}_B}\big)\big) = \mathcal{G}_{AB}\big(\mathbb{H}\big(H^A_{\mathbf{z}_A}, H^B_{\mathbf{z}_B}\big)\big). \quad (8)$$

Proof. This follows directly from criteria I and II. ■

Hence, finding a $\mathcal{G}$ that satisfies Eq. (8) allows one to calculate the equilibrium distribution in Eq. (3). See Supplemental Material [52] for a simple example. In general, finding $\mathcal{G}$ is difficult; however, the next theorem supplies a solution for an important class of situations.

TABLE I. A summary of appropriate choices for $\{\mathcal{F}, \mathcal{H}\}$ to reproduce well-established classes of thermostatistics, which allowed us to also indicate their potential limitations. We have included the conventional partition-funtion-type normalization constants in some cases for completeness. Note, however, that such constants can be reexpressed as a and b or β and H_o, i.e., $Z_{\text{BG}} = e^{-\beta H_o}$ and $Z_q = e_q^{-\beta_q H_o}$ (where $\beta_q = \beta[1 + (1-q)\beta H_o]^{-1}$).

Type of thermostatistics	Correlations	Hamiltonian	$\mathcal{F}(w)$	$\mathcal{H}(H)$	Distribution	Fails to describe:
This work	$\mathbb{C}(w^A, w^B)$ (self-similar)	$\mathbb{H}(H^A, H^B)$ (arbitrary)	$\mathcal{F}(w)$	$\mathcal{H}(H)$	Eq. (10)	systems failing criteria I and II
Conventional thermostatistics [5–7]	$w^A w^B$ (independent)	$H^A + H^B$ (noninteracting)	$\ln(w)$	H	$\frac{1}{Z_{\text{BG}}} e^{-\beta H}$ (exponential class)	correlations, nonadditive Hamiltonians
Tsallis' (q-) thermostatistics [4,8]	$w^A \otimes_q w^B$ (correlated)	$H^A + H^B$ (noninteracting)	$\ln_q(w)$	H	$\frac{1}{Z_q} e_q^{-\beta_q H}$ (q-deformed class)	nonadditive Hamiltonians
An exactly solvable example exhibiting both correlations and nonextensive energies	$w^A \otimes_q w^B$ (correlated)	$H^A \oplus_p H^B$ (interacting)	$\ln_q(w)$	$\ln\left(e_p^H\right)$	$\exp_q\left[-\beta \ln\left(e_p^H\right) + H_o\right]$	

Theorem 2. Given single-variable invertible maps $\mathcal{F}$ and $\mathcal{H}$ satisfying the functional equations

$$\mathcal{F}_{AB}(\mathbb{C}(w^A, w^B)) = \mathcal{F}_A(w^A) + \mathcal{F}_B(w^B), \tag{9a}$$

$$\mathcal{H}_{AB}(\mathbb{H}(H^A, H^B)) = \mathcal{H}_A(H^A) + \mathcal{H}_B(H^B), \tag{9b}$$

then there exists a family of equilibrium distributions given by

$$w_{\mathbf{z}}^X \equiv \mathcal{G}_X\left(H_{\mathbf{z}}^X\right) = \mathcal{F}_X^{-1}\left[a^X \mathcal{H}\left(H_{\mathbf{z}}^X\right) + b^X\right] \ \forall \mathbf{z}, \tag{10}$$

where a^X and b^X are constants obeying the system composition rules

$$a^{AB} = a^A = a^B, \quad b^{AB} = b^A + b^B. \tag{11}$$

Note that a^X and b^X are generalizations of a common inverse-temperature-like quantity $\beta = a^X$ and an extensive average-energy-like quantity $H_o^X = -b^X/\beta$ in the more familiar form of Eq. (10), $w^X = \mathcal{F}^{-1}\{\beta[\mathcal{H}(H^X) - H_o^X]\}$.

Proof. We defer the proof and a nontrivial example to the Supplemental Material [52].

In our generalized thermostatistic formalism, the solutions to Eq. (8) give the most general form of the equilibrium distribution and Eq. (10) provides a recipe for finding it for the cases satisfying Eq. (1). Solutions to Eq. (1) can be guessed for a number of cases of practical interest, as shown below. However, the analytical forms of $\mathcal{F}$ and $\mathcal{H}$ are expected to be difficult to find, in general. Nevertheless, we demonstrate below a systematic numerical method that can find $\mathcal{F}$ and $\mathcal{H}$ for a given $\mathbb{C}$ and $\mathbb{H}$, and thus determine the corresponding equilibrium thermostatistics in the general case.

MaxEnt principle with correlations. We now show that the MaxEnt principle for S^{GS} gives an independent derivation of Eq. (8) when the self-similar correlations, along with the normalization and mean energy, are treated as prior data conditions [25,29]. For composite system AB, the constraints for the normalization and mean energy are the conventional ones, i.e., $I(\{w_{\mathbf{z}_{AB}}^{AB}\}) = \sum_{\mathbf{z}_{AB}} w_{\mathbf{z}_{AB}}^{AB} - 1 = 0$ and $E(\{w_{\mathbf{z}_{AB}}^{AB}\}) = \sum_{\mathbf{z}_{AB}} w_{\mathbf{z}_{AB}}^{AB} H_{\mathbf{z}_{AB}}^{AB} - \bar{H}^{AB} = 0$, respectively, where $\bar{H}^{AB}$ is the average energy. The prior knowledge of the self-similar correlations is represented by Eq. (7b) as a *functional constraint* over the phase space. Thus, the constrained maximization of $S^{\text{GS}}(\{w_{\mathbf{z}}^{AB}\})/k$ leads to

$$\frac{\partial}{\partial w_{\mathbf{z}'_{AB}}^{AB}}\left\{-\sum_{\mathbf{z}_{AB}} \ln w_{\mathbf{z}_{AB}}^{AB} + aI\left(\{w_{\mathbf{z}_{AB}}^{AB}\}\right) + bE\left(\{w_{\mathbf{z}_{AB}}^{AB}\}\right) + \sum_{\mathbf{z}_{AB}} c_{\mathbf{z}_{AB}}\left[w_{\mathbf{z}_{AB}}^{AB} - \mathbb{C}\left(w_{\mathbf{z}_A}^A, w_{\mathbf{z}_B}^B\right)\right]\right\} = 0 \tag{12}$$

with Lagrange multipliers a, b, and $\{c_{\mathbf{z}_{AB}}\}$, where $c_{\mathbf{z}_{AB}}$ is a function over the phase space.

In [52], we show Eq. (12) yields

$$\frac{1}{b^{AB}}\left[\ln \mathbb{C}\left(w_{\mathbf{z}_A}^A, w_{\mathbf{z}_B}^B\right) - a^{AB} - c_{\mathbf{z}_{AB}}^{AB}\right] = \mathbb{H}\left(\frac{1}{b^A}\left[\ln w_{\mathbf{z}_A}^A - a^A - c_{\mathbf{z}_A}^A\right], \frac{1}{b^B}\left[\ln w_{\mathbf{z}_B}^B - a^B - c_{\mathbf{z}_B}^B\right]\right), \tag{13}$$

where equilibrium distributions are given by $\ln w_{\mathbf{z}_X}^X = a^X + b^X H_{\mathbf{z}_X}^X + c_{\mathbf{z}_X}^X$ for phase-space functions $c_{\mathbf{z}_{AB}}^{AB}$, $c_{\mathbf{z}_A}^A$, and $c_{\mathbf{z}_B}^B$ that satisfy the above equation. Setting $\mathcal{G}^{-1}(w_{\mathbf{z}_X}^X) = \frac{1}{b^X}[\ln w_{\mathbf{z}_X}^X - a^X - c_{\mathbf{z}_X}^X]$ shows that Eq. (13) is equivalent to Eq. (8), and so the solutions found here are equivalent to those given by the solutions of Eqs. (3) and (8) for corresponding values of the Lagrange multipliers a^X and b^X.

Relationship with previously studied thermostatistic classes. Table I compares the forms of $\mathcal{F}$ and $\mathcal{H}$, and limitations of various classes of distributions.

An interesting result demonstrated in the third row of the table is that the Tsallis distribution, $\frac{1}{Z_q} e_q^{-\beta_q H}$, corresponds to systems satisfying criteria I and II that have *additive* Hamiltonians. This effectively rules out the validity of the Tsallis distribution for systems with an interaction term in the Hamiltonian. This is also true for examples of multifractal and ϕ-exponential-class thermostatistics [3,4] that are characterized by the Tsallis distribution. Although the Tsallis distribution is known to exhibit nonadditive average energy [4], evidently the nonadditivity is due to correlations forming between subsystems and not interactions [54]. Nevertheless, the distribution derived for nonadditive Hamiltonians of the type given in the fourth row of Table I, may give better fits to experimental data for systems that have long-range interactions compared to the Tsallis distribution (e.g., see [39]).

Moreover, our formalism covers thermostatistics of extreme cases of correlations, most notably the well-studied case of one-dimensional Ising ferromagnets at vanishing temperature (see [52] for details). Aside from such trivial maximally correlated cases and the long-range Ising models [23,40,44–46,55] (corresponding to the *third* row of Table I as long as subsystems are macroscopic), we are not aware of any previous thermostatistic formalism that can describe nontrivial long-range-interacting systems, as in the *last* row, by finding the equilibrium distribution.

Numerical example. The well-studied examples discussed above all have analytic solutions. Next, we demonstrate the versatility of our approach by numerically evaluating the statistics of a complex long-range-interacting model (an extreme case of correlations and nonadditivity). To demonstrate how our approach might handle a practical problem, we intentionally choose composition rules,

$$\mathbb{C}(w^A, w^B) = w^A w^B \frac{(3.3 - w^A)(3.3 - w^B)}{2.3^2}, \qquad (14)$$

$$\mathbb{H}(H^A, H^B) = 0.7(H^A + H^B), \qquad (15)$$

which have *no* known analytical solution for w within our formalism, and which would require extremely long-range interactions for the energy composition rule.

In Fig. 1, we show $\mathcal{F}$ and $\mathcal{Q}$, where $\mathcal{H}(H) = \mathcal{Q}(e^{-H})$ [in BG, $\mathcal{F} \propto \mathcal{Q} \propto \ln(x)$], for Eqs. (14) and (15), and w versus H, where $w(H) = \mathcal{F}^{-1}\{\beta[\mathcal{H}(H) - H_o]\}$. Here, H_o ensures normalization $\int w(H)dH = \bar{1} = 1$ and $\beta = -1$ ensures $\int H\bar{w}(H)dH = \bar{H} = 1$, which corresponds, in BG thermostatistics, to having an inverse temperature of $\beta^{\mathrm{BG}} = -1$ in unitless parameters. It is interesting to see the significant differences between the generalized distribution $\bar{w}(H)$ and the normalized $w^{\mathrm{BG}} = e^{-H}$ (bottom plot), being flatter for small energies and decaying more rapidly for larger energies. Full details of the numerical implementation are discussed in the Supplemental Material [52].

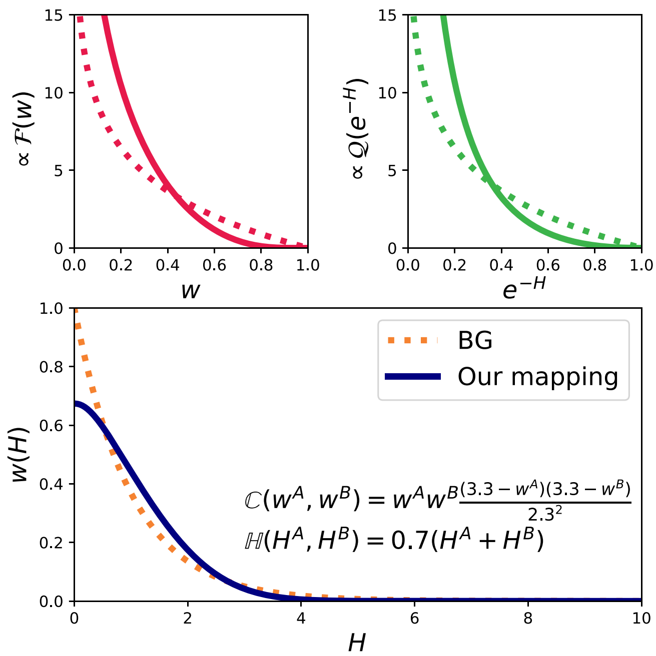

FIG. 1. The top two panels show the mapping functions $\mathcal{F}$ and $\mathcal{Q}$ for $\bar{w}$ and $q = e^{-H}$, respectively (solid lines), and their BG equivalents (dotted lines). The bottom panel shows the distribution $w(H) = \mathcal{F}^{-1}\{\beta[\mathcal{H}(H) - H_o]\}$ (solid line) for the mappings in Eqs. (14) and (15), and the normalized BG distribution with the same average energy (dotted line).

Conclusions. We employed an approach based on Liouville's theorem for equilibrium conditions obeying a thermodynamic limit and self-similarity criterion, to provide an alternative derivation of consistent generalized thermostatistics for systems with correlations and nonadditive Hamiltonians (this is in comparison to the conventional MaxEnt formulations [3,4,6,7]). In our formalism, the equilibrium distributions of such systems are fully characterized by $\mathcal{G}$ in Eq. (3) or by $\{\mathcal{F}, \mathcal{H}\}$ maps in Eq. (1) for the special cases. Upon appropriate choices of $\{\mathcal{F}, \mathcal{H}\}$, our generalized thermostatistic class recovers well-established families, i.e., the standard Jaynes and Tsallis q thermostatistics as demonstrated in Table I. Interestingly, our formalism implies that, for systems satisfying our criteria, the latter family of thermostatistics can *only* capture the thermodynamics of systems with additive Hamiltonians.

Our extension of the MaxEnt principle with S^{GS} to include self-similar correlations as priors, gives an independent derivation of the same equilibrium distributions derived using Liouville's theorem. This independent derivation confirms the central role of the MaxEnt principle applied to S^{GS} as a basis for statistical inference in thermostatistics [29]. Moreover, it also clarifies the controversy surrounding the heuristic application of the MaxEnt principle to generalized entropy functionals, such as the Tsallis entropy. Our derivation of the Tsallis distribution from Liouville's theorem and the MaxEnt principle applied to S^{GS}, with a self-similar correlation prior, provides it with the mathematical support it previously lacked.

It would be interesting to examine the thermodynamics of low-dimensional long-range Ising-type models [23,40,44–46,55–58], which exhibit phase transitions under certain conditions [56–58]. In the context of our formalism, such phase transitions are driven by the set of control parameters given above as $\{a^X, b^X\}$ (and which include the temperature through a global function [54]).

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[1] F. Reif, *Fundamentals of Statistical and Thermal Physics* (McGraw-Hill, New York, 1965).
[2] S. Abe and Y. Okamoto, *Nonextensive Statistical Mechanics and Its Applications*, Lecture Notes in Physics Vol. 560 (Springer-Verlag, Berlin, 2001).
[3] J. Naudts, *Generalised Thermostatistics* (Springer, London, 2011).
[4] C. Tsallis, *Introduction to Nonextensive Statistical Mechanics* (Springer, New York, 2009).
[5] J. W. Gibbs, *Elementary Principles in Statistical Mechanics* (Charles Scribner's Sons, New York, 1902).
[6] E. T. Jaynes, Phys. Rev. **106**, 620 (1957).
[7] E. T. Jaynes, Phys. Rev. **108**, 171 (1957).
[8] C. Tsallis, J. Stat. Phys. **52**, 479 (1988).
[9] A. R. Plastino and A. Plastino, Phys. Lett. A **193**, 140 (1994).
[10] C. Tsallis, R. Mendes, and A. Plastino, Physica A **261**, 534 (1998).
[11] A. P. S. Martinez and F. Pennini, Nonextensive Statistical Mechanics and its Applications, Physica A Lecture Notes in, 295 (2001).
[12] S. Abe, Physica A **300**, 417 (2001).
[13] A. Plastino, C. Giordano, A. Plastino, and M. Casas, Physica A **336**, 376 (2004).
[14] S. Abe, Physica A **368**, 430 (2006).
[15] H. Suyari, Physica A **368**, 63 (2006).
[16] F. Caruso and C. Tsallis, Phys. Rev. E **78**, 021102 (2008).
[17] C. Tsallis, Braz. J. Phys. **39**, 337 (2009).
[18] C. Tsallis, Entropy **13**, 1765 (2011).
[19] E. Akkermans, Statistical mechanics and quantum fields on fractals, arXiv:1210.6763.
[20] A. Deppman, Phys. Rev. D **93**, 054001 (2016).
[21] T. Oikonomou and G. B. Bagci, Phys. Lett. A **381**, 207 (2017).
[22] P. Jizba and J. Korbel, Phys. Rev. Lett. **122**, 120601 (2019).
[23] *Dynamics and Thermodynamics of Systems with Long Range Interactions*, edited by T. Dauxois, S. Ruffo, E. Arimondo, M. Wilkens (Springer, New York, 2002).
[24] E. T. Jaynes, *Papers on Probability and Statistics and Statistical Physics*, edited by R. Rosenkrantz (Kluwer Academic, Berlin, 1983), Vol. 118.
[25] S. Pressé, K. Ghosh, J. Lee, and K. A. Dill, Rev. Mod. Phys. **85**, 1115 (2013).
[26] S. Pressé, K. Ghosh, J. Lee, and K. A. Dill, Phys. Rev. Lett. **111**, 180604 (2013).
[27] S. Pressé, K. Ghosh, J. Lee, and K. A. Dill, Entropy **17**, 5043 (2015).
[28] C. Tsallis, Entropy **17**, 2853 (2015).
[29] J. Shore and R. Johnson, IEEE Trans. Inf. Theory **26**, 26 (1980).
[30] S. Presse, K. Ghosh, and K. A. Dill, J. Phys. Chem. B **115**, 6202 (2011).
[31] H. Uys, H. Miller, and F. Khanna, Phys. Lett. A **289**, 264 (2001).
[32] C. Beck, Physica A **277**, 115 (2000).
[33] C. Beck, Phys. Rev. Lett. **87**, 180601 (2001).
[34] B. M. Boghosian, Phys. Rev. E **53**, 4754 (1996).
[35] C. Anteneodo and C. Tsallis, J. Mol. Liq. **71**, 255 (1997).
[36] J. A. S. Lima, R. Silva, and J. Santos, Phys. Rev. E **61**, 3260 (2000).
[37] U. Tirnakli, Phys. Rev. E **62**, 7857 (2000).
[38] U. Tirnakli, C. Tsallis, and M. L. Lyra, Phys. Rev. E **65**, 036207 (2002).
[39] A. Lavagno, G. Kaniadakis, M. Rego-Monteiro, P. Quarati, and C. Tsallis, Astrophys. Lett. Commun. **35**, 449 (1998).
[40] V. Latora, A. Rapisarda, and C. Tsallis, Physica A **305**, 129 (2002).
[41] I. Zborovsky, Int. J. Mod. Phys. A **33**, 1850057 (2018).
[42] A. Plastino and A. Plastino, Physica A **222**, 347 (1995).
[43] T. Frank and A. Daffertshofer, Physica A **272**, 497 (1999).
[44] R. Salazar and R. Toral, Phys. Rev. Lett. **83**, 4233 (1999).
[45] R. Salazar, A. Plastino, and R. Toral, Eur. Phys. J. B **17**, 679 (2000).
[46] M. Portesi, A. Plastino, and C. Tsallis, Phys. Rev. E **52**, R3317 (1995).
[47] S. Abe, Phys. Lett. A **224**, 326 (1997).
[48] E. P. Borges and I. Roditi, Phys. Lett. A **246**, 399 (1998).
[49] P. T. Landsberg and V. Vedral, Phys. Lett. A **247**, 211 (1998).
[50] J. Naudts, Physica A **365**, 42 (2006).
[51] T. Oikonomou and G. B. Bagci, Phys. Rev. E **99**, 032134 (2019).
[52] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevE.101.060101 for restrictions on the normalization of Tsallis distribution, a simple example of finding $\mathcal{G}$ satisfying Eq. (8), the proof of Theorem 2 and an associated nontrivial example, how to find Eq. (13) using Eq. (12), how our formalism recovers $w^{\rm BG}$ for the trivial case of Ising ferromagnets, and the details of our numerical approach to find $\mathcal{F}$ and $\mathcal{H}$ maps.
[53] K. P. Nelson and S. Umarov, The relationship between Tsallis statistics, the Fourier transform, and nonlinear coupling, arXiv:0811.3777.
[54] F. Tanjia, S. N. Saadatmand, and J. A. Vaccaro (unpublished).
[55] S. S. Apostolov, Z. A. Mayzelis, O. V. Usatenko, and V. A. Yampol'skii, J. Phys. A: Math. Theor. **42**, 095004 (2009).
[56] D. Ruelle, Commun. Math. Phys. 9, 267 (1968).
[57] D. J. Thouless, Phys. Rev. **187**, 732 (1969).
[58] F. J. Dyson, Commun. Math. Phys. **21**, 269 (1971).