Erratum: Critical behavior of $O(n)$-symmetric systems with reversible mode-coupling terms: Stability against detailed-balance violation [Phys. Rev. E 55, 4120 (1997)]

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(Received 14 March 2020; published 8 May 2020)

DOI: 10.1103/PhysRevE.101.059901

In this paper, field-theoretic renormalization group (RG) methods [1] are utilized to study the universal dynamical critical scaling properties of $O(n)$-symmetric systems with reversible mode-coupling terms subject to detailed-balance violations, implemented through distinct heat bath temperatures for the critical order parameter S and dynamically coupled noncritical conserved fields M.

Unfortunately, in the perturbative evaluation of the three-point vertex functions for the reversible mode couplings g for the $O(n)$-symmetric Sásvari-Schwabl-Szépfalusy (SSS) model [2], we erroneously attached the combinatorial factors $n-1$ to the first-order fluctuation corrections [3] on the right-hand sides of Eqs. (A11) and (A12) listed in the Appendix. Interestingly, away from thermal equilibrium, this in fact invalidates the renormalization constant relation $Z_g = Z_{\tilde{M}}^{1/2} = Z_M^{-1/2}$ that follows from the Ward identity (3.17) [3]. It should also be noted that in generic nonequilibrium circumstances, the field renormalizations obtained from the two-point vertex functions are not the same as those in the corresponding dynamical response functions [4,5].

Consequently, the ensuing renormalization constants are not uniquely determined. Remarkably, there appear to exist (at least) two different consistent prescriptions to fix the resulting redundancies:

(1) One may exploit the structure of the Poisson brackets underlying the system's reversible dynamics, in symbolic shorthand $\{S, S\} = 0$, $\{S, M\} = gS$, and $\{M, M\} = g'M$, which imply the identities $Z_g = Z_M^{1/2} = Z_{g'}$ [6] (where the distinction between the nonlinear mode couplings g and g' appearing in the Langevin equations for the order parameter S and the conserved quantity M has been made for book-keeping purposes only).

(2) One can rescale the fields in the original Langevin equations (2.27) and (2.28) or in the Janssen–De Dominicis response functional (2.31), (2.33), and (2.35) precisely with the prescriptions (2.21) and (2.32), whereupon the noise strengths $\widetilde{\lambda} \to \lambda$ and $\widetilde{D} \to D$ in the correlators (2.28) and (2.29), and $u \to \widetilde{u}$, $g \to \widetilde{g}$, and $g' \to \widetilde{g}'$ as defined in Eqs. (2.34) and (2.36). Formally, this rescaling procedure restores the Einstein relations between noise strengths and relaxation rates, at the price of introducing distinct mode-coupling constants; their ratio yields the nonequilibrium parameter $\Theta = \widetilde{g}'/\widetilde{g} = \widetilde{\lambda} D g'/\lambda \widetilde{D} g = T_S/T_M$ or effective temperature ratio for the heat baths coupled to the order parameter and conserved fields, respectively.

Naturally, procedures 1 and 2 above lead to different intermediate results for renormalization constants and RG flow functions, yet after straightforward analysis they ultimately result in the following identical RG β functions for the couplings $w = \lambda/D$, Θ, $f = \widetilde{g}^2/\lambda D = g^2 \widetilde{D}/\lambda D^2 = w\bar{f}$, and $\widetilde{u}$:

$$\beta_w = wf\left[\left(\frac{1}{2} - \frac{n-1}{1+w}\right)\Theta - (n-1)\,\frac{w^2}{(1+w)^3}\,(1-\Theta)\right], \tag{3.60}$$

$$\beta_\Theta = -\frac{f}{2}\,\Theta\,(1-\Theta)\left[\Theta - \frac{1-2w}{(1+w)^2} + 2(n-1)\,\frac{1+3w+w^2}{(1+w)^3}\right], \tag{3.61}$$

$$\beta_f = f\left[-\epsilon + \left(\frac{2-\Theta}{2} + \frac{n-1}{1+w}\right) f\,\Theta + \frac{w}{(1+w)^2}\left(2 - (n-1)\,\frac{2+w}{1+w}\right) f\,(1-\Theta)\right], \tag{3.62}$$

$$\beta_{\widetilde{u}} = \widetilde{u}\left[-\epsilon + \frac{n+8}{6}\,\widetilde{u} - 2\,(n-1)\,\frac{1+3w+w^2}{(1+w)^3}\,f\,(1-\Theta)\right] + \frac{6\,(n-1)}{1+w}\,f^2\,\Theta\,(1-\Theta), \tag{3.63}$$

where $d = 4 - \epsilon$ is the spatial dimension. Note that β_w and $\beta_{\widetilde{u}}$ are as previously listed.

The modifications for β_Θ and β_f do not alter the ensuing RG fixed point structure discussed, nor do they affect the stability of the equilibrium fixed point [6]. While the nonequilibrium fixed point with $\Theta^* = \infty$ and its properties remain unmodified as well, there are pertinent changes for the other nonequilibrium fixed point with $\Theta^* = 0$: While still $w^* = \infty$, the correct mode-coupling fixed point value is $\bar{f}^* = \epsilon/2$, and consequently the n dependence changes as well for $\widetilde{u}^* = nu_H^* = 6n\epsilon/(n+8)$. At this weak dynamic scaling fixed point, one now obtains to first order in the ϵ expansion $z_S = 2 - (n-1)\epsilon/2 < z_M = 2$ [6].

Finally, similar corrections need to be implemented [3] for the analysis of the SSS model with spatially anisotropic nonequilibrium perturbations: There should be no factor $n-1$ in Eq. (3.14) of Ref. [4], which invalidates (3.15) and several additional intermediate results. Yet the ensuing fixed point structure and stability are not affected; only the isotropic nonequilibrium fixed point with $T^* = 1/\Theta^* = \infty$ needs to be altered as stated above. Correspondingly, the dynamic critical exponents at the anisotropic nonequilibrium fixed point with $T^{\parallel *} = \infty$ and $T^{\perp *} = 0$ in the longitudinal sectors are $z_S^{\parallel} = 2 - (n-1)\epsilon/2 < z_M^{\parallel} = 2$, while those in the transverse sector remain $z_S^{\perp} = 2 = z_M^{\perp}$.

I am deeply indebted to Luca Di Carlo and Giulia Pisegna at the University of Rome Sapienza for spotting and pointing out my error to me; I am also grateful to Luca Di Carlo, Giulia Pisegna, Andrea Cavagna, Irene Giardina, and Tomas S. Grigera for subsequent very fruitful discussions.

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