Viscous Rayleigh-Taylor and Richtmyer-Meshkov instabilities in the presence of a horizontal magnetic field

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We first derive the exact dispersion relation for viscous Rayleigh-Taylor instability in the presence of a horizontal magnetic field using a decomposition method, and we find that the horizontal magnetic field contributes to the generation of vorticity inside the flow, thereby further distorting the velocity field. This differs from the previous view of the horizontal magnetic field behaving as a surface-tension-like force that does not produce any vorticity in inviscid flow. Vorticity transport is also investigated. The well-known approximate dispersion relation yields growth rates based on an irrotational approximation with a maximum error of 19% in comparison with the exact rates. Furthermore, we investigate the physics of the viscous Richtmyer-Meshkov instability in the presence of a magnetic field, and we find that the presence of the magnetic field leads to the generation of more eigenvalues, thereby modifying the motion of the interface. Comparisons confirm that the viscosity and magnetic field both play fundamental roles in interface behavior, and it is clarified that the behaviors of the interface for viscous Richtmyer–Meshkov instability become in agreement with the numerical simulations. The dependences of the eigenvalues on the viscosities and densities of the fluids, as well as on the magnetic field, are also discussed. Finally, we analyze the evolution of the decay modes to investigate the rotationality of the velocity fields.

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I. INTRODUCTION

Rayleigh-Taylor (RT) instability arises when a heavy fluid is supported by a light one in the gravity field [1,2], or when a light fluid is accelerating a heavy one, while Richtmyer-Meshkov (RM) instability arises when a perturbed interface is accelerated impulsively [3,4]. Comprehensive descriptions of these hydrodynamic instabilities can be found in the literature, together with more specific discussions of their relevance to a broad range of applications [5–10]. These instabilities have attracted much attention in the context of inertial confinement fusion (ICF), where they play a key role in determining the uniformity of compression of fusion targets [11,12]. Additionally, both RT and RM instabilities in the presence of magnetic fields have been extensively explored owing to their importance in a variety of astrophysical phenomena, including, for example, supernova explosions [13–15], material properties under extreme conditions [16–22], interactions of shocks with the interstellar medium [23,24], and the physics of accretion disks [25–27].

Given its scientific and technological significance, the inviscid RT instability in linear, nonlinear, and turbulent regimes and in different geometries has been studied in great detail for many years [1,2,28–31]. In contrast, the effects of viscosity on RT instability were not investigated until 1954, when Bellman and Pennington [32], using a decomposition method, found that the growth rate of the instability was slowed down by viscosity and obtained an approximate analytical expression. Soon afterward, using the variational method of Chandrasekhar, Hide also analyzed instability in the presence of viscosity [33]. Although the exact analysis of Bellman and Pennington was criticized by Plesset and Whipple [34], the convenience and simplicity of their approximate expression for viscous RT instability have led to its extensive application in many circumstances [35,36]. Furthermore, in ICF plasmas, viscosity has been found to play a significant role in determining the evolution of hydrodynamic instabilities and the evolution of shock wave propagation [37,38].

The presence of a magnetic field will modify the behavior of the RT instability in an ionized plasma, where a horizontal magnetic field is found to act as a surface-tension-like force, as shown by Kruskal [39] and generalized by Chandrasekhar [40]. Lau *et al.* discussed the magneto-RT instability in a multilayer system in the context of Z pinches, and they also considered the effect of feedthrough in a finite slab [41]. Sun and Piriz analyzed the effect of a horizontal magnetic field on a Z pinch, assuming that part of the target remains solid, and found that the magnetic field imparted elastic properties to the fluid and enhanced the region of stability [42].

RM instability in viscous fluids has also been explored because of its significance in determining initial perturbation amplitudes, which can clearly seed subsequent RT instability, with an effect on the uniformity of compression of ICF targets [43,44]. Such viscous effects on RM instability were studied by Mikaelian [45–47], who derived an approximate dispersion relation based on the momentum equation and found that the presence of viscosity would lead to damping of interfaces. A similar method has also been employed by Piriz *et al.* to study RM instability in solid materials, which

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was proposed to detect the shear modulus and yield strength [48–50]. Carlés and Popinet presented a quantitative model of the effect of viscosity on RM instability and extended it to the weakly nonlinear regime [51,52], where they found that their model and simulations based on it gave results for the interface evolution that were in disagreement with the results of Mikaelian's method [45]. However, by using numerical simulations, Mikaelian found that the approximate dispersion relation yielded better results for the evolution of the interface, with errors below 16% [46]. He then proposed an experimental method to determine the viscosity and yield strength of materials by detecting the damping of a perturbed interface after a shock wave, which has been successfully employed to explore the yield strength under shock wave propagation [46]. However, Mikaelian also pointed out that the exact analytical approach given by the impulsively accelerated model yields worse results for RM instability at a fluid/vacuum interface, twice as large as those from numerical simulations [46,53,54]. In contrast, the results given by the approximate dispersion relation are in better agreement with numerical simulations. Sun and Wang have recently demonstrated that the disagreement may be due to the multiple-eigenvalues effect [55], since Mikaelian considered only one of the eigenvalues with respect to the evolution of the perturbed interfaces. By taking the other mode into account, Sun and Wang have found an interface behavior that is in better agreement with the numerical simulations.

A magnetic field will suppress the growth rate of the interface after shock wave propagation. Samtaney performed a numerical study of the suppression of the RM instability in the presence of a magnetic field and found that the baroclinic generation of vorticity was not affected by the field [56]. Wheatley *et al.* investigated RM instability in the presence of horizontal and transverse magnetic fields by both theoretical modeling and numerical simulations, and they found that the magnetic field caused vorticity to be transported away from the interface, thereby preventing growth of RM instability [57–59]. They were also able to predict interface behavior over a broad range of parameters using an incompressible linearized model derived by solving the corresponding impulsively driven, linearized initial value problem, and they carried out numerical simulations to compare their results with those of the impulsively accelerated model over a broad range of parameters [59]. The role of the magnetic field is to prevent the deposition of circulation on the interface. According to the numerical analysis, when the magnetic field is parallel to the shock surface, the instability is stabilized by the Lorentz force, and vorticity is transported at the Alfvén speed [59].

Jun *et al.* performed a numerical analysis to investigate the effect of magnetic fields on mixing by the RT instability, and they found that in both the linear and nonlinear regimes, the growth of the perturbed amplitude was suppressed by the presence of a magnetic field in a single-mode perturbation [60]. Motivated by a desire to understand magnetic field amplification in supernova remnants, Sano *et al.* simulated the two-dimensional magnetohydrodynamic (MHD) RM instability for three different initial magnetic field orientations: normal, parallel, and at an angle $\pi/4$ to the interface [61]. Inoue *et al.* performed a linear analysis of RM instability of a current sheet in the relativistic regime and found that the instability could be responsible for rapid magnetic energy release in high-energy astrophysical phenomena [62]. Dong and Stone performed numerical simulations to study RT instability in viscous plasmas, taking account of the effects of horizontal and vertical magnetic fields, and found that the magnetic tension force and viscous transport tended to suppress instabilities parallel, but not perpendicular, to field lines [63]. Treating a shock as an impulsive acceleration, Sun *et al.* found that in the presence of both viscosity and surface tension, perturbations at the interface underwent damped oscillations [64]. Cao *et al.* studied the effects of shear flow and a horizontal magnetic field on RM instability and found that the magnetic field suppressed the instability and the shear flow exacerbated it [65]. Qiu *et al.* investigated the combined effects of viscosity and a horizontal magnetic field on RM instability using the viscous irrotational theory, and they found that viscosity provided damping, and the magnetic field provided damping and oscillation [66].

Although the viscous irrotational theory yields good predictions about the growth rates of RTI in infinite layers, it is noted that the exact velocity field is required to obtain the growth rates of RT instability in materials when depth of the flows under study is finite, since the velocity fields appears far from irrotational. Still, to obtain such expressions is quite essential to understanding the behaviors of the interfaces after shock wave propagations [46,48,49,53]. Craik pointed that the irrotational approximation was justified in cases where the depth of the fluids was greater than the wavelength of the perturbation, but may give incorrect results for fluids with thickness less than the wavelength of the perturbations [67]. Sun *et al.* presented a general approach to obtain exact dispersion relation of the RT instability in viscous fluids with thickness effects, where they confirmed that the irrotational approximation should not be applicable for the thin case [68], which may also provide the possibility to explore RM instability in materials [64]. More recently, Piriz *et al.* found a competition phenomenon for the thin slab which makes them less stable as the magnetic field increases, which means that the magnetic field reduces the slab relative deformation in detriment to the elasticity stabilizing efficiency and the slab becomes more unstable than when a single stabilizing mechanism is present [69–71]. In summary, the irrotational flow is suitable for RT instability in materials of thickness much greater than the wavelength of the perturbations; however, it cannot evaluate the behaviors accurately when the thickness effect is considered. Therefore, it is necessary to develop a method to obtain the exact dispersion relations, which makes it clear to understand the role of vorticity inside the flow.

The aim of this paper is to present a combined mechanism for the effects of plasma viscosity and horizontal magnetic field to study the asymptotic behavior of the instability. We will first derive the dispersion relations for magneto-RT instability in viscous fluids based on the decomposition method [64,68,71–76], which yields the exact dispersion relations about viscous RT instability in the presence of a horizontal magnetic field. It will be seen that in the presence of viscosity, the role of the horizontal magnetic field is totally different from its role in the inviscid case. The growth rates predicted by this theory are in more or less good agreement with the results given by the irrotational approximation, with errors

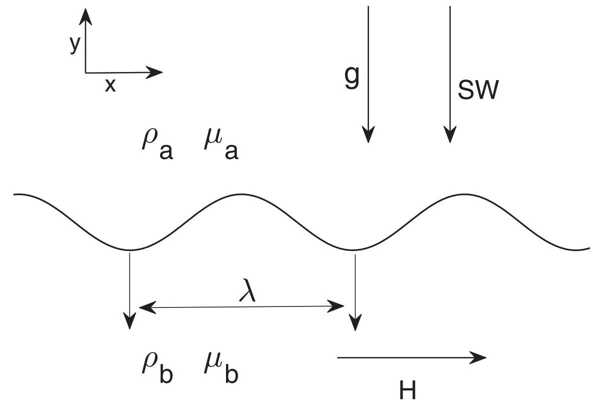

FIG. 1. Schematic description of the physical model. A magnetic field H is directed along the x axis, and SW denotes the vertical motion of a shock wave.

around 19%. In addition, we find that the presence of a magnetic field may complicate the evolution of the magneto-RM instability in a viscous fluid by generating more eigenmodes controlling the motion of the interface. We also investigate the behavior of the vorticity.

II. MATHEMATICAL FORMULATION

The physical model is shown in Fig. 1. There is an interface separating two materials with different densities ρ_α and viscosities μ_α in the y direction, with $\alpha = a, b$ indicating the upper and lower materials, respectively. $A_T = (\rho_a - \rho_b)/(\rho_a + \rho_b)$ is the Atwood number indicating the density difference of the materials. The perturbation wavelength is $\lambda = 2\pi/k$, where k is the wave number. $\mathbf{H}$ is the strength of the horizontal magnetic field in the x direction, SW represents a shock wave traveling perpendicular to the magnetic fields, and g is the downward acceleration due to gravity. The shock wave traveling downward or upward leads to a positive or negative Atwood number. Throughout this paper, we assume the viscosity to be constant and isotropic in the flow, and in particular to be independent of the directions and magnitudes of the magnetic fields. Therefore, when viscosity and the horizontal magnetic field together are taken into account, the complete two-dimensional incompressible momentum equation in the linear regime turns out to be

$$\rho\frac{\partial \mathbf{U}_i}{\partial t} = -\nabla p - \rho g + \frac{\partial p_{ik}}{\partial x_k} + \nu_0 \mathbf{J}\times\mathbf{H}, \tag{1}$$

where $\mathbf{U}_i$ is the velocity of the flow, p is the pressure, ν_0 is the magnetic permeability, $p_{ik} = \mu(\partial \mathbf{U}_k/\partial x_i + \partial \mathbf{U}_i/\partial x_k)$ is the viscous stress tensor, $\mathbf{J}$ is the electric current density, and $i = 1, 2$ depending on whether $x_i = x$ or $x_i = y$.

Below, we will focus on understanding the velocity field of the fluids in the simultaneous presence of a magnetic field and fluid viscosity. If the velocity of the fluid element under consideration is $\mathbf{U}$, then the electrical field that it experiences is not E, as measured by a stationary observer, but rather $\mathbf{E} + \nu_0\mathbf{U}\times\mathbf{H}$. According to Ohm's law, the current density is given by $\mathbf{J} = \sigma(\mathbf{E} + \nu_0\mathbf{U}\times\mathbf{H})$, where σ is the coefficient of electrical conductivity.

From the Maxwell equations, we have the Lorentz force $\mathbf{F} = (\nu_0/4\pi)\nabla\times\mathbf{H}\times\mathbf{H}$, where the magnetic field strength $\mathbf{H} = \mathbf{H}_0 + \mathbf{h}$, with $\mathbf{h}$ being the magnetic perturbation. Assuming that the conductivity is infinite, we can linearize the equation for the magnetic perturbation as

$$\frac{\partial \mathbf{h}}{\partial t} = (\mathbf{H}\cdot\nabla)\mathbf{U}_i. \tag{2}$$

We then obtain the perturbed force due to the horizontal magnetic field as

$$\mathbf{f}_y = \frac{\nu_0}{4\pi}\mathbf{H}\left(\frac{\partial h_y}{\partial x} - \frac{\partial h_x}{\partial y}\right). \tag{3}$$

Using the Helmholtz decomposition theorem and following the approach of Bellman and Pennington [32], we find that the potential functions and the stream functions are given by

$$\phi = A_\alpha e^{\pm ky+nt}\cos kxe^{nt}, \qquad \psi = B_\alpha e^{q_\alpha y+nt}\sin kxe^{nt}, \tag{4}$$

where n is the growth rate, k is the perturbed wave number of the irrotational velocity fields (which is determined by the Laplace equation), q_α is the decay mode determined by the viscosity and magnetic field together.

The perturbed velocity fields of the irrotational part are

$$\mathbf{u}_x = -\frac{\partial\phi}{\partial x}, \qquad \mathbf{u}_y = -\frac{\partial\phi}{\partial y}, \tag{5}$$

along the x and y directions, respectively, and those of the rotational part are

$$\mathbf{v}_x = -\frac{\partial\psi}{\partial y}, \qquad \mathbf{v}_y = \frac{\partial\psi}{\partial x}. \tag{6}$$

Equations (5) and (6) satisfy the requirement of incompressibility automatically.

Taking the curl of Eq. (1) and substituting Eqs. (4)–(6) into Eq. (1), we obtain the expressions for the decay modes:

$$q_\alpha^2 = \frac{\rho_\alpha n}{\mu_\alpha} + k^2 + \frac{\nu_0 H^2 k^2}{4\pi\mu_\alpha n}, \tag{7}$$

from which it can be seen that the horizontal magnetic field contributes to enhance the vorticity, in contrast to the previous conclusion that the horizontal magnetic field in the inviscid region, being equivalent to a surface-tension-like force, does not change velocity fields into irrotational ones.

By integrating Eq. (1) along the y direction from the initial amplitude $y = 0$ to the perturbed amplitude $y = \eta_{x,t} \ll \lambda$, we obtain an expression for the pressure at the perturbed interface:

$$P_\alpha = P_0 + \rho_\alpha n\phi + \frac{\rho_\alpha g}{n}\left(-\frac{\partial\phi}{\partial y} + \frac{\partial\psi}{\partial x}\right) - \frac{\nu_0 H^2 q_\alpha^2}{4\pi n}\int\frac{\partial\psi}{\partial x}\,dy, \tag{8}$$

where P_0 is the pressure at the unperturbed interface.

From the boundary conditions requiring continuity of velocities and stress tensors along the transverse and horizontal directions, we have

$$\begin{aligned} \mathbf{U}_{ax} &= \mathbf{U}_{bx}, \quad \mathbf{U}_{ay} = \mathbf{U}_{by}, \\ \tau_{axy} &= \tau_{bxy}, \quad \tau_{ayy} = \tau_{byy}, \end{aligned} \tag{9}$$

where $\mathbf{U}$ is defined as the total velocity including both irrotational and rotational parts (so, e.g., $\mathbf{U}_x = \mathbf{u}_x + \mathbf{v}_x$), and τ is the stress tensor including the contributions from both the viscosity and the magnetic field. Therefore, we have the following four equations:

$$A_a + B_a + A_b - B_b = 0, \tag{10}$$

$$A_a k + B_a q_a - A_b k + B_b q_b = 0, \tag{11}$$

$$2\mu_a k^2 A_a + \mu_1 (q_a^2 + k^2) B_a + 2\mu_b k^2 A_b - \mu_b (q_b^2 + k^2) B_b = 0, \tag{12}$$

$$\left(\rho_a n + \frac{\rho_a - \rho_b}{n} gk + 2\mu_a k^2 + \frac{\nu_0 H^2 k^2}{4\pi n}\right) A_a + \left(\frac{\rho_a - \rho_b}{n} gk + 2\mu_a q_a k\right) B_a - \left(\rho_b n + 2\mu_b k^2 + \frac{\nu_0 H^2 k^2}{4\pi n}\right) A_b + 2\mu_b k q_b B_b = 0. \tag{13}$$

These have nontrivial solutions if and only if the determinant of the coefficients vanishes; then we have the dispersion relation

$$\left(n^2 - A_T gk + \frac{2}{\rho_a + \rho_b} \frac{\nu_0 H^2 k^2}{4\pi}\right) \times \left(\frac{1}{\mu_a k + \mu_b q_b} + \frac{1}{\mu_a q_a + \mu_b k}\right) + \frac{4nk}{\rho_a + \rho_b} = 0, \tag{14}$$

From Eq. (14), it is not hard to obtain the cutoff wave number by assuming growth rate to be zero ($n = 0$), which gives

$$k_c = \frac{2\pi(\rho_a - \rho_b)}{\nu_0 H^2}. \tag{15}$$

We can now retrieve the results of the irrotational theory by taking $q_\alpha = k$ (with $\alpha = a, b$) in Eq. (14) [74], which yields the following dispersion relation:

$$n^2 - A_T kg + \frac{2}{\rho_1 + \rho_2} \frac{\nu_0 H^2 k^2}{4\pi} + \frac{2n(\mu_1 + \mu_2)}{\rho_1 + \rho_2} k^2 = 0, \tag{16}$$

which turns out to be the same as that obtained by Qiu *et al.* [66]. Still, it is very interesting to have the cutoff wave number from Eq. (16), which turns out to be exactly the same as what we obtained from Eq. (15).

Before discussing the physical details of magneto-RT and RM instabilities in viscous fluids, we first obtain insights about the evolution of the vorticity by taking the curl of the linearized momentum equation (1). This gives the vorticity equation

$$\rho \frac{\partial \Omega}{\partial t} = \mu \nabla^2 \Omega + \frac{\nu_0 H^2}{4\pi n} \frac{\partial^2 \Omega}{\partial x^2}, \tag{17}$$

where $\Omega = \partial u_y/\partial x - \partial u_x/\partial y$ is the vorticity inside the fluid. Equation (17) reduces to the diffusion equation by assuming $H = 0$. However, the vorticity vanishes when $H = 0$, since the velocity field now becomes irrotational.

In fact, Sun and Piriz [42] used the approximately irrotational velocity fields to study the magneto-RT instability in solid media, where they found that a horizontal magnetic field acting as a surface-tension-like force will suppress the growth rate and result in a critical wave number. It can be seen that Eq. (14) turns out to be exactly the same as the equation that Bellman and Pennington obtained for viscous RT instability when $\mathbf{H} = 0$ [32]. For viscous RM instability in the presence of a horizontal magnetic field, Qiu *et al.* used Richtmyer's impulsively accelerated model to obtain an approximate dispersion relation [66], which was the same as our Eq. (16). Noted that the impulsive model is valid only for weak shocks with Mach number close to unity. For stronger shocks, the vorticity created by the compressibility plays a major role.

However, neither of the above studies considered the coupled effects of vorticity and magnetic field on the perturbations of the velocity fields, and they were unable to reveal the enhancement of vorticity due to the presence of a horizontal magnetic field. In the present paper, it has been shown that a horizontal magnetic field enhances the vorticity generated by the viscosity, and, furthermore, that the existence of irrotational velocity fields may cause larger errors in the calculated values of the growth rate in comparison with those obtained from an exact analysis. However, there is still a lack of a detailed analysis of the combined effect of a horizontal magnetic field and viscosity on the flow.

In the next section, we will analyze the effects of RT and RM instabilities in an ionized fluid in the presence of a horizontal magnetic field and viscosity.

III. RESULTS AND DISCUSSION

If the horizontal magnetic field is neglected, an approximate approach assuming irrotational velocity fields yields very good predictions about the growth rate of viscous RT instability, with the maximum errors being within 11% [32]. The assumption of irrotational velocity fields still gives exact values for the growth rates of magneto-RT instability in the presence of horizontal magnetic fields, since these fields just act as surface-tension-like forces without altering the velocity fields [40]. In this section, we first analyze the combined dependence of the growth rates on magnetic field and viscosity, to check the validity of the approximate approach to RT instability. Then, we will use the exact dispersion relations to explore RM instability in this case, and we compare the results with those obtained using the impulsively accelerated model [66].

A. Rayleigh-Taylor instability

From Eq. (4), it is straightforward to see that the decay mode is determined by the viscosities of the fluids and the magnetic field strength simultaneously, which seems to indicate that in this case, the irrotational approximation may give rise to growth rates with larger errors in comparison with the exact analysis. To check this, we make some comparisons between the two approaches.

In Fig. 2, we compare the exact results for the growth rate, given by Eq. (14) and shown by the solid lines, with the approximate results, given by Eq. (16) and shown by the dotted lines. For fixed Atwood number $A_T = 0.5$, the dimensionless growth rate $\sigma = n\mu_a^{1/3}/(\rho_a g^2)^{1/3}$ is plotted as a function of the

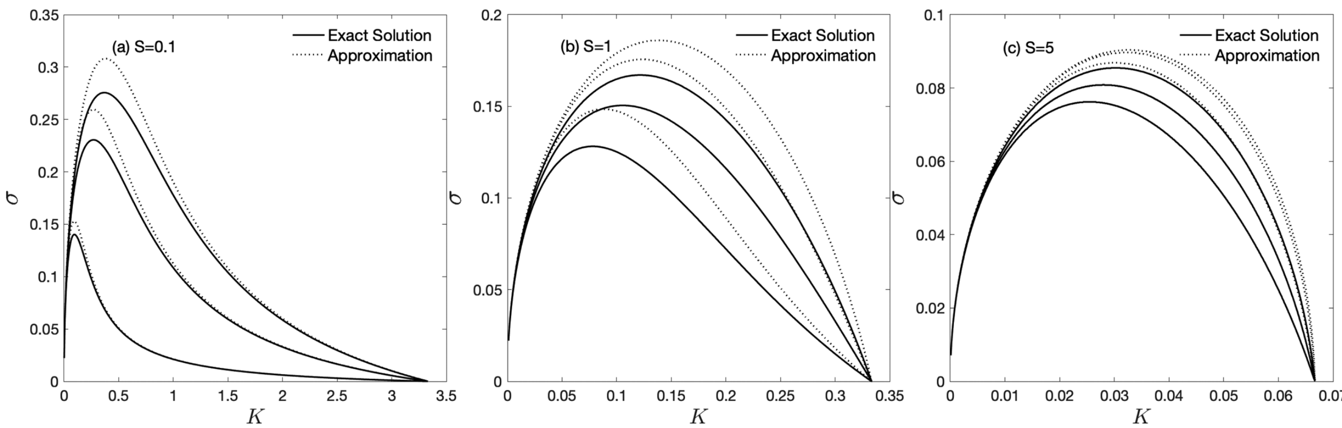

FIG. 2. Dimensionless growth rate σ as a function of dimensionless wave number K for $A_T = 0.5$, $S = 0.1$ (a), 1 (b), and 5 (c), and $m = \mu_b/\mu_a = 0.1, 1, 10$ from top to bottom.

dimensionless wave number $K = k[\mu_a^2/(\rho_a^2 g)]^{1/3}$ for different values of the viscosity ratio $m = \mu_b/\mu_a$ and the dimensionless parameter

$$S = (\nu_0 H^2/4\pi)\left[1/\left(\rho_a g^2 \mu_a^2\right)\right]^{1/3}, \tag{18}$$

which is related to the magnetic field strength and the viscosities of the fluids.

We start by discussing the effects of the parameter S on viscous RT instability in the presence of a horizontal magnetic field. As can be seen, both theories give exactly the same cutoff wave number, which again confirms the validity of the results of Eq. (15). It is well known that for viscous RT instability, the approximate approach yields a maximum error about 11% at the most-unstable wave number [32,40]. However, it can be seen in Fig. 2 that the approximate method always overestimates the growth rate by a maximum error of about 19% for the magneto-RT instability in viscous fluids, which is interesting since the magnetic field contributes to the distortion of the velocity fields by introducing irrotationality. Another interesting finding is that in previous studies of RT instability at solid/solid, solid/liquid, and liquid/liquid interfaces, both theories have tended to give the same growth rates at the smallest and the largest wave numbers, which was taken by Piriz *et al.* to indicate that RT instability is not sensitive to the details of the perturbed velocity fields for the smallest wave numbers and that the velocity field tends to be irrotational for the largest ones [77]. Comparisons of the growth rates from both theories with different S and m are shown in Fig. 2, where first we see that the exact method gives almost the same growth rates at the smallest and largest wave numbers for $m = 0.1$ in Fig. 2(a), since the magnetic field has little effect on the evolution of the growth rate. However, as S increases, a greater discrepancy appears between the growth rates from the two methods, even at the largest wave number, since the effect of the magnetic field becomes dominant, as shown in Figs. 2(b) and 2(c). Note that for sufficiently large S, the effect of m cannot be discriminated well at small m with the approximate method. In general, the approximate method overestimates the growth rate with a maximum error of 17% at the most unstable wave number, and the curves have the same critical wave numbers, since these are independent of the viscosity ratio m.

Figure 3 again plots the exact and approximate results for the growth rate versus wave number K, but this time for a fixed value of the parameter S as shown in Eq. (18) and different values of the viscosity ratio and the Atwood number A_T. It can be seen that the approximate theory still overestimates the growth rates by up to 17% at the most unstable wave number. Both methods again predict the same cutoff wave numbers, since these are always independent of the parameter S. It is also interesting that the approximate theory gives better predictions of the growth rates at the smallest and the largest wave numbers. It can be seen that the Atwood number A_T has a uniform effect on growth rate, with σ increasing with A_T. We also find that as the density ratio $m = \mu_b/\mu_a$ increases, the growth rate decreases for all values of A_T, although this effect is particularly strong effect at large A_T.

It can be seen from Fig. 4 that a large viscous ratio $m = 5$ will significantly suppress the growth rate at larger wave numbers, and therefore both approaches produce more or less the same growth rates at large wave numbers. It is clear that the growth rates increase with A_T and with S.

In this subsection, we have compared the growth rates from the exact analysis with those from the approximation assuming irrotational velocity fields, and we have found a maximum error of 19%, which much larger than the error of 11% obtained in the presence of viscosity alone [32], since the horizontal magnetic field will further distort the velocity field. If we only take the effect of the horizontal magnetic field into account, both theories yield exactly the same growth rates, since a horizontal magnetic field alone does not alter the irrotationality of the velocity field. However, in the case of nonzero viscosity, the horizontal magnetic field does distorts the velocity fields by adding additional vorticity to the flows, leading to larger discrepancies in growth rates between the two theories.

In conclusion, a horizontal magnetic field will behave like a surface-tension force only in the absence of viscosity [39,40]. In this inviscid case, its effects will provide a critical wavelength beyond which the interface will remain stable. However, in the presence of viscosity, a horizontal magnetic field will enhance the vorticity gen-

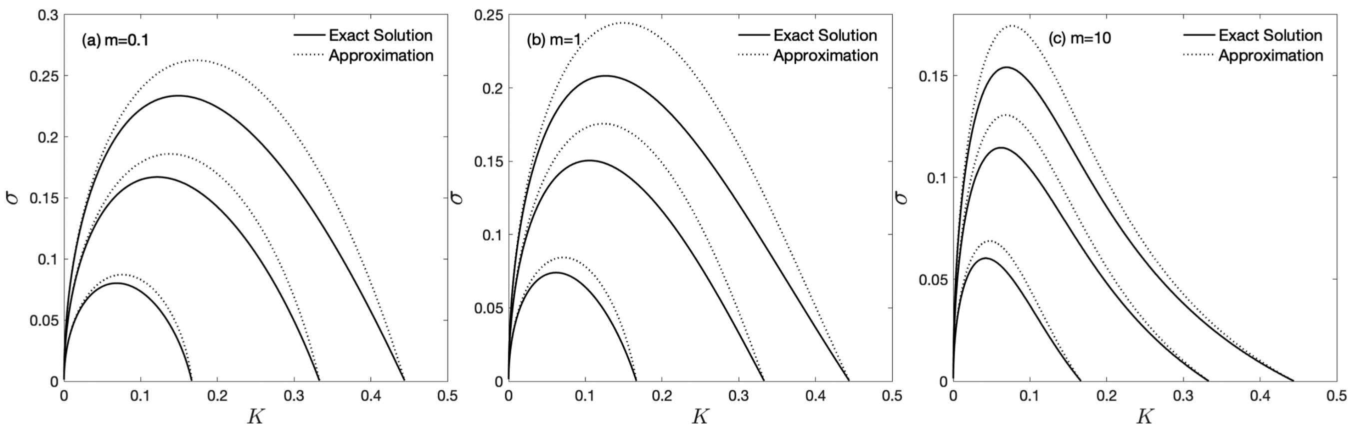

FIG. 3. Dimensionless growth rate σ as a function of dimensionless perturbation wave number K for parameter $S = 1$, viscous parameter $m = \mu_b/\mu_a$ 0.1 (a), 1 (b), 10 (c), and Atwood number $A_T = 0.2, 0.5, 0.8$ from bottom to top.

erated by the viscosity, affecting the growth rate of RT instability.

B. Richtmyer-Meshkov instability

The main purpose of this paper is to investigate MHD instabilities in viscous fluids, especially with regard to the enhancement of vorticity in the presence of a horizontal magnetic field. Wheatley *et al.* carried out a thorough analysis of the effects of magnetic fields on RM instability on the basis of an impulsively accelerated model using a Laplace transform technique [59]. They found that in the presence of a horizontal magnetic field, the perturbation amplitude of the interface oscillates with time. Oscillatory behavior of RM instability was also found in the presence of surface tension [45,59] and ablative pressures [43]. Additionally, Mikaelian performed a series of analyses of RM instability in viscous fluids using the momentum equation and assuming irrotational velocity fields [45–47,53]. He found that the interface exhibited damping behavior. By analyzing interface behavior in the case of RM instability in elastic materials, Ortega *et al.* observed two different patterns of motion: standing waves and oscillatory decay [78]. Recently, however, Sun *et al.* have found that the interface may exhibit damping oscillations, with its motion then being controlled by a multiple-eigenvalue effect after shock wave propagation [55,64]. We have discussed above the effect of vorticity on the dispersion relations of RT instability, from which it is easily seen that vorticity plays a significant role in modifying the growth of the instability. We now focus on the role of the magnetic field in RM instability, since Mikaelian has already discussed some interesting cases related to viscous RM instability [46,53].

1. RM instability in viscous fluids

The behavior of an interfaces under RM instability in the presence of a horizontal magnetic field and viscosity was investigated by Qiu *et al.* [66], who obtained the equation of motion of the interface based on the irrotational approximation and found that damped oscillations were generated. However, as described elsewhere [56–59], vorticity transport may play an essential role in determining interface behavior in this situation. Indeed, the fundamental difference between the irrotational approximation and the present model is the role of vorticity [57–59].

Before discussing RM instability in the presence of magnetic fields and viscosity in fluids, we consider RM instability in viscous fluids, for which the dispersion relation (14) becomes (under the assumption $\mathbf{H} = 0$ and by

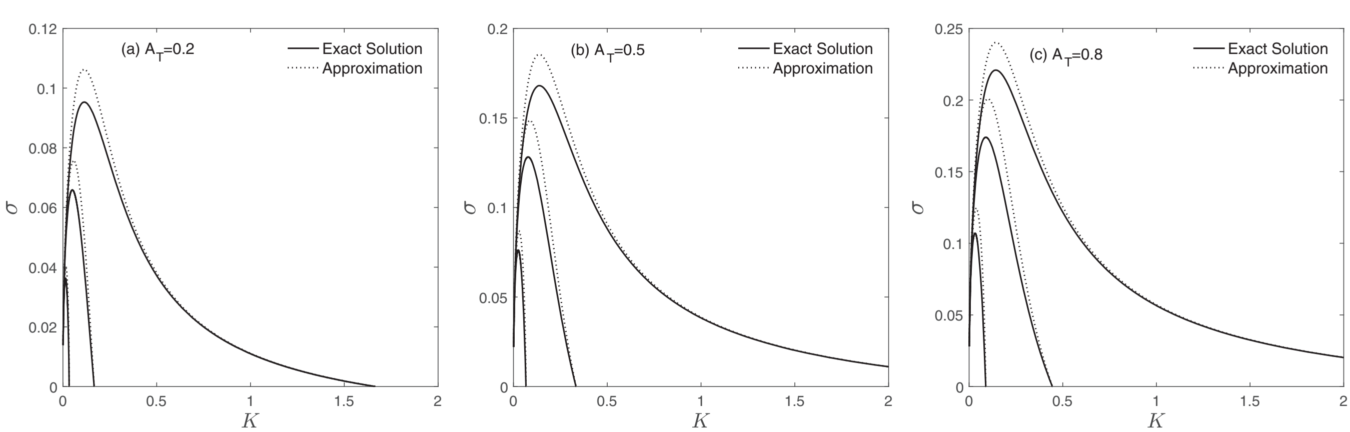

FIG. 4. Dimensionless growth rate σ as a function of dimensionless perturbation wave number K for $m = \mu_b/\mu_a = 5$, $A_T = 0.2$ (a), 0.5 (b), 0.8 (c), and $S = 0.1, 1, 5$ from top to bottom.

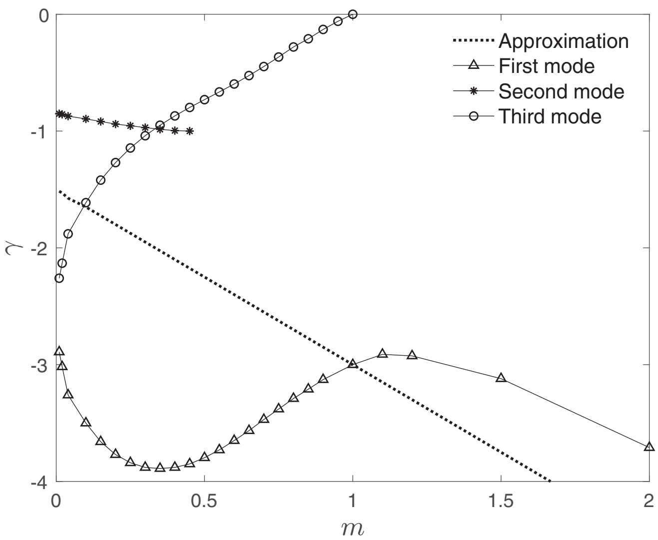

FIG. 5. Dimensionless damping factor $\gamma = \rho_a n/(\mu_a k^2)$ as a function of m for $A_T = 0.5$.

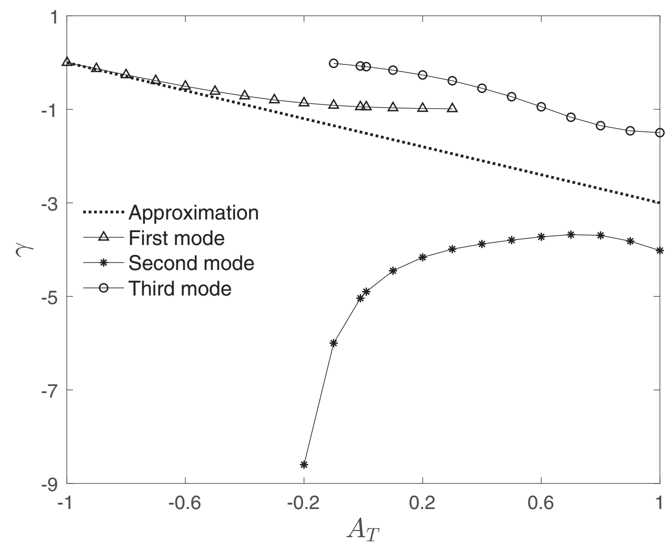

FIG. 6. γ as a function of A_T for viscous parameter $m = \mu_b/\mu_a = 0.5$.

considering $g = 0$)

$$n\left(\frac{1}{\mu_a k + \mu_b q_b} + \frac{1}{\mu_a q_a + \mu_b k}\right) + \frac{4k}{\rho_a + \rho_b} = 0, \quad (19)$$

where in this case the decay modes are defined as $q_\alpha = \sqrt{k^2 + \rho_\alpha n/\mu_\alpha}$. An approximate explicit form of Eq. (19) is then

$$n + 2\frac{\mu_a + \mu_b}{\rho_a + \rho_b}k^2 = 0, \quad (20)$$

which is in agreement with the expression obtained by Mikaelian [45,46]. The typical forms of n in Eq. (19) are complex numbers, whose real parts of negative values contribute mainly to control the behaviors of the interface. To study those dominating factors, we introduce γ representing the real parts of the eigenvalues.

The evolution of the dimensionless damping factor γ in a viscous fluid is shown in Fig. 5, where the dotted line represents the results of the approximate model (20) and the solids lines show one series of damping factors obtained from Eq. (19). A similar result was found for RM instability in elastic solids, for which Ortega *et al.* [78] discovered a discontinuity in the oscillation period at $m = 1$. We will see that for $m > 1$, the evolution of the interface becomes dominated by a single eigenvalue with value close to that predicted by Eq. (20). It should be borne in mind that use of the full expression for the growth rate, both its real and imaginary parts, gives better results for the evolution of the interface than the approximate expression [55,64].

Another topic worth investigating is the dependence of the damping factors on the Atwood number, which has been studied by Qiu *et al.* [66]. Here, we focus mainly on the characteristics of an interface after a shock wave propagates. It is therefore of great interest to examine in detail the evolution of the damping factors as shown in Fig. 6, where the dotted line represents the approximate decay rates of Eq. (20) and closely matches the results of the exact analysis at the initial stage. As A_T increases, more eigenvalues appear to affect the evolution of the interface, while the approximate eigenvalues decrease monotonically and always lie between the different eigenvalues. It should be noted that the values of the approximate growth rates are close to the average of the multiple eigenvalues, which could indicate that the approximate eigenvalues would give quite good predictions about the evolution of the interface in comparison with numerical simulations [46,47,53].

2. Magneto-RM instability in viscous fluids

It is necessary to delve into the RM instability in viscous fluids in the presence of a horizontal magnetic field, which is indeed one of the main motivations for the present paper.

First, we consider the dispersion relation for viscous RM instability in the presence of a horizontal magnetic field by considering $g = 0$, which reads

$$\left(n^2 + \frac{2}{\rho_a + \rho_b}\frac{\nu_0 H^2 k^2}{4\pi}\right)\left(\frac{1}{\mu_a k + \mu_b q_b} + \frac{1}{\mu_a q_a + \mu_b k}\right) + \frac{4nk}{\rho_a + \rho_b} = 0. \quad (21)$$

Equation (21) admits 12 complex solutions to be fully discussed in the proceeding part. It is straightforward to obtain the following approximate dispersion relation by assuming $q_a \approx k$ and $q_b \approx k$:

$$n^2 + \frac{2}{\rho_a + \rho_b}\frac{\nu_0 H^2 k^2}{4\pi} + 2k^2\frac{\mu_a + \mu_b}{\rho_a + \rho_b}n = 0. \quad (22)$$

This turns out to be the same as that obtained by Qiu *et al.* [66].

As can be seen from Fig. 7, the behavior of the interface is dominated by just one eigenvalue $n = -3.71 + 1.25i$ from Eq. (21), with dimensionless magnitudes $A_T = 0.5$ and $m = 2$, while the approximate eigenvalue from Eq. (22) is $n = -4.5$. It should be noted that owing to the presence of the magnetic field, other eigenvalues also appear in the dispersion relation, but, as can be seen from Fig. 7, they have much

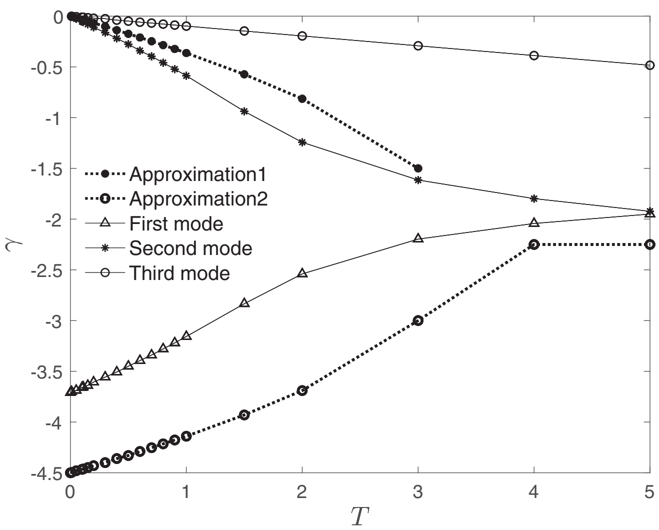

FIG. 7. γ as a function of $T = \rho_a \nu H^2/4\pi\mu_a^2 k^2$ for $A_T = 0.5$ and $m = 2$.

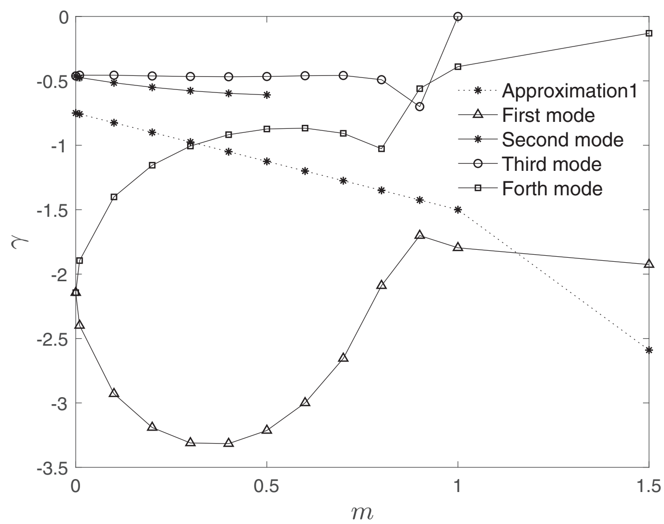

FIG. 8. γ as a function of m for $A_T = 0.5$ and $T = 2$.

smaller decay factors and little impact on the motion of the interface. Only the first two eigenvalues have any significant effect on the interface, and we focus our attention on these. In Fig. 7, the first mode corresponds to the eigenvalue that already arises for viscous RM instability, which means that the damping behavior is weakened by the presence of the horizontal magnetic field, and the approximate eigenvalues with smaller decay factors disappear for $T > 3$. Another feature is captured by the approximate dispersion relation Eq. (22), where the damping factors remain constant as m increases, and in fact the exact dispersion relation Eq. (21) shows the same trend. This seems to indicate that an increase in the magnetic field intensity will lead to a decrease in damping. It should be noted that the behavior of the interface is determined by the full expression for the growth rate, including both the real and imaginary parts, which is why the exact evolution turns out to be in good agreement with the approximate one even though the damping factors are quite different [55].

By solving Eq. (21), it is found that the presence of a magnetic field will generate new eigenvalues (the top three lines in Fig. 7 compared with the case where there is no magnetic field. As can be seen in Fig. 7, two approximate eigenvalues (the increasing one represented by the dotted line with triangles and the decreasing one represented by the dotted line with asterisks) exist in the system, and they converge at $T = 3$, after which only one mode survives and (with account taken of the real part) remains constant. Also of interest are the exact eigenvalues from Eq. (21) represented by the solid lines, the lowest of which can be derived from the case $T = 0$. However, for the other two cases, the eigenvalues are generated from nowhere and determine the evolution of the interface. To sum up, the presence of a horizontal magnetic field will generate more eigenvalues, which will further complicate the motion of the interface.

We have already shown the behavior of the damping factors as a function of the viscosity ratio m. In particular, we analyzed the discontinuity in the parameter γ at $m = 1$, which occurs when the behaviors of the interface becomes dominated by the eigenvalue with least damping power. This phenomenon was first observed by Ortega *et al.* [78] for RM instability in elastic solids. Considering the similarity between the dispersion relations for RM instability in elastic solids and in viscous fluids, Sun *et al.* [55,64] showed that there are a greater number of eigenvalues to complicate the motions in the case of RM in viscous fluids and elastic solids [78] for $m \leqslant 1$. For $m > 1$, the approximate dispersion relations turn out to give good predictions in comparison with the exact ones [48,49,53]. The dimensionless damping factor is shown as a function of m in Fig. 8, where it can be seen that only one series of eigenvalues from Eq. (21) exists, represented by the dotted line with asterisks, and γ decreases monotonically with m. It is very interesting to note that in Fig. 8, there are three eigenvalues $n_1 = -0.46 + 1.48i$, $n_2 = -0.39 + 0.69i$, and $n_3 = -2.14 + 2.70i$ when $m = 0$. The system has six eigenvalues, and each of the modes for $m = 0$ will split into two modes with different rules of evolution for $m > 0$. It is observed that one of the modes disappears at $m = 1$, which corresponds to the viscous and elastic RM instabilities. For convenience, to describe the evolution of the eigenvalues, we choose only four out of the six.

The dimensionless damping factor is shown as a function of the Atwood number in Fig. 9, where Fig. 9(a) shows the first exact mode (solid line) and the approximate mode (dotted line), and Fig. 9(b) shows the other three modes (solid lines) and the other approximate mode (dotted line). It can be seen that in Fig. 9(a), both lines have similar trends of evolution, and this mode will dominate the development of instability, which explains why the approximate mode may give a very good prediction of interface behavior. In Fig. 9(b), it is very interesting to see that the approximate mode appears around $A_T = -0.5$ and the corresponding exact one at around $A_T = -0.6$, and the other two exact modes possess less damping power in comparison with the other modes. However, how those modes interact with each other in the nonlinear regime and what roles they play in determining interface evolution remain unclear, and these are topics that deserve further

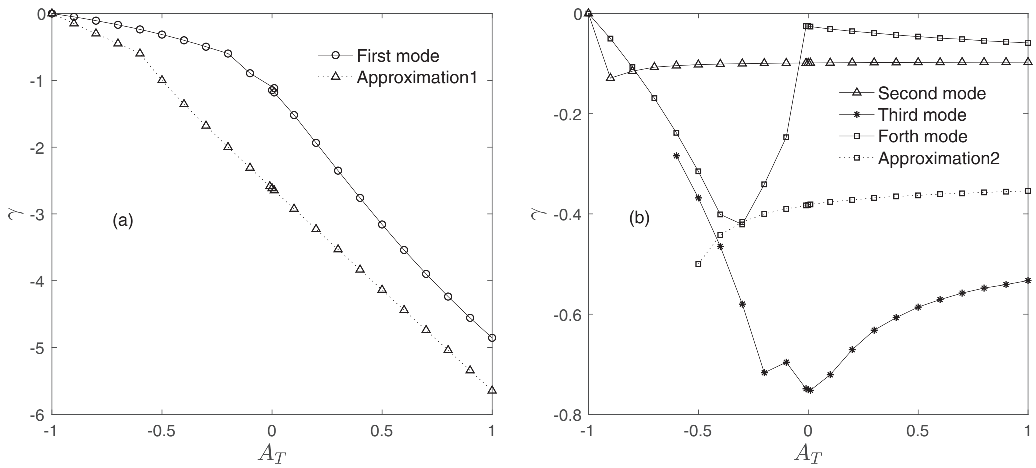

FIG. 9. γ as a function of A_T for $T = 1$ and $m = 2$: (a) first mode and (b) other modes.

theoretical and numerical investigation. Still, it seems to us that the approximate eigenvalue may provide a pretty good prediction of interface evolution.

To sum up, we have discussed here the evolution of RM instability in the presence of a horizontal magnetic field for viscous fluids, which is quite different from the evolution in the inviscid case. It is known that a horizontal magnetic field acts as a surface-tension-like force to suppress interface growth, and we have found here that for nonzero viscosity such a field may modify the vorticity in the flow and further affect the evolution of RM instability.

C. Velocity fields

The viscous RT instability was first discussed by Bellman and Pennington using the decomposition method and continuous boundary conditions on the velocities and stress tensors parallel and perpendicular to the perturbed interface [32]. They gave the dispersion relations and, in particular, derived an approximate dispersion relation that has since been widely utilized to study RT instability in various situations. A similar methodology has been employed to study RM instability in different materials, such as viscous fluids, elastic materials, and elastic-plastic materials, as well as in the presence of magnetic fields [45,46,48,49,53,55,64]. The growth rates yielded by the approximate dispersion relations are in good agreement with the exact ones, with maximum errors around 11% [32], and the same applies to the elastic RT instability with maximum errors around 16% [79]. According to Piriz *et al.*, there is agreement between the results of the approximate and exact methods because RT instability is relatively insensitive to the details of the perturbed velocity field and the growth rates tend to be the classical ones of the inviscid case for the smallest wave numbers k [79]. On the other hand, for the largest values of k, the perturbed velocity field appears to be irrotational.

Figure 10 shows the ratio of the decay mode q_a and the perturbed wave number k as a function of the parameter K for fixed $S = 0$, from which it can be seen that for the smallest K, the ratio turns out to be much larger than the perturbed wave number. This means that the velocity field is far from irrotational; however, as pointed out by Piriz *et al.* [77], RT instability is quite insensitive to the details of the velocity field. By contrast, for sufficiently large $K > 1$, the decay mode approaches the value of the perturbed wave number, which means that the velocity field becomes irrotational.

Figure 11 shows the ratio between the decay mode q_a and the perturbed wave number k as a function of the parameter K for different values of S. In the presence of both viscosity and a horizontal magnetic field velocity field, the behavior of the growth rate of the RT instability is very different from that in the absence of the magnetic field, especially for large S. The decay mode becomes very different from the perturbed wave number, with a ratio that grows rapidly with increasing K and becomes more than ten times larger than unity, which indicates that the velocity field is no longer irrotational at the largest k.

IV. CONCLUSIONS AND REMARKS

In this paper, we first obtained the dispersion relations for viscous RT and RM instabilities in the presence of a magnetic

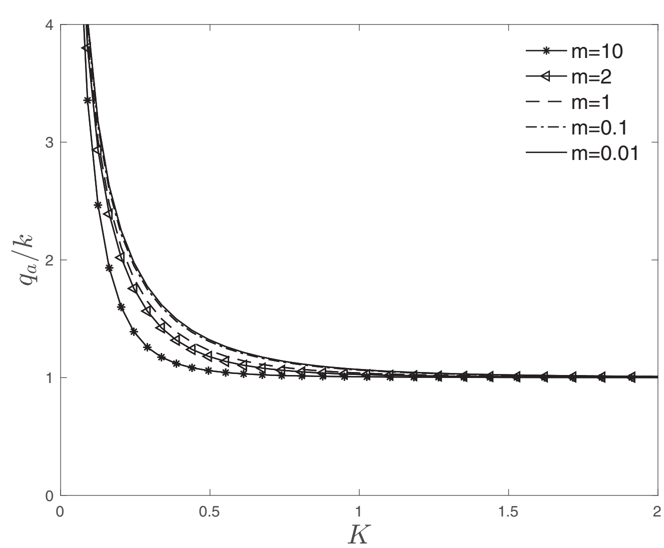

FIG. 10. Ratio of decay mode q_a and wave number k as a function of the parameter K for $S = 0$ and different m.

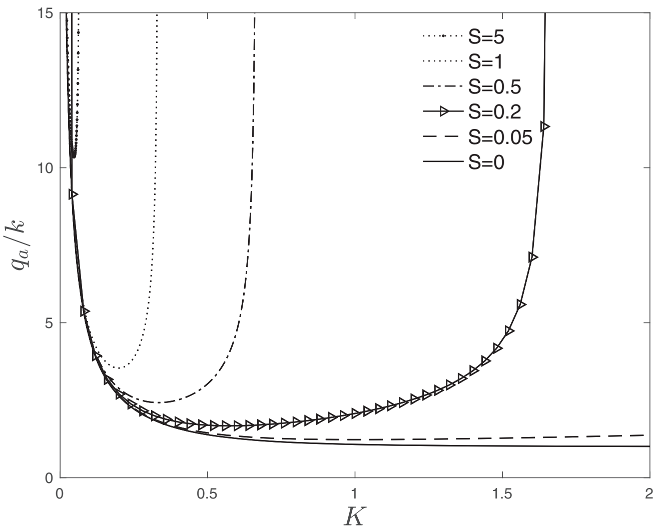

FIG. 11. Ratio of decay mode q_a and wave number k as a function of the parameter K for $m = 1$ and different S.

field by using a decomposition method, and we found that a horizontal magnetic field modifies the vorticity of the flow and the interface behavior. This is in contrast to what happens in the absence of viscosity, when a horizontal magnetic field only suppresses interface growth but does not generate vorticity. It is found that for viscous RT instability in the presence of a horizontal magnetic field, the maximum error in the growth rate obtained using an approximate dispersion relation compared with the growth rate obtained using the exact relation is larger (19%) than the error in the absence of a magnetic field: 19% versus 11%. This is due to the distortion of the velocity fields caused by the magnetic field.

By solving the dispersion relation for RM instability in viscous fluids in the presence of a magnetic field, we obtained complex-valued expressions, although we focused our analysis on the role of the real parts. It can be seen that for RM instability in fluids in the absence of a magnetic field, there exists more than one eigenvalue affecting the motion of the interface, and the approximate eigenvalues are always located between these. This may explain why the approximate method gives quite good predictions of interface behavior. We obtained the growth rates for RM instability in viscous fluids in the presence of a magnetic field, and it can be seen that the effect of the magnetic field is to generate more eigenvalues, thereby complicating the behavior of the system as a whole. The appropriate growth rates then give the average behaviors of all the exact eigenvalues. However, the interactions between the eigenvalues, as well as the nonlinear interface behavior, remain poorly understood, and we hope that future numerical simulations will clarify these aspects.

Equation (17) describes the transportation of vorticity, which further affects the motion of the interface. Moreover, this method enables us to shed light on the behaviors of the interface in three dimensions (3D). In the linear regime, the motion of the interface in two dimensions (2D) is not different from its 3D counterpart, because the vorticity equations have the same form. However, in the nonlinear regime, the instability in 3D starts to grow faster than that in 2D, since the vorticity equations differ.

It was pointed out by Piriz *et al.* [77] that the irrotational approximation for viscous RT instability may yield good approximations of the growth rates of this instability, since it is insensitive to the details of the velocity fields for the smallest k, while the velocity field approaches irrotationality for the largest k. In the present paper, it has been found that the irrotational approximation gives very good predictions for the dispersion relation and the exact cutoff wave numbers. However, we have found that the velocity field is far from irrotational for large S.

In summary, the method presented here provides a powerful tool to investigate the linear regime of RT and RM instability in viscous fluids in the presence of a magnetic field. We have found that the approximate method based on the irrotational flow may give rough predictions for the growth rate within 19% error at the most unstable wavelength for RT instability and that the presence of a magnetic field will generate more eigenvalues. In addition, we have found that the approximate eigenvalues may give good predictions for interface evolution in comparison with an exact analysis. However, the dependence of the initial velocity on the multiple eigenvalues and the effects of the growth rates in the nonlinear regime still remain mysterious. We hope that these issues will be clarified by future numerical analysis. Equally important, further analysis of vorticity transport should clarify the mixing behaviors of RT and RM instabilities. We also hope that future numerical simulations will reveal the effects of multiple eigenvalues on the evolution of RM instability.

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