Hearing Euler characteristic of graphs

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The Euler characteristic $\chi = |V| - |E|$ and the total length $\mathcal{L}$ are the most important topological and geometrical characteristics of a metric graph. Here $|V|$ and $|E|$ denote the number of vertices and edges of a graph. The Euler characteristic determines the number β of independent cycles in a graph while the total length determines the asymptotic behavior of the energy eigenvalues via Weyl's law. We show theoretically and confirm experimentally that the Euler characteristic can be determined (heard) from a finite sequence of the lowest eigenenergies $\lambda_1, \ldots, \lambda_N$ of a simple quantum graph, without any need to inspect the system visually. In the experiment quantum graphs are simulated by microwave networks. We demonstrate that the sequence of the lowest resonances of microwave networks with $\beta \leqslant 3$ can be directly used in determining whether a network is planar, i.e., can be embedded in the plane. Moreover, we show that the measured Euler characteristic χ can be used as a sensitive revealer of the fully connected graphs.

I. INTRODUCTION

The problem of the seven bridges of Königsberg considered by Leonhard Euler in 1736 [1] was one of the most notable mathematical achievements which laid the foundations of graph theory and topology. In 1936 this seeding idea was used by Linus Pauling in physics [2] in order to describe a quantum particle moving in a physical network, the model known today as a quantum graph.

The idea of quantum graphs was further extensively developed in Refs. [3–7]. In the considered model a metric graph $\Gamma = (V, E)$ is formed by the edges $e \in E$ connected together at the vertices $v \in V$. Each edge is seen as an interval on (a separate copy of) the real line $\mathbb{R}$ having the length l_e, then the vertices can be defined as disjoint unions of edge endpoints. Let us consider the Laplace operator $L(\Gamma) = -\frac{d^2}{dx^2}$ acting in the Hilbert space of square integrable functions on Γ satisfying in addition the standard vertex conditions (also called natural, Neumann, or Kirchhoff): the function is continuous at the vertex v, and the sum of oriented derivatives at the vertex v is equal to zero. Such a Laplacian is uniquely determined by the metric graph and is self-adjoint, and its spectrum is purely discrete [6]. Moreover, the operator is non-negative with zero being a simple eigenvalue (provided the graph is connected) with the eigenfunction given by the constant. For more details on quantum graphs, we can refer the reader to the book [6] and the references therein. Quantum graphs were used to simulate, e.g., mesoscopic quantum systems [8,9], quantum wires [10], and optical waveguides [11].

In this article we report breakthrough results on the topology of quantum graphs and microwave networks. We show that measuring several dozen of eigenvalues of the system one may recover its Euler characteristic without seeing a graph, i.e., knowing the number of the graph's vertices and edges. In particular one may even determine structural properties of the network, e.g., whether the graph is planar or fully connected.

The original formula for χ [12] requires knowledge of all eigenenergies of the system and plays a very important role in the study of inverse problems for quantum graphs, but its applicability to laboratory measurements is limited, since only a finite number of eigenenergies can be obtained in any real world experiment.

From the experimental point of view it is important to point out that quantum graphs can be modeled by microwave networks [13–18]. It is attainable because both systems are described by the same equations: the one-dimensional Schrödinger equation appearing in quantum graphs is formally equivalent to the telegrapher's equation for microwave networks [13,16]. Microwave networks, as the only ones, allow for the experimental simulation of quantum systems corresponding to all three classical ensembles in the random-matrix theory (RMT): the systems with T invariance belonging to a Gaussian orthogonal ensemble (GOE) [13–15,17,19] and Gaussian symplectic ensemble (GSE) [20], and the systems without T invariance belonging to a Gaussian unitary ensemble (GUE) [13,18,21–24].

Microwave networks were successfully used, e.g., to demonstrate the usefulness of missing level statistics in variety of applications [23] and to show that there exist graphs which do not obey a standard Weyl's law, called non-Weyl graphs [19].

The most important characteristics of a metric graph $\Gamma = (V, E)$ are the Euler characteristic $\chi = |V| - |E|$ and the total length $\mathcal{L} = \sum_{e \in E} l_e$. The Euler characteristic χ determines the number β of independent cycles in a graph

$$\beta = |E| - |V| + 1 \equiv 1 - \chi, \quad (1)$$

while the total length $\mathcal{L}$ determines the asymptotics of a graph's eigenvalues λ_n via the Weyl's formula

$$\lambda_n = \left(\frac{\pi}{\mathcal{L}}\right)^2 n^2 + O(n), \quad (2)$$

where $O(n)$ is a function which in the limit $n \to +\infty$ is bounded by a constant. The number of independent cycles measures how different a graph is from a tree and is equal to the number of edges that have to be deleted to turn the graph into a tree.

It might seem that the determination of both characteristics would require the knowledge of the whole sequence of eigenvalues. Such an assumption is natural in mathematics and allows one to derive the precise formulas for $\mathcal{L} = \pi \lim_{n \to +\infty} \frac{n}{k_n}$ and $\chi = X(t)|_{t \geqslant t_0}$ [12,25], where

$$X(t) := 2 + 2\pi \sum_{n=1}^{\infty} \cos(k_n/2t) \left[\frac{\sin(k_n/4t)}{k_n/4t} \right]^2, \quad (3)$$

and k_n are the square roots of the eigenenergies λ_n and $t_0 = \frac{1}{2l_{\min}}$ with $l_{\min}$ being the length of the shortest edge of a simple graph. While derivation of the formula for $\mathcal{L}$ is elementary, formula (3) can be obtained either from the trace formula [26–28] connecting the spectrum to the set of periodic orbits on Γ [12] or by analyzing the heat kernels [26].

The knowledge of the whole spectrum allows one to reconstruct the metric graph, provided the edge lengths are rationally independent (see, e.g., Refs. [27–29]) thus providing an affirmative answer to the classical question asked by Mark Kac [30] adopted to quantum graphs as "Can one hear the shape of a graph?" [15].

However, in the real world experiments there is no chance to determine the entire spectrum. For example, in microwave networks because of openness of the systems and the existence of internal absorption one can measure up to several hundreds of eigenfrequencies. Moreover, one cannot guarantee that the edge lengths are rationally independent, therefore it is natural to investigate the question whether the total length $\mathcal{L}$ and the Euler characteristic χ can be reconstructed directly from the spectrum without determining a precise form of the graph. Formulas for $\mathcal{L}$ and $X(t)$ provide such a possibility, but their character is completely different. The total length $\mathcal{L}$ is a positive real number, and hence to determine it with a high precision one needs to know high-energy eigenvalues λ_n. More eigenvalues are determined the better approximation of $\mathcal{L}$ is obtained. The Euler characteristic χ is an integer number (often negative), and hence to determine it precisely it is enough to know the right-hand side of (3) with an error less than $1/2$. Therefore, knowing that in the experiment only a limited number of the eigenvalues can be measured, we shall concentrate in this article on determining the Euler characteristic χ.

II. AN EFFECTIVE FORMULA FOR THE EULER CHARACTERISTIC

The series in formula (3) for the Euler characteristic is slowly converging. Its application requires the measurements of several hundreds or even more of eigenenergies which in most cases is not achievable. Therefore, we derived a function

$$X(t) := 2 + 8\pi^2 \sum_{k_n \neq 0} \frac{\sin(k_n/t)}{(k_n/t)[(2\pi)^2 - (k_n/t)^2]}, \quad (4)$$
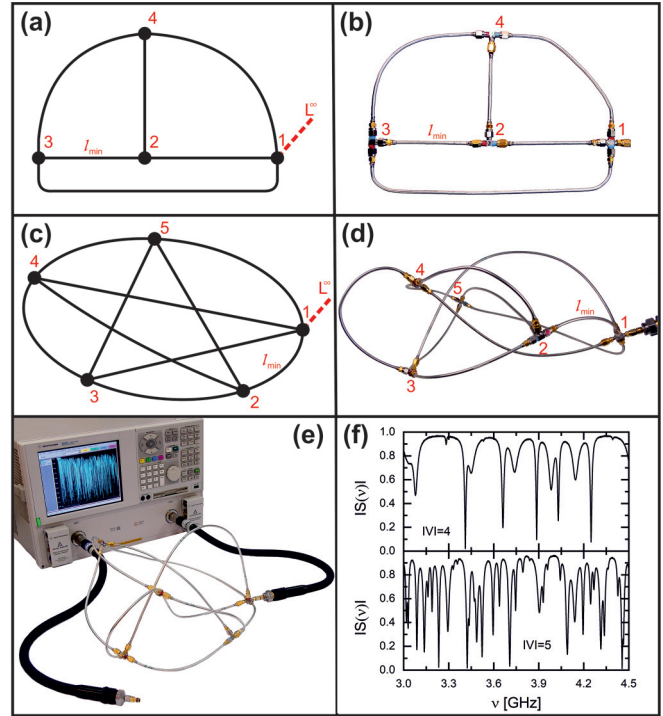

FIG. 1. Panels (a) and (b) show the schemes of a planar quantum graph with $|V| = 4$ vertices and $|E| = 6$ edges and a microwave network with the same topology. Panels (c) and (d) show the schemes of a nonplanar quantum graph with $|V| = 5$ vertices and $|E| = 10$ edges and a microwave network with the same topology. The microwave networks were connected to the vector network analyzer with the flexible microwave cable, which is equivalent to attaching an infinite lead to a quantum graph [panel (e)]. Panel (f) shows the examples of the moduli of the scattering matrix $|S(\nu)|$ of the microwave networks with $|V| = 4$ and $|V| = 5$ vertices, respectively, measured in the frequency range $\nu = 3.0$–4.5 GHz.

which gives the Euler characteristic $\chi = X(t)|_{t \geqslant t_0}$ and is characterized by a much better convergence. The details of derivation are given in the Appendix.

III. EXPERIMENTAL IMPLEMENTATION

Let us assume that in the experiment the K lowest resonances (eigenvalues) are measured. We shall calculate the Euler characteristic χ by evaluating the function $X(t)$ by substituting the infinite series with a finite sum and assuming that $t \geqslant t_0$. Let us introduce the function $X_K(t)$ corresponding to formula (4)

$$X_K(t) = 2 + 8\pi^2 \sum_{n=1}^{K} \frac{\sin(k_n/t)}{(k_n/t)[(2\pi)^2 - (k_n/t)^2]}. \quad (5)$$

We are going to analyze whether this function gives a good approximation for the Euler characteristic χ when $t = t_0 = \frac{1}{2l_{\min}}$. Comparing (4) with (A10) we obtain $\epsilon = |X(t_0) - X_K(t_0)|$. In order to guarantee the difference ϵ is less than $1/2$, e.g., $1/4$, it is enough to take the first K eigenvalues evaluated
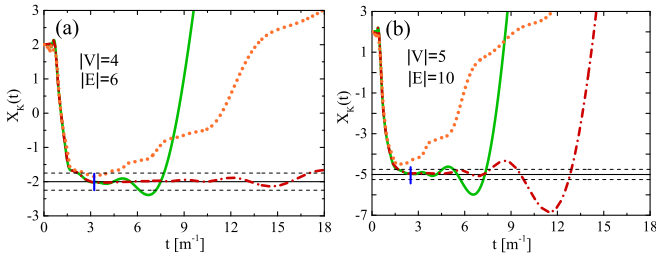
FIG. 2. The approximation function for the Euler characteristic $X_K(t)$ calculated for a planar microwave network with $|V| = 4$ vertices and $|E| = 6$ edges [panel (a)] and for a non-planar fully connected microwave network with $|V| = 5$ vertices and $|E| = 10$ edges [panel (b)]. The full green and red dash-dotted lines show the function $X_K(t)$ calculated from the Eq. (A10) for the first $K = 28$ and $K = 106$ resonances [panel (a)], and $K = 74$ and $K = 132$ resonances [panel (b)], respectively. The blue vertical mark shows the value of $t_0 = \frac{1}{2l_{\min}}$ used for the evaluation of the required number of resonances $K = 28$ [see the formula (A11)] [panel (a)] and $K = 74$ [panel (b)]. For the comparison we show the function $X_K(t)$ (orange dotted line) calculated from the Eq. (3) using the first $K = 28$ [panel (a)] and $K = 74$ [panel (b)] resonances. The black full line shows the expected value of the Euler characteristic $\chi = -2$ [panel (a)] and $\chi = -5$ [panel (b)]. The black broken lines show the limits of the expected errors $\chi \pm 1/4$.

by the following formula:

$$K \simeq |V| - 1 + 2\mathcal{L}t_0 \left[1 - \exp\left(\frac{-\epsilon\pi}{\mathcal{L}t_0} \right) \right]^{-1/2}. \quad (6)$$

The details of the proof are given in the Appendix.

The formula for the Euler characteristic (4) was tested experimentally using planar and nonplanar microwave networks for which the counting function of the number of resonances satisfies the Weyl's law [19]. For such networks the Euler characteristic is the same as for the corresponding closed quantum graphs.

In Figs. 1(a) and 1(b) we present the schemes of a planar quantum graph Γ with $|V| = 4$ vertices and $|E| = 6$ edges and a planar microwave network with the same topology as Γ. The total optical length of the microwave network is $\mathcal{L} = 1.494 \pm 0.006$ m, and the optical length of the shortest edge is $l_{\min} = 0.155 \pm 0.001$ m. The optical lengths l_i^{opt} of the edges of the network are connected with their physical lengths l_i^{ph} through the relationship $l_i^{\text{opt}} = \sqrt{\varepsilon} l_i^{\text{ph}}$, where $\varepsilon = 2.06$ is the dielectric constant of the Teflon used for the construction of the microwave cables. The quantum graph is a closed dissipationless system for which according to the definition of the Euler characteristic $\chi = |V| - |E| = -2$. One should point out that the lack of dissipation is a standard assumption considered in the mathematical analysis of graphs.

In Fig. 2(a) we show that formula (A10) can be easily used to reconstruct the Euler characteristic of the microwave network in Fig. 1(b) and obtain the correct result $\chi = -2$. As in all real life systems, this system is open and is characterized by small dissipation [21]. The resonances $\nu_1, \ldots, \nu_N$ of the microwave network required for the evaluation of the Euler characteristic were determined from the measurements of

a one-port scattering matrix $S(\nu)$ of the network using the vector network analyzer (VNA) Agilent E8364B.

One should note that it is customary for microwave systems to make measurements of the scattering matrices as a function of microwave frequency ν. Then the real parts of the wave numbers k_n are directly related to the positions ν_n of the resonances $\text{Re } k_n = \frac{2\pi}{c} \nu_n$. The VNA was connected to the microwave network with the flexible HP 85133-616 microwave cable which is equivalent to attaching an infinite lead to a quantum graph [19]. Before each measurement the VNA was calibrated using the Agilent 4691-60004 electronic calibration module to eliminate the errors in the measurements.

In order to avoid the missing resonances we analyzed the fluctuating part of the integrated spectral counting function $N_{fl}(\nu_i) = N(\nu_i) - N_{av}(\nu_i)$ [17], that is, the difference of the number of identified eigenfrequencies $N(\nu_i) = i$ for ordered frequencies $\nu_1 \leqslant \nu_2 \leqslant \cdots$ and the average number of eigenfrequencies $N_{av}(\nu_i)$ calculated in the considered frequency range. Using this well-known method [17] we were able to identify the first $N = 106$ resonances in the frequency range of 0.31–10.76 GHz. The problem with the resolution of the resonances begins for $N \simeq 100$–150, but then the sensitivity of the Euler characteristic (4) for the missing resonances is very weak.

In Fig. 2(a) we show the approximation function for the Euler characteristic $X_K(t)$ (A10) calculated using the first $K = 28$ (green full line) and $K = 106$ (red dashed-dotted line) experimentally measured resonances of the system, respectively. The value $K = 28$ was estimated from formula (A11) assuming that $\epsilon = 1/4$ and taking into account the optical size of the network $\mathcal{L}t_0 = 4.82 \pm 0.05$. In Fig. 1(f) we show, as an example, the modulus of the scattering matrix $|S(\nu)|$ of the experimentally studied microwave network Γ with $|V| = 4$ measured in the frequency range $\nu = 3.0$–4.5 GHz. Figure 2(a) demonstrates that it is enough to use the first $K = 28$ resonances (green full line) to identify a clear plateau close to the expected value $\chi = -2$. This plateau extends from $3 \text{ m}^{-1} < t < 6 \text{ m}^{-1}$ and includes the parameter $t_0 \simeq 3.23 \text{ m}^{-1}$ which was used for the evaluation of the required number of resonances $K = 28$ [see formula (A11)]. The Euler characteristic calculated for $K = N = 106$ resonances (red dashed-dotted line) displays a very long plateau along the expected value $\chi = -2$. The plateau extends from $3 \text{ m}^{-1} < t < 17 \text{ m}^{-1}$ showing that we actually deal with the excessive number of resonances required for the practical evaluation of the Euler characteristic. Just for the comparison we also show in Fig. 2(a) the Euler characteristic calculated from Eq. (3) using the first $K = 28$ resonances (orange dotted line). As expected, formula (3) shows much worse convergence to the predicted value of $\chi = -2$.

Although for the analysis of the convergence of formula (4) [see Eq. (A11)] we used the graph's parameters $\mathcal{L}$ and t_0 in the real applications we do not need them. The power of formula (4) stems from the fact that the sequence of the lowest resonances allows for the determination of the Euler characteristic without knowing physical parameters of the graph. In practice, if a plateau in $X_K(t)$ along a given integer number is not observed, it means that the number of resonances used in the calculations is insufficient and it ought to be increased.

It is important to point out that formula (4) allows also for the determination whether a system is planar. In the analyzed cases of the graph Γ and the microwave network the number of cycles yielded from formula (1) is $\beta = 1 - \chi = 3$. In accordance with the Kuratowski's theorem [31] every nonplanar graph should contain K_5 (the complete graph on five vertices) or $K_{3,3}$ (the complete bipartite graph on three and three vertices) as subgraphs. These graphs have six and four cycles, respectively, therefore, without even seeing a graph or having a complete information about the number of vertices and edges we found that the graph is planar and the microwave network simulates the planar graph.

Let us now analyze the situation of nonplanar fully connected (complete in the mathematical terminology) graphs and networks. In Figs. 1(c) and 1(d) we present the nonplanar fully connected quantum graph K_5, complete graph on $|V| = 5$ vertices, characterized by the Euler characteristic $\chi = -5$, and the microwave network with the same topology. The total optical length of the microwave network is $\mathcal{L} = 3.949 \pm 0.010$ m, and the optical length of the shortest edge is $l_{\min} = 0.202 \pm 0.001$ m. To perform the measurements of the first $N = 132$ eigenresonances the network was connected to the VNA with the flexible microwave cable [see Fig. 1(e)]. In Fig. 1(f) we show the modulus of the scattering matrix $|S(\nu)|$ of this network ($|V| = 5$) measured in the frequency range $\nu = 3.0$–4.5 GHz.

The approximation function for the Euler characteristic (A10) calculated for the first $K = 74$ (green full line) and $K = 132$ (red dashed-dotted line) experimentally measured resonances of the system, respectively, is shown in Fig. 2(b). The value $K = 74$ was estimated from formula (A11) assuming again that $\epsilon = 1/4$ and taking into account the optical size of the K_5 network $\mathcal{L}t_0 = 9.74 \pm 0.10$. Figure 2(b) shows that using $K = 74$ resonances measured for the nonplanar microwave network in Fig. 1(d) the correct Euler characteristic $\chi = -5$ can be easily evaluated (full green line). In this case a long plateau close to the expected value $\chi = -5$ is seen in the parameter range 2.5 m$^{-1} < t < 4$ m^{-1}. The situation improves even further for the Euler characteristic calculated for $K = N = 132$ resonances measured in the frequency range of 0.19–5.12 GHz (red dashed-dotted line). In this case the plateau is extended in the range 2.5 m$^{-1} < t < 7.5$ m^{-1} clearly indicating that the Euler characteristic can be also properly evaluated using much less resonances. In Fig. 2(b) we also show the approximation function for the Euler characteristic $X_K(t)$ calculated from Eq. (3) using the first $K = 74$ resonances (orange dotted line). It is visible that formula (3) yields the results which are far from the predicted value of $\chi = -5$ and can be used only for a much higher number of resonances $K = 1243$ [see formula (A16) in the Appendix].

For the analyzed graph K_5 and the microwave network the number of cycles calculated from formula (1) is $\beta = 1 - \chi = 6$. Since the number of cycles is higher than three we cannot directly assess whether the system is planar or not since the application of the Kuratowski's theorem requires full information about the topology of the investigated graph, which in principle is not available. In such a situation, in general, we can test only whether graphs and networks analyzed by us are fully connected. The fully connected simple networks and

graphs are especially interesting because there is an explicit link between the number of vertices $|V|$ of a graph and the Euler characteristic

$$|V| = \frac{3 + \sqrt{9 - 8\chi}}{2}. \tag{7}$$

This formula holds for both planar and nonplanar graphs. Applying formula (7) in the case of the microwave network Γ with the measured Euler characteristic $\chi = -2$ we get $|V| = 4$. Since the number of vertices yielded by formula (7) is integer, it means that our planar network is also fully connected. In the case of the network K_5 with the measured Euler characteristic $\chi = -5$ we directly find that the number of vertices of the network is $|V| = 5$, in obvious agreement with the number of the vertices of the network. Therefore, in this case the experimental evaluation of the Euler characteristic χ allowed us to find that we deal with the fully connected nonplanar K_5 network.

In summary, we showed that the Euler characteristic χ can be determined (heard) from a finite sequence of the lowest resonances $\nu_1, \ldots, \nu_N$ of a microwave network. We also demonstrated that the spectrum of a simple microwave network can be used to find the number $\beta = 1 - \chi$ of independent cycles. If $\beta \leqslant 3$ then a studied system is planar. Moreover, the Euler characteristic χ allows us to identify whether the networks and graphs are fully connected. In such cases it is possible to determine the number of vertices and edges of the systems. Thus, we clearly showed that the Euler characteristic χ is a powerful measure of graphs' or networks' properties, including topology, complementing in an important way the inverse spectral methods that require the infinite number of eigenenergies or resonances for their application.

ACKNOWLEDGMENTS

APPENDIX

1. An effective formula for the Euler characteristic

Formula (3) for the Euler characteristic derived in Refs. [12,25] using the trace formula coming from Refs. [26–28] is not effective when the number of known eigenvalues is limited.

Therefore, we derived an effective formula for the Euler characteristic χ with a better convergence of the series. The formula is obtained by applying the distribution $u(k)$ [12,25],

$$u(k) := 2\delta(k) + \sum_{k_n > 0}[\delta(k - k_n) + \delta(k + k_0)] = \chi\delta(k) + \frac{\mathcal{L}}{\pi}$$

$$+ \frac{1}{\pi}\sum_{p \in P} \ell[\mathrm{prim}(p)]S(p)\cos[k\ell(p)], \tag{A1}$$

where the sum is taken over all periodic orbits P on Γ, $\ell(p)$ is the length of the orbit p, and the coefficients $S(p)$ are products

of scattering coefficients along the orbit p, to the test function

$$\varphi(x) = \begin{cases} 1 - \cos(2\pi x), & 0 \leqslant x \leqslant 1; \\ 0, & \text{otherwise,} \end{cases} \quad (A2)$$

which is continuous and has a continuous first derivative.

Formula (A1) alone shows that knowing the spectrum, equivalently, the distribution on the left-hand side of formula (A1), allows one to reconstruct the Euler characteristic χ.

The Fourier transform of the test function $\varphi(x)$ is

$$\sqrt{2\pi}\hat{\varphi}(k) = \int_0^1 [1 - \cos(2\pi x)]e^{-ikx}dx$$

$$= -i(e^{-ik} - 1)\frac{4\pi^2}{k[k^2 - (2\pi)^2]}, \quad (A3)$$

and its real part is given by

$$\text{Re}\sqrt{2\pi}\hat{\varphi}(k) = -\frac{\sin(k)}{k}\frac{4\pi^2}{k^2 - (2\pi)^2}. \quad (A4)$$

The key point of the proof is to use the relation between the Fourier transforms of the distributions and the test functions

$$u[\hat{\varphi}_t(k)] = \hat{u}[\varphi_t(x)]. \quad (A5)$$

Applying $\hat{u}$ to $\varphi_t(x)$ for

$$2t\ell_{\min} \geqslant 1 \Leftrightarrow t \geqslant t_0 = \frac{1}{2\ell_{\min}}, \quad (A6)$$

where $\ell_{\min}$ is the length of the shortest edge of the graph and therefore $2\ell_{\min}$ is the length of the shortest periodic orbit, we get

$$\hat{u}[\varphi_t(x)] = \chi, \quad t \geqslant t_0. \quad (A7)$$

Calculating $u[\hat{\varphi}_t(k)]$ we obtain the formula (4) for the Euler characteristic

$$\chi = X(t)|_{t \geqslant t_0}, \quad X(t) = 2 + 8\pi^2 \sum_{k_n \neq 0} \frac{\sin(k_n/t)}{(k_n/t)[(2\pi)^2 - (k_n/t)^2]}, \quad (A8)$$

improving formula (3). The possible zeros in the denominator are not dangerous since they cancel with the zeros in the numerator.

2. The error estimate for the effective formula

We are interested in estimating how many resonances are needed to determine the Euler characteristic χ, in other words how many terms in the series are enough to evaluate $X(t)$. Since the Euler characteristic χ takes integer values it is enough to require that the error ϵ is less than $1/2$:

$$\epsilon = |X(t) - X_K(t)||_{t=t_0}$$

$$= \left| 8\pi^2 \sum_{n=K+1}^{\infty} \frac{\sin(k_n/t_0)}{(k_n/t_0)[(2\pi)^2 - (k_n/t_0)^2]} \right|, \quad (A9)$$

where

$$X_K(t) = 2 + 8\pi^2 \sum_{n=1}^{K} \frac{\sin(k_n/t)}{(k_n/t)[(2\pi)^2 - (k_n/t)^2]}. \quad (A10)$$

Our claim is that it is enough to take

$$K > |V| - 1 + 2\mathcal{L}t_0\left[1 - \exp\left(\frac{-\epsilon\pi}{\mathcal{L}t_0}\right)\right]^{-1/2}, \quad (A11)$$

where $\mathcal{L}t_0 = \frac{\mathcal{L}}{2l_{\min}}$. For $\frac{\mathcal{L}}{2l_{\min}} \gg 1$ condition (A11) can be approximated by

$$K > |V| - 1 + \frac{2}{\sqrt{\epsilon\pi}}\left(\frac{\mathcal{L}}{2l_{\min}}\right)^{3/2}. \quad (A12)$$

To prove (A11) we assume first that K is sufficiently large to guarantee that the denominator in (A9) is negative $k_{K+1} > 2\pi t_0$. Taking into account the elementary lower estimate for the eigenvalues

$$k_n^2 \geqslant \left(\frac{\pi}{\mathcal{L}}\right)^2 (n + 1 - |V|)^2, \quad (A13)$$

where $|V|$ is the number of vertices, we arrive at the following sufficient condition for the denominator to be negative:

$$K > |V| - 1 + \frac{\mathcal{L}}{\ell_{\min}}. \quad (A14)$$

Then the series can be estimated as

$$|X(t_0) - X_K(t_0)|$$

$$\leqslant 8\pi^2 \sum_{n=K+1}^{\infty} \frac{|\sin(k_n/t_0)|}{(k_n/t_0)[(k_n/t_0)^2 - (2\pi)^2]}$$

$$\leqslant 8\frac{(\mathcal{L}t_0)^3}{\pi} \sum_{n=K+1}^{\infty} \frac{1}{(n+1-|V|)[(n+1-|V|)^2 - 4\mathcal{L}^2t_0^2]}$$

$$\leqslant 8\frac{(\mathcal{L}t_0)^3}{\pi} \int_K^{\infty} \frac{dx}{(x+1-|V|)[(x+1-|V|)^2 - 4\mathcal{L}^2t_0^2]}$$

$$= \frac{\mathcal{L}t_0}{\pi} \log\frac{(K+1-|V|)^2}{(K+1-|V|)^2 - 4\mathcal{L}^2t_0^2}, \quad (A15)$$

where we again used (A13) and substituted series with an integral on the last step. Requiring that the error is less than ϵ leads to (A11).

3. The error estimate for the original formula

Using similar arguments we may derive a rigorous estimate for the number of necessary resonances K required in the case of formula (3):

$$K > |V| - 1 + \frac{32\mathcal{L}^2}{\epsilon\pi^2}t_0^2 \equiv |V| - 1 + \frac{32}{\epsilon\pi^2}\left(\frac{\mathcal{L}}{2\ell_{\min}}\right)^2. \quad (A16)$$

Since the ratio $\frac{\mathcal{L}}{2\ell_{\min}}$ in formula (A16) is raised to the second power the above estimate for $\frac{\mathcal{L}}{2l_{\min}} \gg 1$ is definitely much worse than (A12), which clearly explains why the old formula (3) for the Euler characteristic is ineffective in the real world applications.

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