Stochastic master stability function for noisy complex networks

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In this paper, we broaden the master stability function approach to study the stability of the synchronization manifold in complex networks of stochastic dynamical systems. We provide necessary and sufficient conditions for exponential stability that allow us to discriminate the impact of noise. We observe that noise can be beneficial for synchronization when it diffuses evenly in the network. On the contrary, an excessively large amount of noise only acting on a subset of the node state variables might have disruptive effects on the network synchronizability. To demonstrate our findings, we complement our theoretical derivations with extensive simulations on paradigmatic examples of networks of noisy systems.

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I. INTRODUCTION

Elucidating the underlying mechanisms that drive a group of coupled individual toward a synchronized motion has been the focus of an intense research effort in the last decades [1,2]. Scientists from diverse disciplines have wondered how a school of fish may move at unison to escape from predators [3,4], birds migrate together in flocks [5], and honeybee workers perform coordinated wagging to indicate the direction of the food source [6]. This surge of interest for synchronization crossed the borders of natural sciences, due to the countless applications on engineered systems, ranging from cryptography and communications [7,8], swarm robotics [9], vehicle coordination [10], and electric power networks [11], to name a few.

The study of dynamical complex networks has highlighted that the interplay between the topological properties of the graph describing the interaction between individuals and their own (decoupled) dynamics is crucial to determine the emergence of a synchronous behavior. In particular, the so-called master stability function (MSF) approach provided necessary and sufficient conditions for the exponential stability of the synchronization manifold [12], shedding light on the role played by the eigenvalues of the Laplacian matrix for network synchronizability. Later works then focused on guaranteeing global convergence of the node trajectories, and alternative tools were employed to achieve this goal. Specifically, contraction theory has been used to assess convergence of neighboring trajectories, thus resulting in conditions on appropriately defined measures of the Jacobian matrix associated to the network dynamics [13,14]. Lyapunov-based methods were also employed to provide conditions for global synchronizability, which was shown to depend both on network connectivity and on the properties of the vector fields describing the individual dynamics, see, e.g., Refs. [15,16].

Most of the studies on synchronization in complex networks are framed in a deterministic setting, where the dynamics are described in terms of ordinary differential equations. However, the dynamical evolution of coupled systems may be affected by noise, and a deterministic model would not faithfully reproduce the behavior of the network. Indeed, the presence of noise and uncertainty may hinder synchronizability, and the convergence toward complete synchronous solution is made more difficult by sudden bursts of desynchronized behavior [17,18].

The use of ODEs and difference equations to model complex networks affected by noise has been explored in continuous- and discrete-time settings, respectively, see, e.g., Refs. [19–25]. These works mainly focused on the adverse effect of noise on synchronizability, by (i) illustrating that mean-square convergence is hindered by high levels of noise [19–22] or (ii) applying the deterministic MSF to a set of networks corresponding to the possible realizations of the noise [23–25]. Following the observation that, in several domains of application, noise may even induce synchronization of decoupled systems [26–32], recent literature decided to model noisy complex networks as coupled stochastic differential equations (SDEs). More specifically, researchers provided sufficient conditions for global synchronizability [33,34] and observed that a common noise acting on all the network nodes may induce synchronization. However, since all the conditions for convergence of networks of noisy nonlinear systems provided in the literature were sufficient but not necessary, determining whether noise can be an opportunity or an additional challenge toward synchronizing a complex network is still an open theoretical research problem [34,35].

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In this work, we aim at providing a more complete picture of the impact of noise, illustrating both its potential benefits and hindrance for synchronizability. To this goal, rather than looking for unavoidably conservative global conditions, we broaden the master stability function approach to deal with the presence of noise affecting the individual dynamics of the network. Indeed, our approach allows to derive necessary and sufficient conditions for almost sure local exponential stability of the synchronization manifold. An extensive numerical exploration of these conditions on paradigmatic testbeds of complex networks highlights how generally low noise intensities are beneficial for synchronizability, the effect on larger noise signals depends on the way it diffuses in the network, on the individual dynamics of the nodes, and on the connectivity of the network topology.

The outline of the manuscript is as follows. In Sec. II, we provide the necessary preliminaries on stochastic systems and then review the traditional master stability function approach for deterministic complex networks. Then, in Sec. III, we introduce our stochastic complex network model and derive the stochastic master stability function (SMSF) to provide conditions for almost sure local exponential synchronizability of the synchronization manifold. The beneficial and detrimental effects of noise are then discussed in Sec. IV, and conclusions are finally drawn in Sec. V.

II. MATHEMATICAL PRELIMINARIES

A. Stochastic differential equations

Let us consider the following stochastic Itô equation:

$$dz(t) = \varphi(z,t)dt + \gamma(z,t)db(t), \tag{1}$$

where $z \in \mathbb{R}^m$, ϕ, and γ are nonlinear vector fields commonly denoted as drift and diffusion functions, respectively, and $b(t)$ is a Wiener process [36]. Let $(\Omega, \mathcal{F}, P)$ be a complete probability space with a filtration $\{\mathcal{F}_t\}$ satisfying

(i) $\emptyset \in \mathcal{F}$, where $\emptyset$ denotes the empty set;
(ii) $A \in \mathcal{F} \Rightarrow \Omega - A \in \mathcal{F}$;
(iii) $A_1 \in \mathcal{F},\ A_2 \in \mathcal{F} \Rightarrow A_1 \bigcup A_2 \in \mathcal{F}$.

The following Lemma provides sufficient conditions on φ and γ ensuring the existence and uniqueness of the solution $z(t)$ of Eq. (1) on every finite subinterval $[t_0, T]$ of $[t_0, +\infty]$:

Lemma 1 (Existence and uniqueness of a global solution). [36 Thm. 2.3.6] If

(1) (Lipschitz condition) for every real number $T > t_0$ and integer $\delta \geqslant 1$, there exists a positive constant $K_{T,\delta}$ such that

$$\max(\|\varphi(x,t) - \varphi(y,t)\|^2, \|\gamma(x,t) - \gamma(y,t)\|^2) \leqslant K_{T,\delta}\|x-y\|^2$$

for all $t \in [t_0, T]$ and $x, y \in \mathbb{R}^n$ fulfilling $\max(\|x\|, \|y\|) \leqslant \delta$, and

(2) (Growth condition) for every $T > t_0$ there exists a positive constant K_T such that

$$z^T\varphi(x,t) + \tfrac{1}{2}\|\gamma(z,t)\|^2 \leqslant K_T(1 + \|z\|^2)$$

for all $z \in \mathbb{R}^n$ and $t \in [t_0, T]$,

then there exists a unique global solution $z(t)$ to Eq. (1), with $z(t)$ being a real-valued measurable $\{\mathcal{F}_t\}$-adapted process with finite variance.

Definition 1 (Equilibrium of a stochastic equation). $z(t) = \bar{z}$ is the equilibrium of the stochastic Itô Eq. (1) if both

$$\varphi(\bar{z}, t) = 0 \text{ and } \gamma(\bar{z}, t) = 0 \quad \forall t.$$

If $\bar{z} = 0$, then the equilibrium is called the *trivial solution* of the stochastic system (1).

Definition 2 (Sample Lyapunov exponent and almost sure stability). Let us consider a stochastic Itô process of the form (1) having the trivial solution $\bar{z} = 0$. The sample Lyapunov exponent associated to $\bar{z}$ is

$$\text{Lyap}(t, z(t_0)) := \frac{1}{t}\log(\|z(t; t_0, z(t_0))\|). \tag{2}$$

The trivial solution is *locally almost sure exponentially stable* if and only if there exists ε such that

$$\limsup_{t\to+\infty} \text{Lyap}(t, z(t_0)) < 0 \text{ almost surely}, \tag{3}$$

for all $z(t_0) : \|z(t_0)\| < \varepsilon$, while it is *globally almost sure exponentially stable* if and only if (3) holds for all $z(t_0) \in \mathbb{R}^n$. Note that the left-hand side of inequality (3) can be used as an estimate of the exponential rate of convergence toward the trivial solution of (1).

B. Master stability function

Traditionally, a complex network has been modeled as the following set of coupled ordinary differential equations:

$$\dot{x}_i(t) = f(x_i, t) + \sigma \sum_{j=1}^{N} a_{ij}[h(x_j, t) - h(x_i, t)], \tag{4}$$

for all $i = 1, \ldots, N$, where N is the number of nodes, $x_i \in \mathbb{R}^n$ is the state of the ith node, $f : \mathbb{R}^n \times \mathbb{R}^+ \to \mathbb{R}^n$ is a nonlinear vector field, $h : \mathbb{R}^n \times \mathbb{R}^+ \to \mathbb{R}^n$ is the inner coupling function, σ is the overall coupling strength, and a_{ij} is the ijth element of the adjacency matrix describing the interconnections among the nodes.

Network (4) admits an invariant set $x_i(t) = x_s(t)$ for all $i = 1, \ldots, N$, denoted as *synchronization manifold* $\mathcal{S}$, where x_s is a solution of the individual uncoupled dynamics. Therefore, to assess the local stability of the synchronous state it suffices to analyze the dynamics that are transverse to the synchronization manifold. In Ref. [12], the authors showed that this can be done by studying the lower-dimensional parametric equation

$$\dot{\zeta}(t) = [f_x(x_s, t) - \eta h_x(x_s, t)]\zeta(t), \tag{5}$$

where $\zeta \in \mathbb{R}^n$, $\eta \geqslant 0$, and the subscript x stands for the derivative with respect to the first argument. For each value of η, n Lyapunov exponents can be extracted. The largest of these exponents MSF(η) is called master stability function. Now if we sort the eigenvalues of the Laplacian matrix associated to network (4) in ascending order, that is, $0 = \lambda_1 \leqslant \lambda_2 \leqslant \ldots \leqslant \lambda_N$, then we can provide the following result on the local stability of the synchronization manifold:

Lemma 2. The synchronization manifold is locally exponentially stable if and only if MSF($\sigma\lambda_i$) < 0 for $i = 2, \ldots, N$.

Depending on the individual dynamics f and on the coupling function h, three types of MSF have been classified [37]. Specifically, type I is monotone increasing, type II is monotone decreasing, while type III is nonmonotone. For type

II MSF, the larger the Fiedler eigenvalue λ_2 (also known as algebraic connectivity), the easier is to synchronize the network. Type III MSF is of particular interest, since it might admit finite ranges of negative values, thus requiring the ratio λ_N/λ_2 to be small enough to guarantee the existence of a coupling strength σ for which $\mathcal{S}$ is locally stable.

III. STOCHASTIC MASTER STABILITY FUNCTION

A. Stochastic complex network model

Here we consider a complex network of $N > 1$ stochastic dynamical systems, diffusively coupled through an undirected and connected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V}$ and $\mathcal{E}$ are the set of the network nodes and edges, respectively. To model the presence of multiple sources of uncertainty acting on the individual dynamics of each node, we assume that the network is subject to p independent noises. Therefore, the traditional deterministic model (4) is replaced by the following nonlinear stochastic differential equation of Itô type:

$$dx_i(t) = \left\{ f(x_i, t) + \sigma \sum_{j=1}^{N} a_{ij}[h(x_j, t) - h(x_i, t)] \right\} dt + \sum_{k=1}^{p} \sigma_k^* g_k(x_i, t) db_k(t), \tag{6}$$

for all $i \in \mathcal{V}$, where the diffusion function $g_k : \mathbb{R}^n \times \mathbb{R}^+ \to \mathbb{R}^n$ models how the kth noise propagates through the network, σ_k^* modulates its variance, and $b = [b_1, \ldots, b_p]^T$ is a standard p-dimensional Wiener process that acts as a disturbance on all the network nodes. We assume that all the functions we have introduced satisfy the hypotheses of Lemma 1, so that a unique global solution of the SDEs exists.

The assumption that the network is made of identical systems, i.e., the functions f and g_k, $k = 1, \ldots, p$, are identical for all the nodes in $\mathcal{V}$, see Eq. (6), grants that also in this stochastic setting the invariant synchronization manifold $\mathcal{S}$ exists [38]. The synchronous trajectory $x_s(t)$ is a solution of the stochastic differential equation

$$dx_s(t) = f(x_s, t)dt + \sum_{k=1}^{p} \sigma_k^* g_k(x_s, t) db_k(t).$$

Introducing the synchronization error

$$e(t) = [e_1(t)^T, \ldots, e_N(t)^T]^T,$$

with $e_i(t) := x_i(t) - x_s(t)$, we can give the definition of exponential convergence toward $\mathcal{S}$ in a stochastic sense. In fact, the error dynamics is ruled by

$$de_i(t) = dx_i(t) - dx_s(t) = \left\{ f(x_s + e_i, t) - f(x_s, t) + \sigma \sum_{j=1}^{N} a_{ij}[h(x_s + e_j, t) - h(x_s + e_i, t)] \right\} dt + \sum_{k=1}^{p} \sigma_k^* [g_k(x_s + e_i, t) - g_k(x_s, t)] db_k(t), \tag{7}$$

for $i = 1, \ldots, N$, and the trivial solution $e(t) = 0$ of (7) corresponds to the invariant synchronization manifold $\mathcal{S}$ of (6).

Definition 3. The synchronization manifold $\mathcal{S}$ is almost sure (a.s.) locally exponentially stable if and only if there exist $\varepsilon > 0$ such that

$$\limsup_{t \to +\infty} \frac{1}{t} \text{Lyap}(t, e(0)) < 0 \text{ almost surely} \tag{8}$$

for all $e(0) : \|e(0)\| < \varepsilon$ and $\mathbb{1}_N^T e(0) = 0$.

B. Local stability analysis

To study the local stability of the synchronization manifold we would need to focus on the dynamics of (7) that are transversal to $\mathcal{S}$. However, this could be computationally prohibitive for large networks, as the the transversal dynamics would be $n(N-1)$-dimensional. As in the deterministic case, our objective is then to infer the stability properties of $\mathcal{S}$ by computing the sample Lyapunov exponents of a n-dimensional system, thus making computations feasible even for large networks. Toward this goal, we proceed by observing that, in an infinitesimal neighborhood of the synchronization manifold, the error dynamics can be written in compact matrix form as

$$de(t) = [\mathbb{I}_N \otimes f_x(x_s(t), t) - \sigma \mathcal{L} \otimes h_x(x_s(t), t)]e(t)dt + \sum_{k=1}^{p} \sigma_k^* \big(\mathbb{I}_N \otimes g_x^k(x_s(t), t)\big) e(t) db_k(t), \tag{9}$$

where $\otimes$ is the Kronecker product, g_x^k is the Jacobian of g_k with respect to the first argument, and the matrix $\mathcal{L}$ is the Laplacian matrix associated to the graph $\mathcal{G}$.

Let us now consider the transformation T that diagonalizes $\mathcal{L}$ (i.e., $T\mathcal{L}T^{-1} = \Lambda$, with Λ being the diagonal matrix containing the eigenvalues of $\mathcal{L}$) and define a transformed variable $\xi(t) = (T \otimes \mathbb{I}_n)e$. Notice that this transformation does not act on the terms $\mathbb{I}_N \otimes f_x(x_s(t), t)$ and $\mathbb{I}_N \otimes g_x^k(x_s(t), t)$, $k = 1, \ldots, p$, that are already in block diagonal form, since it only affects the identity matrix $\mathbb{I}_N$. Hence, the dynamics of ξ can be obtained by applying the Itô's formula from (9), and is given by

$$d\xi(t) = [\mathbb{I}_N \otimes f_x(x_s(t), t) - \sigma \Lambda \otimes h_x(x_s(t), t)]\xi(t)dt + \sum_{k=1}^{p} \sigma_k^* \big[\mathbb{I}_N \otimes g_x^k(x_s(t), t)\big] \xi(t) db_k(t). \tag{10}$$

System (10) is in block diagonal form, with $(n \times n)$-dimensional blocks, and each block is associated to an eigenvalue of $\mathcal{L}$. Being $\mathcal{G}$ a connected graph, the eigenspace spanned by the eigenvalue 0 has dimension 1 and is orthogonal to all the other eigenvectors [39]. As in the deterministic case, this transformation allows to separately studying the motion along the synchronization manifold $\mathcal{S}$ and the one transverse to it. Indeed, introducing the vector $\xi_i = [\xi_{1,i}, \ldots, \xi_{n,i}]^T$, Eq. (10) can be rewritten as

$$d\xi_i(t) = [f_x(x_s(t), t) - \sigma \lambda_i h_x(x_s(t), t)]\xi_i(t)dt + \sum_{k=1}^{p} \sigma_k^* g_x^k(x_s(t), t) \xi_i(t) db_k(t) \tag{11}$$

for $i = 1, \dots, N$. Replacing λ_i in (11) with the parameter η, we obtain the following family of stochastic linear equations:

$$d\zeta(t) = [f_x(x_s(t), t) - \eta h_x(x_s(t), t)]\zeta(t)dt + \sum_{k=1}^{p} \sigma_k^* g_x^k(x_s(t), t)\zeta(t)db_k(t) \quad (12)$$

that we call *stochastic master stability equation*. Now, we define the stochastic master stability function SMSF(η) as the function that associates to $\eta \geqslant 0$ the sample Lyapunov Exponent associated to the trivial solution $\zeta(t) = 0$ of (12). We are now ready to provide our main stability result.

Theorem 1. The synchronization manifold is locally exponentially stable if and only if SMSF($\sigma\lambda_i$) < 0 for all $i = 2, \dots, N$.

Proof. The thesis follows from Definition 3. ■

Our theoretical derivations allow to highlight the contribution of the individual dynamics and of the network topology to the stability of the synchronization manifold. Indeed, the SMSF is a property of the drift function f, the output function h, the diffusion functions g_ks, and the noise strengths σ_k^*s. The topology, instead, impacts on the local stability of $\mathcal{S}$ through the eigenvalues of the Laplacian matrix and the coupling strength σ.

Remark 1. The diffusion functions g_ks appear in the stochastic master stability function (12) only through their partial derivative g_x^k with respect to the state. This means that an additive noise, that is, a noise whose intensity does not depend on the value of the state, acting on all the nodes of the network, has no impact on the local stability properties of the synchronization manifold.

Remark 2. In the deterministic case, the value of the MSF when $\eta = 0$ is the maximum Lyapunov exponent on the synchronization manifold, that is, the maximum Lyapunov exponent of the uncoupled system. This implies that if the MSF is negative at the origin, then the system will converge to an equilibrium point. Indeed, only converging toward the same equilibrium point two uncoupled systems can synchronize when starting from different initial conditions. In the stochastic case, this consideration does not hold anymore. A negative value of the SMSF at $\eta = 0$ means that two uncoupled systems eventually synchronize, but does not imply a trivial asymptotic behavior, as will be illustrated in Sec. IV A.

C. Convergence toward a stationary point

A specific instance of synchronization is the *consensus* toward a point $\bar{x}$ of the state space, that is, $x_s(t) = \bar{x}$ for all t. Notice that, as a necessary condition for synchronization is that x_s is a solution of the uncoupled dynamics, this implies that

$$f(\bar{x}, t) = 0, \text{ and } g_i(\bar{x}, t) = 0, \ i = 1, \dots, p.$$

For the sake of simplicity, in what follows we study the consensus problem in the presence of a single noise, i.e., when $p = 1$. In this case, the stochastic master stability equation (12) becomes

$$d\zeta(t) = (F - \eta H)\zeta(t)dt + \sigma_1^* G\zeta(t)db(t), \quad (13)$$

where $F = f_x(\bar{x}, t)$, $H = h_x(\bar{x}, t)$, and $G = g_x^1(\bar{x}, t)$. We can now give the following corollary of Theorem 1.

Corollary 1. Let $\bar{x}$ be an equilibrium of the uncoupled dynamics, and $p = 1$. The synchronization manifold $x_i(t) = \bar{x}$ for all i is locally exponentially stable if $(F - \eta H)$ and G commute for all η, and

$$\max\left(\text{Re}\left\{\text{eig}\left[F - \sigma\lambda_i(\mathcal{L})H - \tfrac{1}{2}(\sigma_1^* G)^2\right]\right\}\right) < 0, \quad (14)$$

for all $i = 2, \dots, N$.

Proof. Let us introduce

$$Y(t) = \left(F - \eta H - \tfrac{1}{2}G\right)t + G[b(t) - b(0)]. \quad (15)$$

Noting that

$$dY(t) = \left(F - \eta H - \tfrac{1}{2}G\right)dt + Gdb(t), \quad (16)$$

we can then show that

$$Z(t) = \exp[Y(t)] \quad (17)$$

is the fundamental matrix of the stochastic master stability Eq. (13). Indeed, by applying Itô's formula [36] to (17), we can write

$$dZ(t) = \exp[Y(t)]dY(t) + \tfrac{1}{2}\exp[Y(t)][dY(t)]^2. \quad (18)$$

As $(F - \eta H)$ and G commute for all η, we obtain

$$dZ(t) = (F - \eta H)Z(t)dt + GZ(t)db(t), \quad (19)$$

thus proving that $Z(t)$ satisfies Eq. (13) and therefore is its fundamental matrix. We then have that the unique solution of (13) is

$$\zeta(t) = \exp\{(F - \eta H - G/2)t + G[b(t) - b(0)]\}\zeta(0). \quad (20)$$

We can then conclude that the stochastic master stability function is

$$\text{SMSF}(\eta) = \max\left(\text{Re}\left\{\text{eig}\left[F - \eta H - \tfrac{1}{2}(\sigma_1^* G)^2\right]\right\}\right).$$

The thesis trivially follows. ■

When the noise diffuses proportionally to the node state, the previous corollary can be further specified.

Corollary 2. Let $\bar{x}$ be an equilibrium of the uncoupled dynamics, and $p = 1$. The synchronization manifold $x_i(t) = \bar{x}$ for all i is locally exponentially stable if $g_1(x_i, t) = x_i$ and

$$\max(\text{Re}\{\text{eig}[F - \sigma\lambda_i(\mathcal{L})H]\}) - \tfrac{1}{2}(\sigma_1^*)^2 < 0, \quad (21)$$

for all $i = 2, \dots, N$.

Proof. As $g_1(x_i, t) = x_i$, we have $G = \mathbb{I}_n$. This means that (i) $(F - \eta H)$ commutes with G and (ii) Eq. (21) is equivalent to (14). From Corollary 1, the thesis follows. ■

These two corollaries demonstrate that when the network is in the neighborhood of an equilibrium point of the uncoupled dynamics, and the noise acts evenly on each state variable, it can only be beneficial for synchronizability. Indeed, its effect is to shift downward the SMSF, the shift being proportional to the noise intensity. In the following section, we will notice how this is not always true for nonlinear systems synchronizing onto a nontrivial trajectory.

IV. NUMERICAL EXAMPLES

Here we illustrate how our theoretical findings can be used to assess the impact of noise in networks of nonlinear systems. Specifically, we start by studying a network of Van der Pol oscillators, where we show an instance of the so-called noise-induced synchronization [26–32,34]. Then, we consider the example of coupled Chua's circuits to compare our local necessary and sufficient conditions for exponential synchronization with the global sufficient conditions available in the literature [34]. Finally, we consider a network of Rössler systems, where these global conditions cannot be used to study synchronizability. On the contrary, our local analyses allow for a thorough characterization of the effect of noise, showing that it may also destabilize the synchronization manifold.

In all our numerical analyses, we employ the standard Euler-Maruyama weak integrator [40] to simluate the SDEs, and the time step is fine-tuned to ensure convergence. Further, we employ an optimized version of the discrete QR method given in Ref. [41] to compute the transversal sample Lyapunov exponent. To increase the robustness of the numerical analysis, in all the plots of this section, each sample Lyapunov exponent is computed as the maximum value reached over 10 computations. For ease of illustration, we focus on the case of $p = 1$ noise acting on the network, since the case $p > 1$ would not be qualitatively different.

A. Network of noisy Van der Pol oscillators

Here we consider a network of Van der Pol oscillators [42] with damping coefficient $\mu = 0.2$ and coupled through springs sharing the same elastic coefficient σ. In our model (6), this means that

$$f(x_i, t) = \begin{bmatrix} x_{i2} \\ \mu(1 - x_{i1}^2)x_{i2} - x_{i1} \end{bmatrix}, \quad h(x_i, t) = [0, x_{i1}]^T,$$

where x_{i1} and x_{i2} are the angular position and velocity of the ith oscillator, respectively. Further, we assume that a turbulent wind acts on the system, which we model as a noise acting on the second state variable, that is, we set $g_1(x_i, t) = [0, x_{i2}]^T$. In this setting, the stochastic network model (6) becomes

$$\begin{aligned} dx_{i1} &= x_{i2}dt \\ dx_{i2} &= \left[\mu(1 - x_{i1}^2)x_{i2} - x_{i1} + \sigma \sum_{j=1}^{N} a_{ij}(x_{j1} - x_{i1}) \right] dt \\ &\quad + \sigma^* x_{i2} db(t). \end{aligned} \tag{22}$$

Notice that, considering the physical interpretation of network (22), noise can only act on the second state variable, since by definition the angular velocity x_{i2} is the derivative of the angular position x_{i1}.

In Fig. 1, we report the SMSF for different values of the noise intensity σ^*. In the absence of noise ($\sigma^* = 0$), we observe that the (deterministic) MSF of the system is zero when $\eta = 0$ (the uncoupled dynamics converge toward a limit cycle), and then linearly decreases until $\eta = 0.43$, where a negative plateau is reached. When noise affects the dynamics, its effect is not trivial as in the consensus case studied in

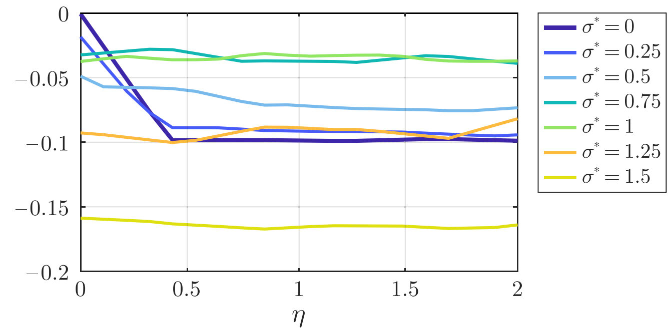

FIG. 1. Stochastic master stability function of the network of Van der Pol oscillators (22) for selected values of the noise intensity σ^*.

Corollary 2. Although local exponential synchronization is always guaranteed in coupled Van der Pol oscillators, the presence of noise may reduce or increase the exponential convergence rate. Indeed, introducing a small amount of noise improves the rate of convergence for sparse networks [i.e., networks with a small Fiedler eigenvalue $\lambda_2(\mathcal{L})$], which is instead reduced for densely connected networks [i.e., large $\lambda_2(\mathcal{L})$]. In simple terms, the fact that all the network nodes are subject to the same noise (e.g., the turbulent wind) makes synchronization easier if the oscillators are weakly coupled, while acts as a disturbance when they are densely connected. Finally, when the noise is excessively large, its impact on synchronizability dominates that of the coupling. Indeed, the SMSF tends to becomes flat, that is, the rate of convergence becomes independent of the network connectivity, and shifts downward as the noise further increases. As a result, independent of the coupling configuration, the oscillators will converge toward the same highly noisy trajectory, thus achieving synchronization.

This paradigmatic examples also allow us to discuss an instance of *noise-induced synchronization*, a phenomenon that is well-known in the literature on stochastic systems, see, e.g., [26–32,34]. While decoupled deterministic systems can only synchronize at a point of the state space that is an asymptotically stable equilibrium point of the individual dynamics, this is not true in the presence of noise, as noted in Remark 2. To demonstrate this point, we simulate the time evolution of two uncoupled Van der Pol systems subject to the same noise with strength $\sigma^* = 0.25$. Figure 2 illustrates how the error norm exponentially converges to zero, with the individual trajectories converging toward a stochastic version of the Van der Pol attractor, and this is consistent with the fact that, when $\sigma^* = 0.25$, we have SMSF(0) $= -0.018$.

B. Network of noisy Chua's circuits

Here, in the general Eq. (6), we consider as individual dynamics a stochastic version of the well-known Chua's circuit [43]. Specifically,

$$f(x_i, t) = \begin{bmatrix} \alpha[x_{i2} - \gamma(x_{i1})] \\ x_{i1} - x_{i2} + x_{i3} \\ -\beta x_{i2}, \end{bmatrix}, \tag{23}$$

where $\alpha = 9$, $\beta = 100/7$, and

$$\gamma(x_{i1}) = m_1 x_{i1} + \tfrac{1}{2}(m_0 - m_1)(|x_{i1} + 1| - |x_{i1} - 1|),$$

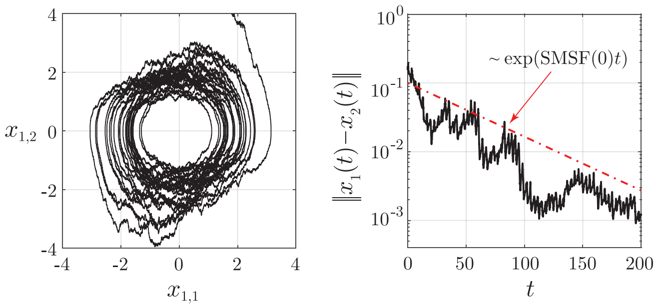

FIG. 2. Simulation of $N = 2$ uncoupled stochastic Van der Pol systems (22) when $\sigma^* = 0.25$. Phase-portrait of node 1 (left panel) and time evolution of the norm of the synchronization error (right panel).

with $m_0 = -1/7$ and $m_1 = 2/7$ selected so that, in the absence of noise and coupling, the dynamics exhibit the double scroll chaotic attractor. Note that the system is piecewise linear, and therefore (23) can be rewritten as

$$f(x_i, t) = \begin{cases} A_1 x_i - b, & \text{if} \quad x_{i1} > 1, \\ A_2 x_i, & \text{if} \quad |x_{i1}| < 1, \\ A_1 x_i + b, & \text{if} \quad x_{i1} < -1, \end{cases}$$

with $b = [m_0 - m_1, 0, 0]^T$, and

$$A_1 = \begin{bmatrix} -\alpha m_1 & \alpha & 0 \\ 1 & -1 & 1 \\ 0 & -\beta & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -\alpha m_0 & \alpha & 0 \\ 1 & -1 & 1 \\ 0 & -\beta & 0 \end{bmatrix}.$$

Thus, it satisfies the QUAD assumption [44], that is, for all $(x, y) \in \mathbb{R}^3$ and $t \in \mathbb{R}^+$, it fulfills the following inequality:

$$(x - y)^T [f(x, t) - f(y, t)] \leqslant k_f \|x - y\|^2,$$

where $k_f = \max(\lambda_N(A_1^{\mathrm{sym}}), \lambda_N(A_2^{\mathrm{sym}})) = 8.1$, with $A_i^{\mathrm{sym}} = (A_i + A_i^T)/2$. In Ref. [34], the authors showed that, if the systems are coupled through the identity function, i.e., $h(x_i, t) = x_i$, only one noise affect the system ($p = 1$), with $g_1(x, t) = G(t)x$, and $\sigma_1^* = \sigma^*$, then the stochastic synchronization manifold is globally stable when

$$\sigma\lambda_2(\mathcal{L}) > k_f + \frac{(\sigma^*)^2}{2}[\bar{d}(t)^2 - 2\underline{d}(t)^2], \tag{24}$$

where $\bar{d}(t)$ and $\underline{d}(t)$ are the eigenvalues of $G(t)$ with the largest and the smallest absolute value, respectively [45]. As the diffusion function considered in our numerical analysis is $g_1(x, t) = x_i$, we have $\bar{d}(t) = \underline{d}(t) = 1$. Figure 3 depicts the stochastic master stability function computed for different values of the noise intensity σ^*, from which we observe that the stability region, i.e., the region where SMSF < 0, is unbounded.

We notice how the global condition given in Ref. [34] is unavoidably conservative. Indeed, in Fig. 3, the green area on the left of the white line corresponds to coupling configurations and noise intensities for which condition (24) is not fulfilled, while the SMSF is negative, thus implying local exponential stability. To further illustrate this point, we consider a network of $N = 100$ Chua's circuits, whose stability is characterized by the black point in Fig. 3. Indeed, the network topology, illustrated in the left panel of Fig. 4, is

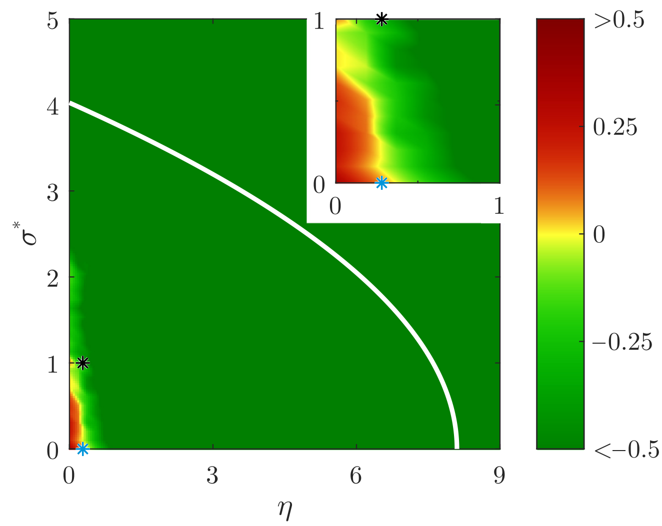

FIG. 3. Stochastic master stability function of the network of stochastic Chua's circuits (6)–(23) for $\sigma^* \in [0, 5]$. The white line is the locus $\eta - (\sigma^*)^2/2 = k_f$. The stars identify the points $(\sigma\lambda_2(\mathcal{L}), 0)$ and $(\sigma\lambda_2(\mathcal{L}), 1)$ for the network depicted in Fig. 4.

characterized by $\lambda_2(\mathcal{L}) = 2.84$, and we selected $\sigma^* = 1$ and $\sigma = 0.1$. From Theorem 1, we have that, although the black point $(\sigma\lambda_2(\mathcal{L}), \sigma^*)$ in Fig. 3 is far from the white line, almost sure local exponential synchronization is achieved. This is consistent with the performed numerical simulation, whose outcome is depicted in the right panel of Fig. 4.

C. Network of noisy Rössler oscillators

Here we consider the case when the individual dynamics do not fulfill the QUAD assumption, and therefore the global conditions for synchronizability presented in Ref. [34] cannot be employed. Indeed, we select a stochastic version of the Rössler system [47] as individual dynamics in network

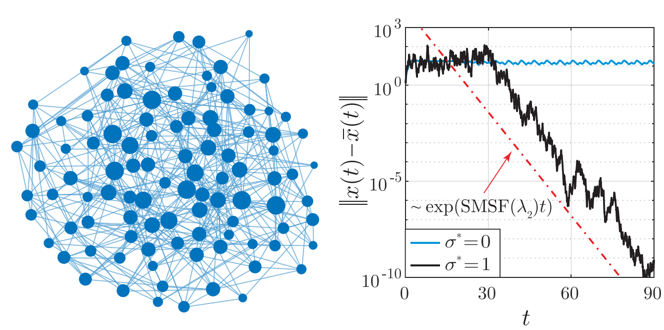

FIG. 4. Simulation of a network of $N = 100$ stochastic Chua's circuits (6)–(23) when g and h are both the identity function, and $\sigma = 0.1$. The left panel depicts the network topology, generated through the Watts-Strogatz model [46], while the norm of the error with respect to the average trajectory $\bar{x}(t) = \mathbb{1}_N \otimes \sum_{i=1}^N x_i(t)/N$ is plotted in the right panel when $\sigma^* = 1$ (black line) and in the absence of noise (blue line).

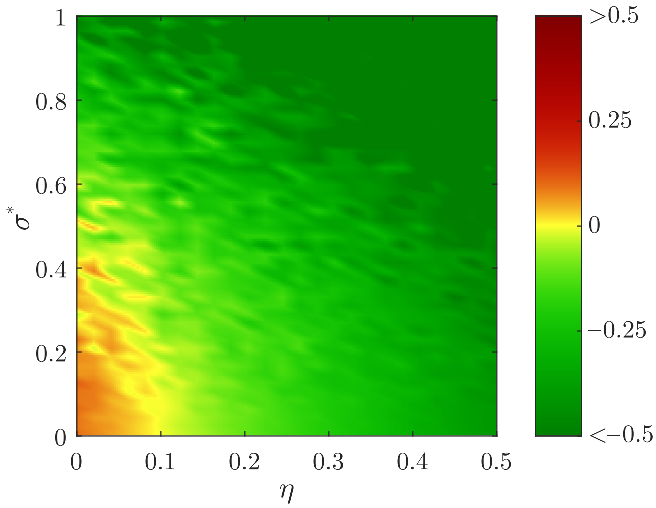

FIG. 5. Stochastic master stability function of the network of stochastic Rössler systems (6)–(25), with $h(x_i, t) = g(x_i, t) = x_i$, for $\sigma^* \in [0, 1]$.

model (6). Namely,

$$f(x_i, t) = \begin{bmatrix} -x_{i2} - x_{i3} \\ x_{i1} + a x_{i2} \\ b + x_{i3}(x_{i1} - c) \end{bmatrix}, \tag{25}$$

where $a = b = 0.2$ and $c = 5.7$ are selected so that, when the noise is absent, the uncoupled dynamics of each node admits a chaotic attractor. Furthermore, we select the coupling and diffusion functions as the identity, i.e., $h(x_i, t) = x_i$ and $g_1(x_i, t) = x_i$. The SMSF of the network is reported in Fig. 5 to illustrate that, in this case, noise helps the systems to synchronize, and, when its intensity overcomes a certain threshold (numerically identified as $\sigma^* = 0.63$), it induces synchronization between decoupled systems. The addition of noise in the network shifts downward the SMSF, as clearly highlighted in Fig. 6, where the SMSFs of the network for increasing noise intensities are reported. This behavior is qualitatively similar to what has been theoretically proved for consensus when the noise diffusion function is the identity, see Corollary 2.

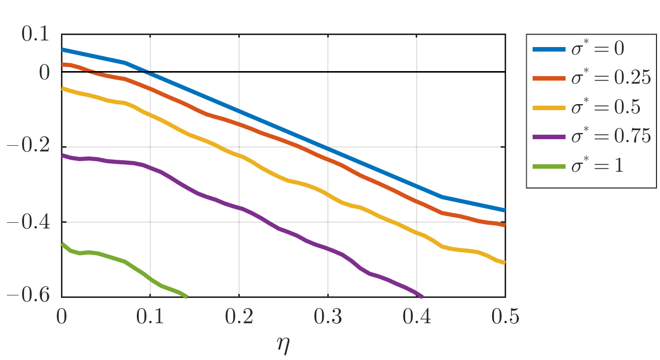

FIG. 6. Stochastic master stability function of the network of stochastic Rössler systems (6)–(25), with $h(x_i, t) = g(x_i, t) = x_i$, for selected values of the noise intensity σ^*.

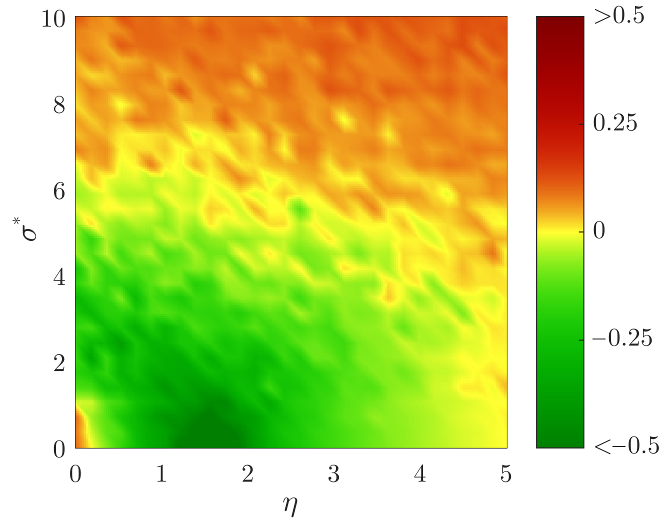

FIG. 7. Stochastic master stability function of the network of stochastic Rossler systems (6)–(25) with $h(x_i, t) = g(x_i, t) = [x_{i1}, 0, 0]^T$ for $\sigma^* \in [0, 10]$.

Next, we make a different selection of the coupling and diffusion functions, and specifically we set

$$g(x_i, t) = [x_{i1}, 0, 0]^T, \quad h(x_i, t) = [x_{i1}, 0, 0]^T.$$

This yields to a qualitative change of the SMSF of the network, which is reported in Fig. 7. While for small values of σ^* noise still has a stabilizing effect, when its intensity excessively increases, its impact on the stability of the synchronization manifold dramatically changes. Indeed, for $\sigma^* > 7$ we have that the SMSF is positive for all values of η, thus making synchronization unfeasible for all possible coupling configurations.

A closer look at this transition toward instability can be given in Fig. 8, which shows how the noise changes the qualitative behavior of the SMSF, eventually turning the type III deterministic MSF (when $\sigma^* = 0$) into a type I SMSF (when $\sigma^* = 8$).

V. CONCLUSIONS

This paper contributes to improve our understanding of the effect of noise on synchronizability of complex networks.

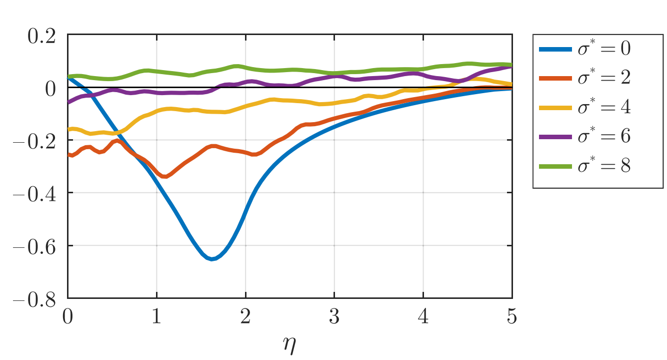

FIG. 8. Stochastic master stability function of the network of stochastic Rossler systems (6)–(25) with $h(x_i, t) = g(x_i, t) = [x_{i1}, 0, 0]^T$ for selected values of the noise intensity σ^*.

Specifically, we broaden the master stability function approach, originally developed for deterministic networks, to study networks of coupled stochastic systems. This allows to provide necessary and sufficient conditions for the almost sure local exponential stability of the synchronization manifold. Different from existing work that only provided sufficient conditions, our analysis allows to paint a wide picture of the impact of noise. We show that, when it evenly affects all of the node state variables, noise facilitates the emergence of synchronization, acting as an entraining signal for the network. In this case, when the noise overcomes a certain threshold, almost sure synchronization is achieved even in the absence of coupling, a phenomenon that has been classified in the literature as noise-induced synchronization. However, when the noise only enters in a subset of the node state variables, its impact on synchronizability is less trivial. Indeed, noise can also have the effect of reducing the exponential convergence rate toward the synchronization manifold, or even making it unstable. Assessing whether the noise is detrimental or beneficial requires taking into account its intensity, the coupling configuration of the network, and the individual node dynamics. In our numerical analysis, we observed that low noise intensities are typically beneficial for weakly coupled systems, while the impact of high noise levels is very dependent on the individual dynamics and coupling configuration of the network.

The analysis performed in this manuscript can be extended in several directions. First, it would be important to investigate the impact on synchronization of parameter mismatches among the coupled oscillators. In this case, although the synchronization manifold would not be invariant, thereby preventing complete synchronization, a relevant research problem would be to assess whether the distance among the trajectories of the oscillators would remain almost surely bounded. So far, this problem has only been addressed in a deterministic setting [48]. Second, currently the noise only enters at the nodes individual dynamics, but in general it may also affect the communication protocol, as illustrated, e.g., in Refs. [23,24,33]. Ongoing works are devoted to extend the stochastic master stability approach to also encompass these cases [49].

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