Resetting processes with noninstantaneous return

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We consider a random two-phase process which we call a reset-return one. The particle starts its motion at the origin. The first, displacement, phase corresponds to a stochastic motion of a particle and is finished at a resetting event. The second, return, phase corresponds to the particle's motion toward the origin from the position it attained at the end of the displacement phase. This motion toward the origin takes place according to a given equation of motion. The whole process is a renewal one. We provide general expressions for the stationary probability density function of the particle's position and for the mean hitting time in one dimension. We perform explicit analysis for the Brownian motion during the displacement phase and three different types of the return motion: return at a constant speed, return at a constant acceleration with zero initial speed, and return under the action of a harmonic force. We assume that the waiting times for resetting events follow an exponential distribution or that resetting takes place after a fixed waiting period. For the first two types of return motion and the exponential resetting, the stationary probability density function of the particle's position is invariant under return speed (acceleration), while no such invariance is found for deterministic resetting, and for exponential resetting with return under the action of the harmonic force. We discuss necessary conditions for such invariance of the stationary PDF of the positions with respect to the properties of the return process, and we demonstrate some additional examples when this invariance does or does not take place.

I. INTRODUCTION

Random motion under resetting [1] represents a process in which a stochastically moving particle returns from time to time to its initial position and starts its motion anew from the very beginning. Examples of such processes are found in many fields such as biochemistry [2–4], biology [5,6], and computer science [7]. In computer science, random walks with stochastic restarts represent a useful strategy to optimize search algorithms in computationally hard problems [7]. The organisms can use stochastic resetting or switching between different phenotype states to adapt to fluctuating environments [5]. Also, the motion of foraging animals, when they are searching for food and return home from time to time, is one example of resetting. One can also consider a robotic vacuum cleaner performing stochastic motion while cleaning the apartment and returning to its base to recharge.

Early works concentrated on the case when the stochastic displacement process is a simple Brownian motion [8], i.e., a Markovian process with stationary increments. Later on, one has also studied other types of motion between the resetting events, such as Lévy flights [9,10], continuous-time random walks with or without drift [11–14], and scaled Brownian motion [15,16]. The waiting times between the resetting events are considered to be distributed according to an exponential distribution [8], δ distribution (resetting at a fixed time after starting the stochastic motion) [17,18], power law [19], and other types [20,21].

The main attention was always paid to the probability density function (PDF) of displacements as measured at a given time or in a stationary state, and to the mean first-passage time (MFPT)—the average time necessary to hit a specified target [22,23]. With respect to the first-hitting properties, early work concentrated on searchers performing one-dimensional Brownian motion with Poissonian resetting [8,24,25]. While the MFPT to a target for a diffusing particle in the absence of resetting may diverge, in the presence of resetting it turns finite, and there exists an optimal rate of resetting which minimizes the MFPT. The discussion has been extended to two and higher dimensions in Refs. [26,27]. It has been shown that the search process is most effective in the case of the deterministic resetting, occurring with constant time intervals between the resetting events [17,18,28].

Typically, the return to the initial position (assumed at the origin of the coordinate system in what follows) is considered to be instantaneous. In some studies, a random refractory period after the resetting event has been introduced [29–31], but the return to the origin was still described as an instantaneous jump. However, in many situations, especially when the process considered corresponds to a motion of a material object in space, such an assumption is nonphysical. In what follows we discuss what happens when the return to the origin follows a given equation of motion and takes finite time for completion. The exemplary trajectories of such a process are depicted at Fig. 1 for the case of Brownian motion with return at a constant speed in one dimension (1D). This
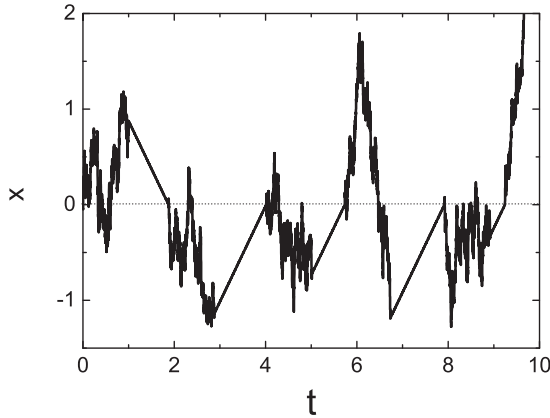
FIG. 1. A trajectory of a particle performing Brownian motion under resetting with the return at a constant speed $v = 1$. The resetting events follow at a fixed time $t_r = 1$ after the beginning of the displacement phase. The diffusion coefficient in the displacement phase is $D = 1$.
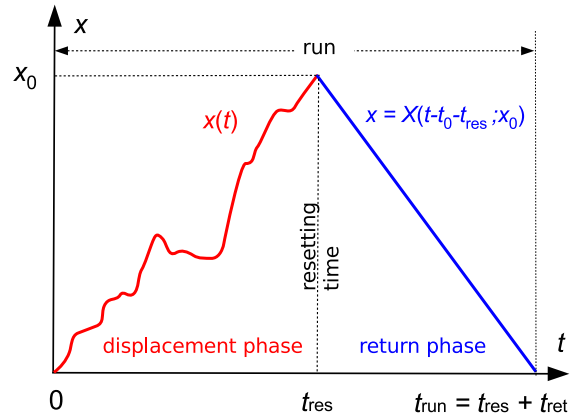

FIG. 2. The notation used. A run stars at time t_0 with a renewal event and consists of two motion phases: the displacement phase and the return phase. The time axis corresponds to the time elapsed from the beginning of the run $t - t_0$. The total duration of the return phase $t_{\text{ret}}(x_0)$ depends on the particle's position x_0 at the instant of resetting and on the equation of motion $x = X(t; x_0)$.

situation was just recently considered by Pal, Kuśmierz, and Reuveni [32–34] and also Masó-Puigdellosas, Campos, and Méndez [35], work that was performed parallel to ours, and the comparison to their work is given below.

In the current study we derive the stationary PDF and the mean squared displacement (MSD) as well as the MFPT for the noninstantaneous return to the origin. We proceed as follows. In the next Sec. II we define the model of our system and give general analytic expressions for the stationary PDF. In Secs. III and IV we consider the normal diffusion under exponential and deterministic resetting, correspondingly. In Sec. V we discuss the conditions under which the PDF remains invariant with respect to the return speed and consider the examples of situations when such an invariance takes place for any resetting time distribution, or when no such resetting time distribution exists. Section VI is devoted to the investigation of the MFPT. Finally, we give our conclusions in Sec. VII.

II. PROBABILITY DENSITY FUNCTION

A. Model

The reset-return process consists of subsequent runs. Each run is a sequence of two processes, the stochastic displacement process $x(t)$ which is interrupted by the resetting event, and the deterministic return process, which ends when the particle returns to the origin, as depicted in Fig. 2. The waiting time density for a resetting event is given by a function $\psi(t_{\text{res}})$, where t_{res} is the time elapsed from the beginning of the run. The motion during the return phase takes place according to the deterministic equation of motion $x = X(t; x_0)$, where t now is the time elapsed since the beginning of the return phase, and x_0 is the particle's coordinate at the beginning of this phase (i.e., at the end of the preceding displacement one). The equation of motion $X(t; x_0)$ is chosen such, that the return from each point x_0 to the origin takes place in a finite time $t_{\text{ret}}(x_0)$, given by the solution of the equation

$$X[t_{\text{ret}}(x_0); x_0] = 0. \qquad (1)$$

The full run finishes by the return to the origin. Then the process restarts anew with the displacement phase of the next run. Therefore, the whole process is a renewal one. The total duration of the run is $t_{\text{run}} = t_{\text{res}} + t_{\text{ret}}$.

B. Duration of the run

Let $\phi(t)$ denote the probability density that the duration of the run is equal to t. It is convenient to work in the Laplace domain. The Laplace transform of the resetting PDF is

$$\tilde{\phi}(s) = \int_0^\infty \phi(t) \exp(-ts) dt. \qquad (2)$$

The probability density $\phi_n(t)$ for the particle to return to the origin during the nth run at time t satisfies the renewal equation [36]

$$\phi_n(t) = \int_0^t \phi_{n-1}(t')\phi(t - t')dt'. \qquad (3)$$

Here we assume that the previous run has ended at time t' and the duration of the very last run is $t - t'$. The sum of all $\phi_n(t)$ gives the renewal rate $\kappa(t)$.

$$\kappa(t) = \sum_{n=1}^\infty \phi_n(t), \qquad (4)$$

which defines that any run has been terminated at time t. The Laplace transform of $\kappa(t)$ yields

$$\tilde{\kappa}(s) = \sum_{n=1}^\infty \tilde{\phi}_n(s) = \frac{\tilde{\phi}(s)}{1 - \tilde{\phi}(s)}. \qquad (5)$$

In the present work we will concentrate only on the situation when the first moment $\langle t_{\text{run}} \rangle = \int_0^\infty t\phi(t)dt$ of $\phi(t)$ does exist and its Laplace transform is given by

$$\langle \tilde{t}_{\text{run}} \rangle = -\lim_{s \to 0} \frac{d}{ds} \int_0^\infty \phi(t) \exp(-ts)dt = -\frac{d\tilde{\phi}(s)}{ds}\Big|_{s=0}. \qquad (6)$$

In such a way we get $\tilde{\phi}(s) = 1 - \langle t_{\mathrm{run}} \rangle s + \dots$. Using Eq. (5) and performing the inverse Laplace transform, we obtain

$$\lim_{t \to \infty} \kappa(t) = \frac{1}{\langle t_{\mathrm{run}} \rangle}. \tag{7}$$

If the first moment $\langle t_{\mathrm{run}} \rangle$ does not exist, then $\kappa(t)$ stays explicitly time-dependent. Such general reset-return processes seem to be rich with respect to possible aging phenomena, whose investigation is left for further work.

The distribution $\psi(t_{\mathrm{res}})$ of resetting times t_{res} is given, and the return time $t_{\mathrm{ret}}(x_0)$ is the deterministic function of the coordinate x_0 at the end of the displacement phase. Therefore,

$$\phi(t_{\mathrm{run}}) = \int_0^\infty dt_{\mathrm{res}} \psi(t_{\mathrm{res}})$$
$$\times \int_{-\infty}^\infty dx_0 \delta[t_{\mathrm{run}} - t_{\mathrm{res}} - t_{\mathrm{ret}}(x_0)] p(x_0 | t_{\mathrm{res}}). \tag{8}$$

Here $p(x|t)$ is the PDF of the displacement during the displacement phase at time t.

The mean value of the run duration $\langle t_{\mathrm{run}} \rangle$ is given by the sum of the duration of the resetting $\langle t_{\mathrm{res}} \rangle$ and return $\langle t_{\mathrm{ret}} \rangle$ phases

$$\langle t_{\mathrm{run}} \rangle = \langle t_{\mathrm{res}} \rangle + \langle t_{\mathrm{ret}} \rangle, \tag{9}$$

$$\langle t_{\mathrm{res}} \rangle = \int_0^\infty t' \psi(t') dt', \tag{10}$$

$$\langle t_{\mathrm{ret}} \rangle = \int_0^\infty dt' \int_{-\infty}^\infty dx_0 t_{\mathrm{ret}}(x_0) p(x_0 | t') \psi(t'), \tag{11}$$

where t_{ret} is given by the solution of Eq. (1).

C. General form of the probability density function

Our next task will be to obtain the displacement PDF at time t,

$$P(x, t) = \int_0^t \kappa(t_0) Q(x; t - t_0) dt_0, \tag{12}$$

where $Q(x; t)$ is the PDF of the position at the measurement time t, and $\kappa(t_0)$ accounts for the fact that the last run has started at time t_0. Let us first take the beginning of the displacement phase at t_0 as a new origin of time. The measurement time is now $\Delta t = t - t_0$. For given t_{res} and x_0 the cases when this measurement time Δt falls into the displacement or into the return phases are mutually excluding. Let us first fix the value of t_{res} (i.e., consider all realizations with reset time in the close vicinity of t_{res}) and get the PDF of displacements conditioned on t_{res} and on the corresponding x_0:

$$q(x|\Delta t; t_{\mathrm{res}}, x_0) = q_1(x|\Delta t; t_{\mathrm{res}}) + q_2(x|\Delta t; t_{\mathrm{res}}, x_0),$$
$$q_1(x|\Delta t; t_{\mathrm{res}}) = p(x|\Delta t) \Theta(t_{\mathrm{res}} - \Delta t),$$
$$q_2(x|\Delta t; t_{\mathrm{res}}, x_0) = \delta[x - X(\Delta t - t_{\mathrm{res}}; x_0)]$$
$$\times \Theta(\Delta t - t_{\mathrm{res}}) \Theta[t_{\mathrm{res}} + t_{\mathrm{ret}} - \Delta t]. \tag{13}$$

Here the Θ function in q_1 represents the condition that the measurement time falls into the displacement phase of the run. The first Θ function in q_2 accounts for the measurement performed during the return phase and the last Θ function in q_2 gives the condition that the measurement occurs before the

particle returns to the origin. To lift the conditioning on x_0 and t_{res} we first average over the distribution of x_0 for given t_{res} and then over the distribution of t_{res}:

$$Q(x; \Delta t) = Q_1(x; \Delta t) + Q_2(x; \Delta t),$$
$$Q_1(x; \Delta t) = \int_0^\infty dt_{\mathrm{res}} \psi(t_{\mathrm{res}}) q_1(x|\Delta t; t_{\mathrm{res}}),$$
$$Q_2(x; \Delta t) = \int_0^\infty dt_{\mathrm{res}} \psi(t_{\mathrm{res}})$$
$$\times \int_{-\infty}^\infty dx_0 q_2(x|\Delta t; t_{\mathrm{res}}, x_0) p(x_0 | \Delta t_{\mathrm{res}}). \tag{14}$$

The normalization of $P(x, t)$ given by Eq. (12) is proved in the Appendix. The MSD can be calculated performing the integration:

$$\langle x^2 \rangle = 2 \int_0^\infty x'^2 P(x') dx'. \tag{15}$$

D. Stationary probability density function

In the present work we concentrate on stationary PDF in the asymptotic limit $t \to \infty$,

$$P(x) = \lim_{t \to \infty} P(x, t) = \int_0^\infty \kappa(t_0) Q(x; t - t_0) dt_0. \tag{16}$$

The consideration of nonstationary PDFs is beyond the scope of the current paper.

Although one cannot exclude that a stationary PDF of coordinate does exist also for nonstationary process of renewals, the situation is really simple only if the renewal resetting process does possess the mean waiting time and is stationary. In this case, $\kappa(t_0)$ is given by Eq. (7) for t_0 sufficiently long, and the expression for $P(x)$ reads

$$P(x) = \frac{\rho_1(x) + \rho_2(x)}{\langle t_{\mathrm{res}} \rangle + \langle t_{\mathrm{ret}} \rangle}, \tag{17}$$

with $\rho_1(x) = \int_0^\infty Q_1(x; t') dt'$ and $\rho_2(x) = \int_0^\infty Q_2(x; t') dt'$. Equation (17) represents a mixture of probability density functions of positions in corresponding phases of motion taken with weights proportional to the duration of the phases. $P_1(x) = \rho_1(x)/\langle t_{\mathrm{run}} \rangle$ and $P_2(x) = \rho_2(x)/\langle t_{\mathrm{run}} \rangle$ account for the probability densities governing the displacement and return phases, correspondingly. Thus, $\rho_1(x)$ and $\rho_2(x)$ are the rescaled conditional PDFs of the displacement and return phase, correspondingly (PDF divided by $\langle t_{\mathrm{run}} \rangle$).

The mean duration of the resetting phase is

$$\langle t_{\mathrm{res}} \rangle = \int_0^\infty \rho_1(x) dx, \tag{18}$$

and the mean duration of the return phase is

$$\langle t_{\mathrm{ret}} \rangle = \int_0^\infty \rho_2(x) dx. \tag{19}$$

If the fraction

$$\frac{\rho_2(x)}{\rho_1(x)} = C, \tag{20}$$

where $C = \text{const}$, then the PDF becomes independent of the return process:

$$P(x) = \frac{\rho_1(x)}{\langle t_{\text{res}} \rangle}. \tag{21}$$

Below we will investigate under which conditions does this statement hold.

E. PDF of the displacement phase

The rescaled PDF of the displacement phase $\rho_1(x)$ is given by

$$\rho_1(x) = \int_0^\infty dt_{\text{res}} \psi(t_{\text{res}}) \int_0^{t_{\text{res}}} p(x|t) dt \tag{22}$$

and can be immediately evaluated since $\psi(t)$ and $p(x|t)$ are known. In the following we consider a displacement process corresponding to standard Brownian motion (pure diffusion without drift) with a PDF

$$p(x|t) = \frac{1}{\sqrt{4\pi Dt}} e^{-\frac{x^2}{4Dt}}. \tag{23}$$

We also investigate two different types of the waiting time distributions of the resetting events: exponential waiting time distribution: $\psi(t) = re^{-rt}$ and deterministic resetting with fixed time interval: $\psi(t) = \delta(t - t_r)$.

F. PDF of the return phase

For the rescaled PDF of the return phase $\rho_2(x)$ we obtain

$$\rho_2(x) = \int_0^\infty dt' \int_0^\infty dt_{\text{res}} \psi(t_{\text{res}})$$
$$\times \int_{-\infty}^\infty dx_0 \delta[x - X(t' - t_{\text{res}}; x_0)]$$
$$\times \Theta(t' - t_{\text{res}}) \Theta[t_{\text{res}} + t_{\text{ret}}(x_0) - t'] p(x_0|t_{\text{res}}). \tag{24}$$

We assume that the equation of motion is such, that the distance to the origin in the return phase is a monotonically decreasing function of time. We moreover assume that the equation of motion $x = X(t - t_{\text{res}}; x_0)$ is symmetric with respect to x_0: For $x_0 > 0$ the value of x stays nonnegative all the time until it reaches the value $x = 0$; for $x_0 < 0$ it stays negative all the time, and $X(t - t_{\text{res}}; -x_0) = -X(t - t_{\text{res}}; x_0)$.

Then $\rho_2(x)$ can be presented as sum of $\rho_2^+(x)$ for $x > 0$ and $\rho_2^-(x)$ for $x < 0$. In the following we will discuss the evaluation of $\rho_2^+(x)$, while the evaluation of ρ_2^- follows along the similar lines. Thus, we assume $x > 0$ and interchange the sequence of integration in t' and in t_{res} in Eq. (24). We note that the first Θ function does not depend on x_0, can be safely moved out from the last integral, and then essentially fixes the lower bound of integration in t'. Then we introduce the new variable of integration $\tau = t' - t_{\text{res}}$ and get

$$\rho_2^+(x) = \int_0^\infty dt_{\text{res}} \psi(t_{\text{res}}) \int_0^\infty dx_0 p(x_0|t_{\text{res}})$$
$$\times \int_0^\infty d\tau \delta[x - X(\tau; x_0)] \Theta(t_{\text{ret}}(x_0) - \tau). \tag{25}$$

Now we note that $t_{\text{ret}}(x_0)$ is essentially the solution of the equation $X(\tau; x_0) = 0$ for given x_0. Thus, the last Θ function

is always unity when the argument of the δ function is zero, and therefore can be omitted:

$$\rho_2^+(x) = \int_0^\infty dt_{\text{res}} \psi(t_{\text{res}})$$
$$\times \int_0^\infty dx_0 p(x_0|t_{\text{res}}) \int_0^\infty d\tau \delta[x - X(\tau; x_0)]. \tag{26}$$

From our monotonicity assumption it follows that the function $X(t; x_0)$ is a monotonically decaying function of time t (the particle steadily approaches the origin) and a monotonically growing function of x_0 (the further from the origin the initial position is, the longer the particle needs to return to the origin). This means that the equation $x - X(t' - t_{\text{res}}; x_0) = 0$ does not possess roots for $x > x_0$ and does possess a single root for $0 < x < x_0$. Therefore, the integral given by Eq. (26) is zero for $x > x_0$. Let now $\tau(x, x_0)$ be the solution of the equation $X(\tau; x_0) = x$ for $x < x_0$. This solution is given by the function inverse to $X(\tau; x_0)$. Now we use the rule of variable change in the δ function and denote the speed at which the particle crosses x on return from x_0 by $|v(x; x_0)|$:

$$\int_0^\infty d\tau \delta[x - X(\tau; x_0)] = \begin{cases} 0 & \text{for} \quad x_0 < x, \\ |v(x; x_0)|^{-1} & \text{for} \quad x_0 > x. \end{cases} \tag{27}$$

The substitution of this expression into Eq. (26) gives us the expression for $\rho_2^+(x)$. Changing the sequence of integration, we obtain the final form for the rescaled PDF of the return motion $\rho_2(x)$,

$$\rho_2(x) = \int_{|x|}^\infty \frac{dx_0}{|v(x; x_0)|} \int_0^\infty dt_{\text{res}} \psi(t_{\text{res}}) p(x_0|t_{\text{res}}). \tag{28}$$

Below we consider explicitly three different types of return motion.

G. Return at a constant speed $v = \text{const}$

As a first example we consider the return at constant speed with

$$x = x_0 - vt. \tag{29}$$

The duration of the return phase is linearly proportional to the absolute value of the starting position of the return motion

$$t_{\text{ret}} = \frac{|x_0|}{v}. \tag{30}$$

Equation (28) then attains the form

$$\rho_2(x) = \frac{1}{v} \int_{|x|}^\infty dx_0 \int_0^\infty dt \, p(x_0|t) \psi(t). \tag{31}$$

H. Return at a constant acceleration $a = \text{const}$

We consider the return motion of a particle of unit mass $m = 1$ under the action of a constant force or in a linear potential $U = ax$. We assume that at the beginning of the return motion the initial speed $v_0 = 0$ and constant acceleration $a > 0$ for $x_0 > 0$ both showing in the direction of the origin (the return motion for $x_0 < 0$ corresponds to $a < 0$):

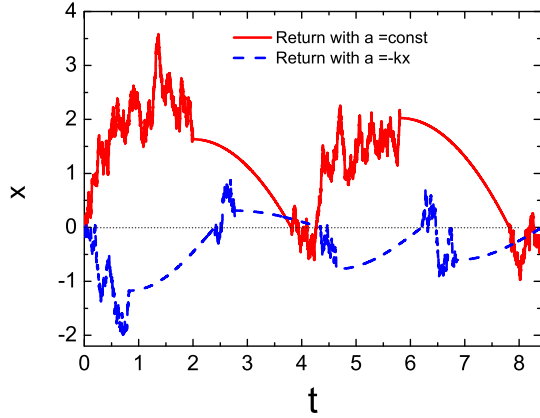$$x(t) = x_0 - at^2/2,$$
$$v(t) = -at. \tag{32}$$

FIG. 3. Typical trajectories for the return with a constant acceleration $a = 1$ (deterministic resetting with $t_r = 2$) and acceleration, proportional to the displacement $a = -kx$, $k = 1$ (exponential resetting with $r = 1$).

Here t is time elapsed from the start of the return motion. The return time is

$$t_{\text{ret}} = \sqrt{\frac{2x_0}{a}}, \tag{33}$$

and the absolute value of the return speed is

$$|v(x; x_0)| = \sqrt{2a(x_0 - x)}. \tag{34}$$

Equation (28) in this case takes the form

$$\rho_2(x) = \sqrt{\frac{1}{2|a|}} \int_{|x|}^{\infty} \frac{dx_0}{\sqrt{|x_0 - x|}} \int_0^{\infty} dt_{\text{res}} \psi(t_{\text{res}}) p(x_0 | t_{\text{res}}). \tag{35}$$

The corresponding trajectories are shown at Fig. 3 with a red line.

I. Return under the action of a harmonic force

Here we consider motion with acceleration proportional to the position of the particle: $a = -kx$ and zero initial velocity. We assume that a particle has a unit mass $m = 1$ and is under the action of the harmonic spring with spring constant k or in a quadratic potential $U = kx^2/2$. The coordinate during the return process is

$$x = x_0 \cos(\sqrt{k}t), \tag{36}$$

where t is time elapsed from the start of the return motion. The absolute value of the velocity of the particle at the position x is

$$v = \sqrt{k(x_0^2 - x^2)}, \tag{37}$$

and the rescaled PDF of the return phase is

$$\rho_2(x) = \int_{|x|}^{\infty} dx_0 \frac{1}{\sqrt{k(x_0^2 - x^2)}} \int_0^{\infty} dt \, p(x_0 | t) \psi(t). \tag{38}$$

The return time attains the value

$$\langle t_{\text{ret}} \rangle = \frac{\pi}{2\sqrt{k}}, \tag{39}$$

which does not depend on the position x_0 in the beginning of the return motion in contrast to the case of return with constant speed [Eq. (30)] and constant acceleration [Eq. (33)]. The corresponding trajectories are shown at Fig. 3 with a blue line.

J. Computer simulations

The analytical predictions for MSD and PDF, as well as for the mean hitting time, discussed later, are compared with numerical simulations directly following from the discretization of Langevin equations in the motion phase and of the equation of motion for the return at a constant speed. The time axis is discretized with the step $dt = t_{i+1} - t_i$, and the time of the first resetting event is generated according to its probability density $\psi(t)$. For the deterministic resetting this resetting time is fixed. During the displacement phase the particle performs stochastic motion according to a finite-difference analogue of the Langevin equation

$$x_{i+1} = x_i + \xi_i \sqrt{2Ddt}. \tag{40}$$

Here $x_i = x(t_i)$ is the coordinate of the particle at time t_i, and ξ_i is the random number distributed according to a standard normal distribution generated using the Box-Muller transform. When the resetting event occurs, the particle starts moving to the origin according to Eq. (29) or Eq. (32) or Eq. (36), depending on the type of the return motion. When the particle crosses the origin, the time of the next resetting event is generated, and the particle starts performing stochastic motion until this resetting event. All simulations are performed with $N = 10^5$ particles. The simulation of the trajectories of the particles is shown in Fig. 1 for the return at constant speed and in Fig. 3 for the return at constant acceleration and under the action of the harmonic force.

III. DIFFUSION WITH EXPONENTIAL RESETTING

Let us consider standard Brownian motion with PDF given by Eq. (23) an exponential waiting time distribution of the resetting events:

$$\psi(t) = r \exp(-rt). \tag{41}$$

The survival probability is $\Psi(t) = e^{-rt}$, the average location at the resetting event is equal to $\langle |x_0| \rangle = \sqrt{D/r}$ and the mean resetting time is

$$\langle t_{\text{res}} \rangle = \frac{1}{r}. \tag{42}$$

The rescaled PDF of the displacement phase $\rho_1(x)$ is given by the Laplace distribution:

$$\rho_1(x) = \frac{1}{2\sqrt{Dr}} \exp\left(-\frac{|x|}{\sqrt{D/r}}\right). \tag{43}$$

A. Return at a constant speed

If the return velocity does not change during the return motion or depends only coordinate x and does not depend on x_0, then ρ_2 is also given by the Laplace distribution

$$\rho_2(x) = \frac{1}{2v} \exp\left(-\frac{|x|}{\sqrt{D/r}}\right), \tag{44}$$
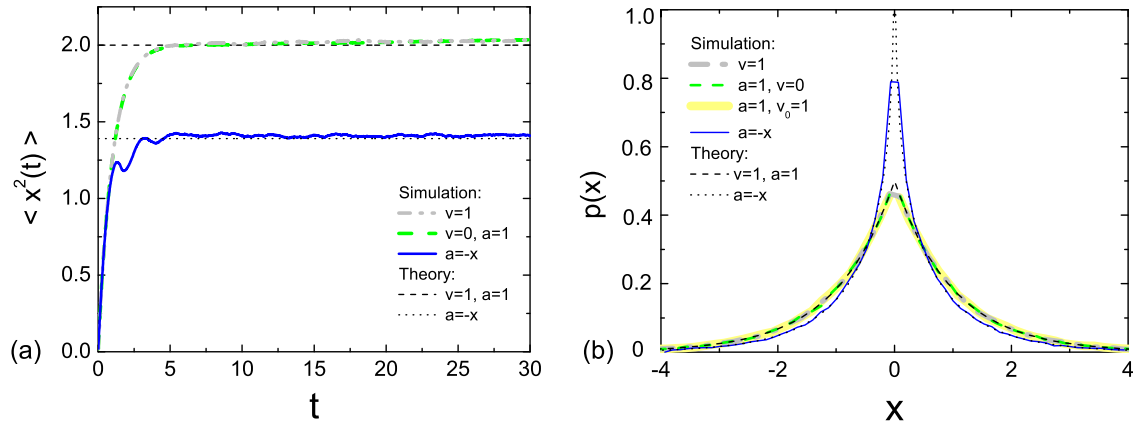
FIG. 4. Mean-squared displacement (a) and probability density function (b) for the exponential resetting with $D = 1$, $r = 1$. The dashed black line in panel (a) shows the analytical asymptotic result for the MSD, Eq. (46), the dashed black line in panel (b) shows the result for the PDF from Eq. (45).

but with a different prefactor compared to the displacement part of the PDF [Eq. (43)]. The condition of the stationarity of the ratio of two integrals given by Eq. (20) is fulfilled with $C = \sqrt{Dr}/v$. The PDF becomes completely invariant with respect to the return process and is the same as PDF of normal diffusion with instantaneous exponential resetting [8]:

$$P(x) = \frac{1}{2\sqrt{D/r}} \exp\left(-\sqrt{\frac{r}{D}}|x|\right). \tag{45}$$

The same effect has been obtained independently in Refs. [33–35], using a different approach. The MSD can be derived according to Eq. (15),

$$\langle x^2 \rangle = \frac{2D}{r}, \tag{46}$$

also independently of the velocity of resetting. The analytical and numerical results for the MSD and the PDF for Brownian motion with exponential resetting are depicted and compared at Fig. 4. Figure 4(b) confirms the result that for the case of the Brownian motion with exponential resetting, the models with instantaneous return and with the return at a constant speed lead to the same stationary probability densities of the coordinate. The MSD rapidly tends to a steady-state value given by Eq. (46) [Fig. 4(a)].

B. Return at $v = v(x_0 - x)$

Let us assume that the velocity of particle depends only on the displacement from the initial position x_0: $v = v(x_0 - x)$. In this case,

$$\rho_2(x) = \frac{\hat{C}}{2} e^{-\sqrt{\frac{r}{D}}|x|}, \tag{47}$$

where

$$\hat{C} = \int_0^\infty \frac{dz}{2v(z)} \sqrt{\frac{r}{D}} e^{-z\sqrt{\frac{r}{D}}}. \tag{48}$$

Therefore, the fraction $\rho_2(x)/\rho_1(x)$ does not depend on the location x, and the PDF has the same form as for the instantaneous resetting [Eq. (45)].

C. Return at a constant acceleration

For the return with a constant acceleration a with zero initial velocity $v_0 = 0$ we obtain

$$\rho_2(x) = \sqrt{\frac{\pi}{8a}} \left(\frac{r}{D}\right)^{\frac{1}{4}} \exp\left(-\frac{|x|}{\sqrt{D/r}}\right). \tag{49}$$

The condition, Eq. (20), is fulfilled again with

$$C = \sqrt{\frac{\pi}{2a}} D^{\frac{1}{4}} r^{\frac{3}{4}}. \tag{50}$$

The average return time can be obtained according to Eq. (19): $\langle t_{\rm ret} \rangle = \sqrt{\frac{\pi}{2|a|}} \left(\frac{D}{r}\right)^{\frac{1}{4}}$. The PDF is again independent on the particular value of the acceleration of the particles and has the same form of the Laplace distribution as PDF for instantaneous resetting [Eq. (45)]. The MSD is the same as well, as shown in Fig. 4(a). If the return motion occurs with constant acceleration a, but the initial return velocity v_0 is nonzero, then the return velocity v is also the function only of the difference $x_0 - x$: $v = 2v_0 - \sqrt{v_0^2 + 2a(x_0 - x)}$. Therefore, the final form of the PDF again remains the same as in the case of the instantaneous resetting [see Fig. 4(b)].

D. Return under the action of a harmonic force

If the velocity explicitly depends on x and x_0, then such an invariance does not hold. Let us consider the return according to the harmonic motion, Eq. (36). The rescaled displacement PDF is in this case equal to

$$\rho_2(x) = \frac{1}{2}\sqrt{\frac{r}{kD}} K_0\left(\frac{|x|}{\sqrt{D/r}}\right), \tag{51}$$

and the PDF is not the same as for the instantaneous resetting [blue line at Fig. 4(b)]:

$$P(x) = \frac{\frac{1}{\sqrt{Dr}} \exp\left(-\frac{|x|}{\sqrt{D/r}}\right) + \sqrt{\frac{r}{kD}} K_0\left(\frac{|x|}{\sqrt{D/r}}\right)}{\frac{2}{r} + \frac{\pi}{\sqrt{k}}}. \tag{52}$$
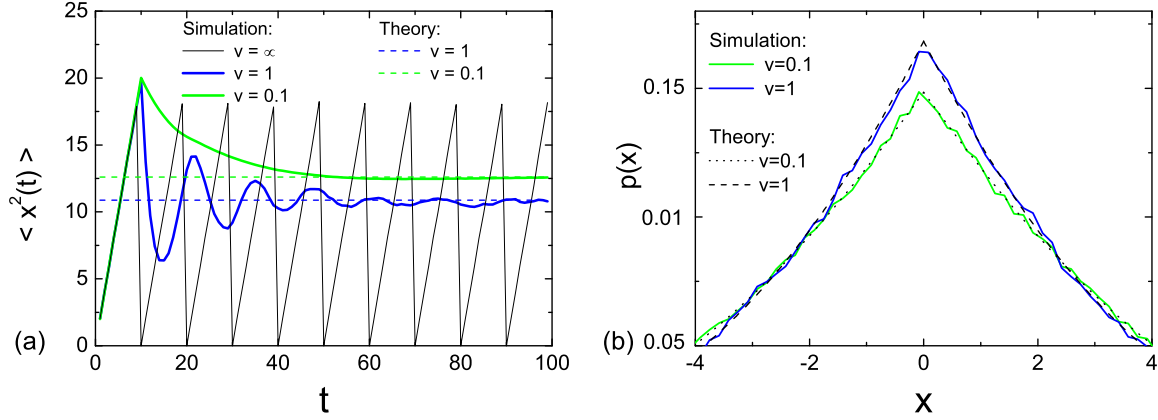
FIG. 5. (a) Mean-squared displacement as a function of time and (b) stationary probability density function at $t = 100$ for the case of fixed-time resetting and the return at constant speed. The dashed lines in panel (a) show the analytical asymptotic results for the MSD, Eq. (58), the dotted and dashed black lines in panel (b) are the results for the PDF from Eq. (57). The resetting time is $t_r = 10$, the diffusion coefficient $D = 1$.

The MSD can be obtained according to Eq. (15):

$$\langle x^2 \rangle = \frac{4D\sqrt{k} + \pi Dr}{2r\sqrt{k} + \pi r^2}. \tag{53}$$

The simulation data for MSD are shown at Fig. 4(a) as a solid blue line. The MSD rapidly tends to the steady state given by Eq. (53).

IV. DIFFUSION WITH DETERMINISTIC RESETTING

Let us now consider normal diffusion with PDF given by Eq. (23) under resetting at fixed time:

$$\psi(t) = \delta(t - t_r). \tag{54}$$

The survival probability for resetting events is now $\Psi(t) = \Theta(t_r - t)$. The mean duration of the displacement phase is $\langle t_{\text{res}} \rangle = t_r$ and the average location at the resetting events $\langle |x_0| \rangle = \sqrt{4Dt_r/\pi}$. The first integral is

$$\rho_1 = \sqrt{\frac{t_r}{\pi D}} \exp\left(-\frac{x^2}{4Dt_r}\right) - \frac{|x|}{2D}\text{erfc}\left(\frac{|x|}{\sqrt{4Dt_r}}\right). \tag{55}$$

A. Return at a constant speed

For the return at constant speed the rescaled return PDF takes the form

$$\rho_2 = \frac{1}{2v}\text{erfc}\left(\frac{|x|}{\sqrt{4Dt_r}}\right), \tag{56}$$

where erfc(x) is the complementary error function. The PDF is

$$P(x) = \frac{\rho_1(x) + \rho_2(x)}{t_r + \frac{1}{v}\sqrt{\frac{4Dt_r}{\pi}}}, \tag{57}$$

with $\rho_1(x)$ given by Eq. (55) and $\rho_2(x)$ given by Eq. (58). The MSD, given by Eq. (15), reads

$$\langle x^2 \rangle = Dt_r\left(1 + \frac{\frac{2}{3}\sqrt{Dt_r}}{t_r v\sqrt{\pi} + 2\sqrt{Dt_r}}\right). \tag{58}$$

The quotient of ρ_2 given by Eq. (56) and ρ_1 given by Eq. (55) is not constant any more and now explicitly depends on x. Consequently, the PDF and the MSD are not invariant with respect to the return velocity, as found for the case of exponential resetting. The results for the PDF and the following MSD are represented in Fig. 5 by dashed lines. The oscillations of the MSD are observed. For instantaneous return these oscillations persist indefinitely. Note that our Eq. (58), by construction, only predicts the time-averaged MSD (and therefore gives a constant value $\langle x^2 \rangle = Dt_r$) but not its oscillations at the pace dictated by the resetting process. For the return at finite velocity the oscillations are damped: If the velocity is high, then the MSD performs several perceptible oscillations when approaching a constant value given by Eq. (58). At small velocities the MSD approaches this value monotonically after the initial overshooting. The final value of the MSD at long times is correctly reproduced by Eq. (58).

B. Return at a constant acceleration

In the case of the return with constant acceleration,

$$\rho_2(x) = \sqrt{\frac{|x|}{16\pi a Dt_r}} e^{-\frac{x^2}{8Dt_r}} K_{1/4}\left(\frac{x^2}{32Dt_r}\right), \tag{59}$$

where $K_{1/4}$ is the modified Bessel function of second kind. The average return time is

$$\langle t_{\text{ret}} \rangle = \sqrt{\frac{2}{\pi a}}(4Dt_r)^{\frac{1}{4}}\Gamma\left(\frac{3}{4}\right). \tag{60}$$

The PDF attains the form

$$P(x) = \frac{\rho_1(x) + \rho_2(x)}{t_r + \sqrt{\frac{2}{\pi a}}(4Dt_r)^{\frac{1}{4}}\Gamma\left(\frac{3}{4}\right)}, \tag{61}$$

with $\rho_1(x)$ given by Eq. (57) and $\rho_2(x)$ given by Eq. (59). The PDF is depicted at Fig. 6(b). It is again not invariant with respect to the acceleration, analogously to the case of the return with constant velocity. The MSD can be obtained by numerical integraton of the PDF given by Eq. (61) according to Eq. (15).
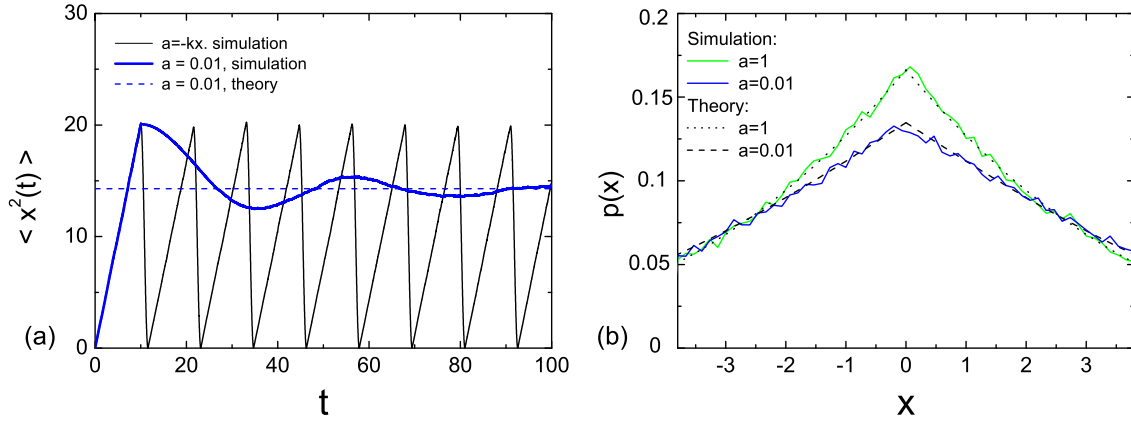
FIG. 6. (a) Mean-squared displacement as a function of time (for the return at constant acceleration and with harmonic motion) and (b) stationary probability density function at $t = 100$ for the case of fixed-time resetting and return with constant acceleration. The dashed lines in panel (a) show the analytical asymptotic results for the MSD, Eq. (58), the dotted and dashed black lines in panel (b) are the analytical results for the PDF obtained by incorporating Eqs. (55) and (59) into Eq. (61). The resetting time is $t_r = 10$, the diffusion coefficient $D = 1$.

C. Return under the action of a harmonic force

For the simple harmonic return motion we obtain

$$\rho_2(x) = \frac{1}{4\sqrt{\pi k D t_r}} K_0\left(\frac{x^2}{8Dt_r}\right) \exp\left(-\frac{x^2}{8Dt_r}\right). \quad (62)$$

The return time is given in terms of Eq. (39). Interestingly, in contrast to the return with constant speed and acceleration, the average return time does not depend on the fixed resetting time t_r and depends only on strength of the spring k. Consequently, the duration of the run remains always the same. The MSD oscillates as in the case of instantaneous deterministic resetting [see Fig. 6(b)]. The oscillations of MSD are not damped due to the fixed duration of the run. Both MSD and PDF never attain a stationary value.

V. THE INVARIANCE OF PDF WITH RESPECT TO THE RETURN VELOCITY

Let us summarize the findings of the previous section. We have observed that the the PDF under exponential resetting is invariant with respect to return speed and acceleration for the case of normal diffusion [33,34]. On the contrary, for resetting at fixed time such invariance does not hold. Therefore, the following interesting question arises: What kind of conditions are necessary for such an invariance to take place? Below we consider only the return at constant speed. The case of the return at a constant acceleration can be treated analogously.

For the return at a constant speed $|v(x; x_0)| = v$ we immediately get from Eq. (31)

$$\rho_2(x) = \frac{\hat{\rho}_2(x)}{v}, \quad (63)$$

$$\hat{\rho}_2(x) = \int_{|x|}^{\infty} dx_0 \int_0^{\infty} dt \, p(x_0|t)\psi(t). \quad (64)$$

The PDF given by Eq. (17) takes the form

$$P(x) = \frac{\rho_1(x) + v^{-1}\hat{\rho}_2(x)}{\langle t_{\text{res}}\rangle + v^{-1}\langle|x_0|\rangle}. \quad (65)$$

We note that

$$\int_0^{\infty} \hat{\rho}_2(x)dx = \langle|x_0|\rangle. \quad (66)$$

Let us start from Eq. (65) and require $P(x)$ on the left-hand side to be independent on v. Taking the derivative of the right-hand side with respect to v^{-1} and setting this derivative equal to zero we get

$$\frac{\hat{\rho}_2(x)}{\rho_1(x)} = \frac{\langle|x_0|\rangle}{\langle t_{\text{res}}\rangle}. \quad (67)$$

The right-hand side of this equation does not depend on x and depends only on the properties of the transport phase, but not on the return speed.

By virtue of Eqs. (18) and (66) we can see that if the quotient $\hat{\rho}_2(x)/\rho_1(x)$ is independent on x, it is equal to the right hand side of the Eq. (67) automatically. Therefore, the only condition for the velocity independence is

$$\frac{\hat{\rho}_2(x)}{\rho_1(x)} = C, \quad (68)$$

where $C = \text{const}$. Hence, we show that the condition given by Eq. (20) is not only sufficient, but also the necessary condition for the invariance of the PDF with respect to the return velocity.

Equation (68), together with Eqs. (22) and (64), defines an awkward equation involving different integrals of $\psi(t)$ and $p(x|t)$. This relation can be reduced to a functional relation between only two functions. Let us fix some $x > 0$ (the case $x < 0$ follows by symmetry). Interchanging the order of integration in t and t_{res} in Eq. (22) we can express the first integral via the survival probability in a displacement phase $\Psi(t) = \int_t^{\infty} \psi(t')dt'$ and the function $F(x|t) = \int_x^{\infty} p(x'|t)dx'$:

$$\rho_1(x) = \int_0^{\infty} dt \, p(x|t) \int_t^{\infty} \psi(t_{\text{res}})dt_{\text{res}}$$

$$= -\int_0^{\infty} \left[\frac{\partial}{\partial x} F(x|t)\right] \Psi(t)dt. \quad (69)$$

In Eq. (64) we first integrate over x_0 and then perform partial integration in time:

$$\hat{\rho}_2(x) = \int_0^\infty \left[\frac{\partial}{\partial t} F(x|t) \right] \Psi(t) dt. \qquad (70)$$

Substituting Eq. (69) and Eq. (70) into Eq. (68), one can formulate the condition of the invariance of the PDF with respect to the return velocity as an orthogonality relation

$$\int_0^\infty \left[C \frac{\partial}{\partial x} F(x|t) + \frac{\partial}{\partial t} F(x|t) \right] \Psi(t) dt = 0. \qquad (71)$$

This means that for given $F(x|t)$ the function $\Psi(t)$ must be orthogonal to the whole family of functions of time $f_x(t) = C \frac{\partial}{\partial x} F(x|t) + \frac{\partial}{\partial t} F(x|t)$ defined on $[0, \infty)$ and parametrized by x for some particular value of C. The fact that $\Psi(t)$ is a survival probability poses considerable restrictions on the possible solution: It has to be a nonnegative, monotonically decaying function of t with $\Psi(0) = 1$. When any of these restrictions is violated, no physical solution $\Psi(t)$ exists.

Equation (71) is quite tricky. Thus, for a Gaussian PDF $p(x|t)$ it is easy to show that it is solved by $\Psi(t) = \exp(-rt)$ and find $C = \sqrt{Dr}$. However, we were not able to show that this exponential solution is unique. In the present work we do not concentrate on general pathway to solution of Eq. (71), but, using the notation adopted in Eqs. (69) and (70), show two simple (somewhat degenerate) examples corresponding to ballistic motion during the displacement phase. These examples demonstrate that there exist situations, in which the PDF $P(x)$ is invariant under return speed for any resetting waiting time PDF, and also situations, in which no waiting time PDF can be found, under which such an invariance takes place.

A. Ballistic motion during the displacement phase with constant velocity

Let us consider the motion taking place at a constant velocity with the probability density function

$$p(x|t) = \delta(x - v_0 t). \qquad (72)$$

The integrals are equal to

$$\rho_1(x) = \frac{1}{v_0} \Psi\left(\frac{x}{v_0} \right), \qquad (73)$$

$$\hat{\rho}_2(x) = \Psi\left(\frac{x}{v_0} \right). \qquad (74)$$

Equation (68) is fulfilled for $C = v_0$ for *any* $\Psi(t)$: This process is invariant with respect to the return velocity for any distribution of the resetting times. The same consideration is valid for the symmetric distribution

$$p(x|t) = \tfrac{1}{2} \delta(x - v_0 t) + \tfrac{1}{2} \delta(x + v_0 t). \qquad (75)$$

B. Ballistic motion during the displacement phase with random velocity

Now we demonstrate an example of $p(x|t)$ for which no matching $\Psi(t)$ can be found. We namely consider a variant of the previous model, but now with the transport velocity which is randomly chosen from the interval $[0, V]$ at each run. The probability density in this case attains the form

$$p(x|t) = (Vt)^{-1} \Theta(Vt - x). \qquad (76)$$

Our integrals are

$$\rho_1(x) = \frac{1}{V} \int_{x/V}^\infty \frac{\Psi(t)}{t} dt, \qquad (77)$$

$$\hat{\rho}_2(x) = \frac{1}{V} \int_{x/V}^\infty \frac{x}{t} \frac{\Psi(t)}{t} dt. \qquad (78)$$

The proportionality of these two integrals means that

$$\frac{C}{V} \int_{x/V}^\infty \frac{\Psi(t)}{t} dt = \int_{x/V}^\infty \frac{x}{Vt} \frac{\Psi(t)}{t} dt. \qquad (79)$$

Introducing a new constant $c = V/C$, and new independent and dependent variables $y = x/V$ and $\eta(t) = \Psi(t)/t$ we get

$$\int_y^\infty \eta(t) dt = cy \int_y^\infty \frac{\eta(t)}{t} dt. \qquad (80)$$

Differentiating both parts with respect to y twice we get

$$\frac{c-1}{c} \eta'(y) = -\frac{\eta(y)}{y}. \qquad (81)$$

The solution of this equation is $\eta(y) = Ay^{-\frac{c}{c-1}}$ so that

$$\Psi(t) = Bt^{-\frac{1}{c-1}}. \qquad (82)$$

This function, however, cannot be a survival probability because if it decays at infinity (which implies $c > 1$) it diverges at zero and therefore $\Psi(0) = 1$ cannot be fulfilled under any circumstances.

VI. MEAN FIRST-PASSAGE TIME

When turning to mean first-hitting (or first-passage) properties of the process, we will consider the situation when the hitting only can take place during the displacement phase. In one dimension this is trivially the case, since the interval of the x axis from the origin to x_0 was already covered by the trajectory of the displacement process. Therefore, if there were any targets on this interval, then they should already be found during the displacement phase. The situation in higher dimensions is richer. Depending on the properties of the searcher, one can assume that, parallel to the 1D situation, the target can only be found during the displacement phase, or during both, displacement and return, phases. We do not consider these high-dimensional situations in the present work. Our approach to the problem will be similar to the one used in Ref. [28].

Let us denote the run as complete, if the target is found and idle or if the target is not found (Fig. 7). The hitting of the target may occur during the first run, during the second run (the first one is idle and the second is complete as shown at Fig. 7) or the third run (two first runs are idle, i.e., finished), etc., and the probability $h(t)$ to reach a target at time t attains the following form:

$$h(t) = \omega(t) + \int_0^t \phi(t')\omega(t - t')dt'$$

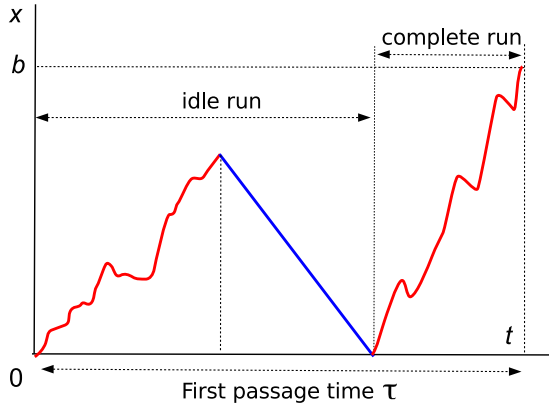$$+ \int_0^t \int_0^{t'} \phi(t'')\phi(t')\omega(t - t' - t'')dt'dt'' + \dots . \qquad (83)$$

FIG. 7. The target is located at $x = b$. The particle starts its motion at the origin $x = 0$. If a target is found during the run, then the run is complete. Otherwise, the run is termed idle.

Here $\omega(t)$ is the probability density of hitting a target at time t after starting a (complete) run, and $\phi(t)$ is a probability density of finishing an idle run at time t after its beginning. Since the series has a form of a sum of multiple convolutions of $\omega(t)$ and $\phi(t)$, this can be easily evaluated in the Laplace domain:

$$\tilde{h}(s) = \tilde{\omega}(s) + \tilde{\phi}(s)\tilde{\omega}(s) + \tilde{\phi}^2(s)\tilde{\omega}(s) + \ldots = \frac{\tilde{\omega}(s)}{1 - \tilde{\phi}(s)}. \tag{84}$$

Here $\tilde{\omega}(s) = \int_0^\infty dt e^{-ts}\omega(t)$ is the Laplace transform of $\omega(t)$. The MFPT τ can be obtained in the following way:

$$\tau = -\frac{d}{ds}\tilde{h}(s)\bigg|_{s=0} = -\frac{\tilde{\omega}'(0)}{1 - \tilde{\phi}(0)} - \frac{\tilde{\omega}(0)\tilde{\phi}'(0)}{[1 - \tilde{\phi}(0)]^2}. \tag{85}$$

Let us introduce the probability $P_c = \tilde{\omega}(0)$ that the run is complete or that the target has been found during the run. $P_i = \tilde{\phi}(0)$ is the probability that a run is finished (idle) and the target has not been found during this run. Each run is either complete or idle: $P_c + P_i = 1$, therefore we get from Eq. (85):

$$\tau = -(\tilde{\omega}'(0) + \tilde{\phi}'(0))/P_c. \tag{86}$$

Taking the derivative of the Laplace transform of ω one can get $\tilde{\omega}'(0) = -\int_0^\infty t\omega(t)dt$. Dividing this by P_c we obtain the mean duration of a complete run: $\langle t_{\text{hit}} \rangle = -\tilde{\omega}'(0)/P_c$. Analogously, for the idle run $\langle t_{\text{run,b}} \rangle = -\tilde{\phi}'(0)/P_i$. Taking into account that $\langle t_{\text{run,b}} \rangle = \langle t_{\text{res}} \rangle + \langle t_{\text{ret,b}} \rangle$, we get

$$\tau = \frac{\langle t_{\text{hit}} \rangle P_c + \langle t_{\text{res}} \rangle P_i + \langle t_{\text{ret,b}} \rangle P_i}{P_c}. \tag{87}$$

Note that the average run and return times, $\langle t_{\text{run,b}} \rangle$ and $\langle t_{\text{ret,b}} \rangle$, differ from the quantities given by Eqs. (9) and (11), correspondingly, because now the motion of the particle is restricted by the adsorbing target, located at $x = b$.

Let us take into account that P_i is the probability that $t_{\text{res}} < t_{\text{hit}}$ and P_c is the probability that the time $t_{\text{hit}} < t_{\text{res}}$. Therefore, Eq. (87) may be rewritten in the following way

$$\tau = \tau_0 + \tau_{\text{ret}}, \tag{88}$$

where

$$\tau_0 = \frac{\langle \min(t_{\text{hit}}, t_{\text{res}}) \rangle}{1 - P_i} \tag{89}$$

is the MFPT for instantaneous resetting, and

$$\tau_{\text{ret}} = \frac{\langle t_{\text{ret,b}} \rangle P_i}{1 - P_i} \tag{90}$$

is additional term, accounting for the return motion. Compared to Eq. (2) of Ref. [32], Eq. (88) does not contain the third term, corresponding to staying home. Eqs. (89) and (90) are, however, awkward for the further usage. Let us perform some modifications of these expressions.

Let $p(x|t; b)$ be the PDF of the particle's positions on a semi-infinite interval with an absorbing boundary (target) at $b > 0$. Then $\Omega(t) = \int_{-\infty}^b p(x|t; b)dx$ is the survival probability of the particle in the interval up to time t, and $\Omega(t_{\text{res}})$ the probability that no hitting took place before resetting at time t_{res}. The hitting time density now can be expressed according to $\omega(t) = -\frac{d}{dt}\Omega(t)$. In Ref. [28] it has been shown that

$$\langle \min(t_{\text{hit}}, t_{\text{res}}) \rangle = \int_0^\infty dt \Psi(t)\Omega(t). \tag{91}$$

The probability of an idle run is

$$P_i = \int_0^\infty dt \psi(t) \int_{-\infty}^b p(x|t; b)dx. \tag{92}$$

Using Eqs. (91) and (92) we get for the MFPT for the instantaneous resetting [28]

$$\tau_0 = \frac{\int_0^\infty dt \Psi(t)\Omega(t)}{1 - \int_0^\infty dt \psi(t)\Omega(t)}. \tag{93}$$

It is easy to see that

$$\langle t_{\text{ret,b}} \rangle P_i = \int_0^\infty \psi(t)dt \int_{-\infty}^b t_{\text{ret}} p(x|t; b)dx. \tag{94}$$

Thus, the additional term, accounting for the return motion, is

$$\tau_{\text{ret}} = \frac{\int_0^\infty \psi(t)dt \int_{-\infty}^b t_{\text{ret}} p(x|t; b)dx}{1 - \int_0^\infty dt \psi(t)\Omega(t)}. \tag{95}$$

The return time t_{ret} is given by Eqs. (30), (33), and (39) for return at constant velocity, constant acceleration and under the action of the harmonic force, correspondingly.

For the case of the Brownian motion our $p(x|t; b)$ reads

$$p(x|t; b) = \frac{1}{\sqrt{4\pi Dt}}\left[\exp\left(-\frac{x^2}{4Dt}\right) - \exp\left(-\frac{(x - 2b)^2}{4Dt}\right)\right] \tag{96}$$

for $x < b$ and vanishes otherwise. The survival probability is

$$\Omega(t) = \text{erf}\left(\frac{b}{2\sqrt{Dt}}\right), \tag{97}$$

where $\text{erf}(b)$ is the error function.

For exponential resetting the first term τ_0 [Eq. (93)] represents the very well known expression for diffusion with instantaneous resetting [8]:
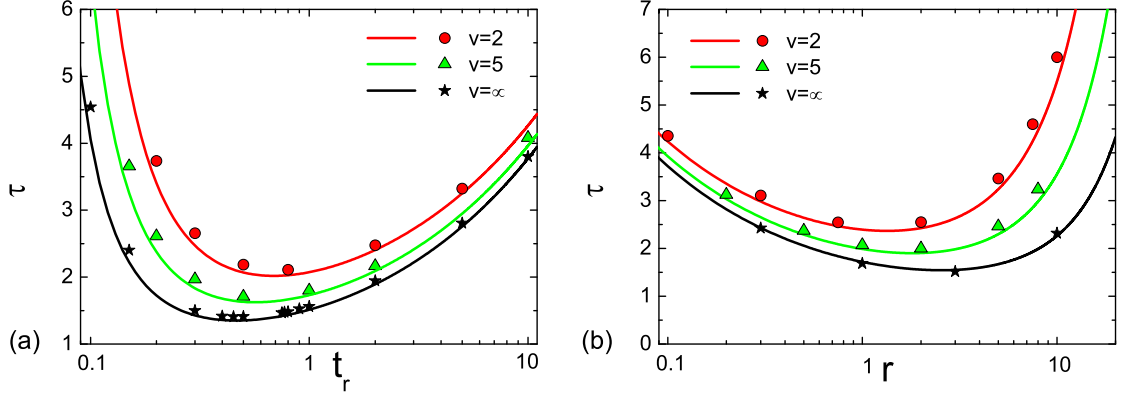
$$\tau_0 = (e^\xi - 1)/r. \tag{98}$$

FIG. 8. Mean first-passage times for the return at constant speed for (a) $\psi(t) = \delta(t - t_r)$, and (b) $\psi(t) = re^{-rt}$. Symbols correspond to the simulation data; solid lines correspond to the analytical result, Eq. (100) with τ_0 and l_v given by Eqs. (99) and (103) for the resetting at fixed time and Eqs. (98) and (102) for the exponential resetting. The target is located at $b = 1$. $v = \infty$ corresponds to the case of instantaneous return.

Here we have introduced a dimensionless parameter $\zeta = b\sqrt{r/D}$, accounting for the ratio between two main length parameters in the system: the distance to the target and the characteristic diffusion length between two subsequent resetting events. The probability, that the target has been found during the run, is $P_i = 1 - e^{-\zeta}$.

The deterministic resetting has been shown to be the most effective search process [17,18,28]. To describe the MFPT, it is useful to introduce a dimensionless parameter $\xi = b/\sqrt{4Dt_r}$. In this case the MFPT for instantaneous resetting [Eq. (93)] takes the form [27]

$$\tau_0 = \frac{t_r(1 + 2\xi^2)\mathrm{erf}(\xi) - 2\xi^2 t_r + \frac{2\xi t_r}{\sqrt{\pi}}\exp(-\xi^2)}{\mathrm{erfc}(\xi)}. \quad (99)$$

The probability that the target has been found during the run is $P_i = \mathrm{erf}(\xi)$.

A. Return at a constant speed

In the case of return with a constant velocity the full return time has the following form:

$$\tau = \tau_0 + \frac{l_v}{v}. \quad (100)$$

Here l_v is the characteristic length of the return paths:

$$l_v = \frac{\int_0^\infty dt\, \psi(t) \int_{-\infty}^b |x| p(x|t; b)dx}{1 - \int_0^\infty dt\, \psi(t)\Omega(t)}. \quad (101)$$

In the case of exponential resetting the parameter l_v is

$$l_v = b\left[\frac{1}{\zeta}(e^\zeta - e^{-\zeta}) - 1\right]. \quad (102)$$

While the PDF for ordinary diffusion with exponential resetting does not depend on the return velocity, the MFPT has an explicit velocity-dependence. The MFPT for exponential resetting has been obtained independently in Ref. [32].

For the deterministic resetting the characteristic length is

$$l_v = \frac{\frac{b}{\sqrt{\pi}\xi}[1 - \exp(-4\xi^2)] + b[1 + \mathrm{erf}(\xi) - 2\mathrm{erf}(2\xi)]}{\mathrm{erfc}(\xi)}. \quad (103)$$

The MFPT τ for both instantaneous return and for return at a constant speed are shown at Fig. 8, demonstrating good agreement between simulation results and theoretical predictions. The MFPT increases with decreasing of the velocity of the ballistic motion toward the origin. The optimal resetting time t_r for the delta distribution of the resetting events increases with decreasing of v: The longer the duration of the return period is, the longer the time period between subsequent resetting events is. Similar effect has been observed for the exponential distribution of the resetting events: The larger the velocity v is, the shorter the optimal average time between resetting events $1/r$ is, which corresponds to larger values of r.

B. Return at a constant acceleration

For the return at constant acceleration and the exponential resetting the correction due to the finite return time is equal to

$$\tau_{\mathrm{ret}} = \sqrt{\frac{\pi b}{2a\zeta}}\left[1 - \frac{1}{\sqrt{\pi}}\Gamma\left(\frac{3}{2}, \zeta\right)\right]e^{-\zeta}$$
$$- \sqrt{\frac{\pi b}{8a\zeta}}e^{-3\zeta}[1 + \mathrm{erfi}(\sqrt{\zeta})] + \sqrt{\frac{b}{2a}}e^{-2\zeta}. \quad (104)$$

Here $\mathrm{erfi}(\zeta)$ is the imaginary error function.

In the case of the deterministic resetting τ_{ret} is given by the following expression:

$$\tau_{\mathrm{ret}} = \left\{\sqrt{\frac{b}{4\pi a\xi}}\left[2\Gamma\left(\frac{3}{4}\right) - \Gamma\left(\frac{3}{4}, \xi^2\right)\right]\right.$$
$$- \frac{\xi\sqrt{b}}{\sqrt{2\pi a}}e^{-\frac{31\xi^2}{8}}\left[K_{\frac{3}{4}}\left(\frac{\xi^2}{8}\right) - K_{\frac{1}{4}}\left(\frac{\xi^2}{8}\right)\right]$$
$$\left. - \frac{\sqrt{2b}\xi}{\sqrt{\pi a}}e^{-4\xi^2}\int_0^1 dz\sqrt{z}\exp(-z^2\xi^2 + z\xi^2)\right\}/\mathrm{erfc}(\xi). \quad (105)$$

Here $K_a(\xi)$ is the modified Bessel function of second kind, and the last integral has to be evaluated numerically.
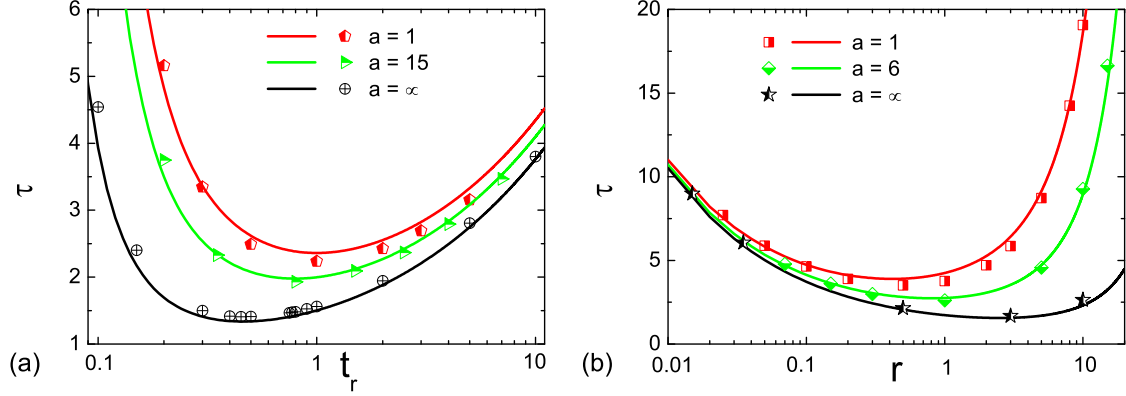
FIG. 9. Mean first-passage times for the return at constant acceleration for (a) deterministic resetting, $\psi(t) = \delta(t - t_r)$, and (b) exponential resetting, $\psi(t) = re^{-rt}$. Symbols correspond to the simulation data, solid lines correspond to the analytical result, Eq. (88) with τ_0 and τ_{ret} given by Eqs. (99) and (105) for the resetting at fixed time and Eqs. (98) and (104) for the exponential resetting. The target is located at $b = 1$. $a = \infty$ corresponds to the case of instantaneous return.

The MFPT for the return at constant acceleration is depicted at Fig. 9. The behavior of MFPT is similar to the case of the resetting at constant velocity. The MFPT grows with increasing of r for exponential resetting and decreasing of t_r for the deterministic resetting. The larger the acceleration the smaller the MFPT. For $a \to \infty$ the MFPT is the same as in the case of the instantaneous resetting.

C. Return under the action of a harmonic force

Let us consider the MFPT for the return with the acceleration proportional to the distance between coordinate and the resetting point: $a = -kx$. The average return time is given by Eq. (39) and does not depend on any parameters of resetting. The correction to MFPT for the exponential resetting it is equal to

$$\tau_{\text{ret}} = \frac{\pi}{2\sqrt{k}}(e^{\zeta} - 1). \qquad (106)$$

For the deterministic resetting,

$$\tau_{\text{ret}} = \frac{\pi}{2\sqrt{k}} \frac{\text{erf}(\xi)}{\text{erfc}(\xi)}. \qquad (107)$$

The comparison between theory and simulations for MFPT is given at Fig. 10. The smaller the spring constant k is, the larger the MFPT is.

VII. CONCLUSIONS

We have considered a model of a resetting process, in which a particle, performing a random motion, returns to its origin not immediately after the resetting event, but, instead, moves toward the origin following a given equation of motion. Once the origin is reached, the particle starts performing the stochastic motion again. In the present work we derived general expressions for the probability density function and mean squared displacement for this process.

In the current study we have specifically discussed the case of the return at a constant speed, at a constant acceleration and under the action of a harmonic force. For a Brownian
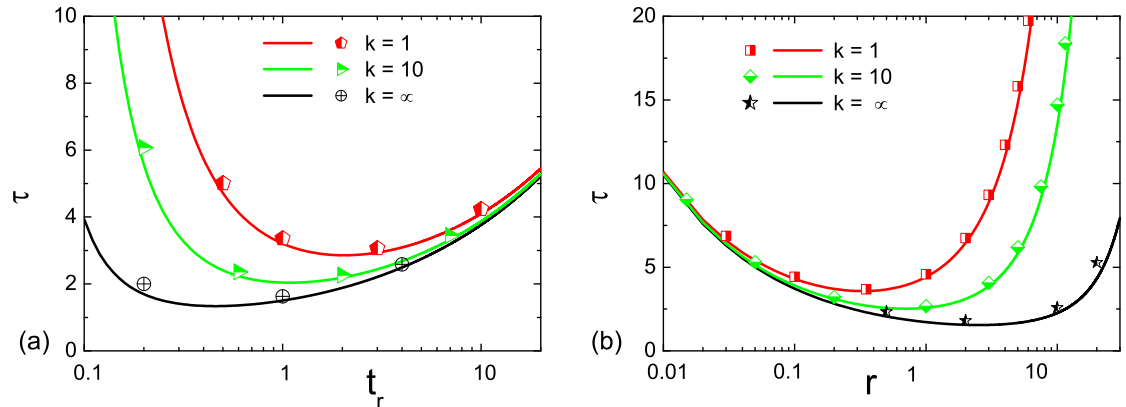

FIG. 10. Mean first-passage times for the return under the action of the harmonic spring. The particle moves with acceleration $a = -kx$ under (a) deterministic resetting, $\psi(t) = \delta(t - t_r)$, and (b) exponential resetting, $\psi(t) = re^{-rt}$. Symbols correspond to the simulation data, solid lines correspond to the analytical result, Eq. (88) with τ_0 and τ_{ret} given by Eqs. (99) and (107) for the resetting at fixed time and Eqs. (98) and (106) for the exponential resetting. The target is located at $b = 1$. $k = \infty$ corresponds to the case of instantaneous return.

motion during the displacement phase, two specific resetting protocols were considered, for which analytical results can be obtained in a closed form: the exponential waiting time distribution between the beginning of the stochastic motion and the reset event, and a deterministic resetting after a fixed time since the beginning of the stochastic motion. Interestingly, for the Brownian motion with the exponential resetting, and for the return at a constant speed, at a constant acceleration, and, in general, with velocity depending only on the distance from the position the particle had at the end of the displacement phase, $v = v(x_0 - x)$, the displacement's PDF does not depend on the corresponding velocity or acceleration. However, if the return velocity explicitly depends on x and x_0 (and not only on their difference), then this invariance does not hold anymore. The invariance also does not hold for the deterministic resetting.

We have addressed the question of the invariance of the PDF with respect to the return speed in a general setting. While using our general expressions it is easy to prove whether a specific pair of displacement PDF in the transport phase and waiting time PDF in the resetting process satisfies the condition for such an invariance, the problem of finding a waiting time PDF under which a stationary PDF under resetting will be invariant with respect to return speed for a given $p(x|t)$ is tricky and worth further investigation. However, we discussed two simple examples showing that there exist situations in which such an invariance takes place under a whatever waiting time PDF, and also situations in which such an invariance can never hold.

We have also considered the mean first-passage times to a given target for Brownian motion under exponential and deterministic resetting. This mean first-passage times depend on the return velocity, on the return acceleration or on the spring constant, correspondingly. The reason is that the first hitting cannot take place during the return phase which passes through a region which was already visited in the stochastic motion. Therefore, the smaller the velocity of return, the acceleration or the spring constant, the longer is the time necessary to find a target. As a consequence, the optimal time between the two resetting events increases with decreasing return speed, acceleration, or spring constant.

APPENDIX: NORMALIZATION OF THE PROBABILITY DISTRIBUTION FUNCTION

The survival probability $\Phi(t)$ that the run did not finish until time t is

$$\Phi(t) = 1 - \int_0^t \phi(t')dt' = \int_t^\infty \phi(t')dt'. \quad \text{(A1)}$$

Performing the integration of Eq. (8) over t_{run}, one can get

$$\Phi(t) = \int_0^\infty dt_{\text{res}}\psi(t_{\text{res}})\int_{-\infty}^\infty dx_0\Theta[t_{\text{res}} + t_{\text{ret}}(x_0) - t]p(x_0|t_{\text{res}}). \quad \text{(A2)}$$

Let t be the measurement time and t_0 the starting time of the last run. Then the probability density of t_0 is given by

$$g(t_0) = \kappa(t_0)\Phi(t - t_0). \quad \text{(A3)}$$

Here $\kappa(t_0)$ is the probability density that the run has started at time t_0 and $\Phi(t - t_0)$ accounts for the fact that it has not been terminated before t. The normalization of $g(t_0)$ [Eq. (A3)] follows by noting that in the Laplace domain $\tilde{\Phi}(s) = s^{-1}[1 - \tilde{\phi}(s)]$. Therefore, $I(t) = \int_0^t g(t_0)dt_0 = \int_0^t \kappa(t_0)\Phi(t - t_0)dt_0$ has a form of the convolution, and its Laplace transform $I(s) = \tilde{\kappa}(s)\tilde{\Phi}(s)$ is equal to s^{-1}, which corresponds to $I(t) = 1$ in the time domain. Hence,

$$\int_0^t g(t_0)dt_0 = \int_0^t \kappa(t_0)\Phi(t - t_0)dt_0 = 1. \quad \text{(A4)}$$

Now from Eqs. (13) and (14) one can explicitly write down the equations for $Q_1(x; \Delta t)$ and $Q_2(x; \Delta t)$:

$$Q(x; \Delta t) = Q_1(x; \Delta t) + Q_2(x; \Delta t),$$

$$Q_1 = \int_0^\infty p(x|\Delta t)\Theta(t_{\text{res}} - \Delta t)\psi(t_{\text{res}})dt_{\text{res}},$$

$$Q_2 = \int_0^\infty dt_{\text{res}}\psi(t_{\text{res}})\int_{-\infty}^\infty dx_0\delta[x - X(\Delta t - t_{\text{res}}; x_0)]$$
$$\times \Theta(\Delta t - t_{\text{res}})\Theta[t_{\text{res}} + t_{\text{ret}}(x_0) - \Delta t]p(x_0|t_{\text{res}}). \quad \text{(A5)}$$

The fact that $P(x, t)$ [Eq. (12)] is normalized can be seen by integrating Eq. (A4) over x. Integrating $Q(x; \Delta t)$, Eq. (A5), over x we get

$$\int_{-\infty}^\infty Q(x; \Delta t)dx = \int_0^\infty dt_{\text{res}}\psi(t_{\text{res}})\bigg[\Theta(t_{\text{res}} - \Delta t)$$
$$+ \Theta(\Delta t - t_{\text{res}})\int_{-\infty}^\infty dx_0\Theta$$
$$\times [t_{\text{res}} + t_{\text{ret}}(x_0) - \Delta t]p(x_0|t_{\text{res}})\bigg]. \quad \text{(A6)}$$

Now we note that $\Theta(-y) = 1 - \Theta(y)$ for any real y. Making this substitution for the last Θ function in the last expression and using that $\int_{-\infty}^\infty dx_0 p(x_0|t_{\text{res}}) = 1$ we get

$$\int_{-\infty}^\infty Q(x; \Delta t)dx = \int_0^\infty dt_{\text{res}}\psi(t_{\text{res}})$$
$$\times \bigg[1 - \int_{-\infty}^\infty dx_0\Theta[\Delta t - t_{\text{res}} - t_{\text{ret}}(x_0)]$$
$$\times p(x_0|t_{\text{res}})\bigg]$$
$$= \int_0^\infty dt_{\text{res}}\psi(t_{\text{res}})\int_{-\infty}^\infty dx_0\Theta$$
$$\times [t_{\text{res}} + t_{\text{ret}}(x_0) - \Delta t]p(x_0|t_{\text{res}}), \quad \text{(A7)}$$

which coincides with our expression for $\Phi(\Delta t)$, Eq. (A2). The overall normalization then follows from Eq. (A4).

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